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DESIGN OF REDUCED ORDER H_∞ FILTERS FOR DISCRETE TIME SYSTEMS

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Abstract: In this contribution the design of reduced order H_∞ filters of order n-κ is investigated for nth order discrete time systems with m measurements of which κ are undisturbed. Assuming a reduced order observer structure for the H_∞ filter, the filter gains achieving H_∞ optimal *a priori* estimates $\hat{z}(k)$ and *a posteriori* estimates $\hat{z}^+(k)$ are derived using the Bounded Real Lemma. A simple example demonstrates the proposed design procedure.

Keywords: H_∞ filtering, discrete time systems, singular filtering problem, Riccati equation approach.

1. INTRODUCTION

The application of H_∞ filters, which estimate some linear combination of the system states in the H_∞ norm minimization sense, is appropriate if there is little knowledge of the statistics of the driving and of the measurement noise signals. When compared to minimum variance estimators (Kalman filters) they are less sensitive to uncertainty in the system parameters (Shaked and Theodor, 1992). Apart from their relevance for optimal estimation problems H_∞ filters are also of importance in the solution of the H_∞ control problem, where for the calculation of the H_∞ output feedback controller an H_∞ estimator has to be found for an H_∞ state feedback control law in the presence of a worst case disturbance (Zhou and Doyle, 1996).

The H_∞ filtering problem was first considered by Grimble (1988) and by Shaked (1990) using a frequency domain approach. A solution of the H_∞ filtering problem in the framework of the Riccati equation approach is given in (Zhou and Doyle, 1996). The corresponding theory has also been developed in the discrete time case (see e.g. (Basar, 1991; Yaesh and Shaked, 1991)).

This paper considers the time domain design of reduced order H_∞ filters for discrete time systems, where κ of the m measurements y of the nth order plant are not affected by disturbances. The resulting filter is of order n-κ, since it suffices to build an (n-κ)th order observer to reconstruct to whole system state. Assuming a reduced order observer structure for the filter, the filter gain which achieves a prescribed H_∞ norm bound for the estimation error is obtained from the Bounded Real Lemma given in (de Souza and Xie, 1992). The H_∞ estimation problem can be solved under various patterns of information. In this contribution *a priori* and *a posteriori* H_∞ filtering are considered. The *a priori* H_∞ filter uses the measurements in a one step delay, whereas the *a posteriori* H_∞ filter uses the current measurements in order to generate the desired estimate. When using such *a posteriori* H_∞ filters the H_∞ norm bound may be lower than the one that is obtained by *a priori* H_∞ filters.

After introducing the reduced order observer schemes employed in the next section, Section 3 gives a short formulation of the underlying H_∞ estimation problems. On the basis of the Bounded Real Lemma (de Souza and Xie, 1992) the *a priori* H_∞ filter is derived in Section 4. Using the results of Section 3 the *a posteriori* H_∞ filter is presented in Section 5. A simple demonstrating example follows in Section 6 and Section 7 contains some concluding remarks.

2. PRELIMINARIES

Consider a time invariant, discrete time, linear system of nth order with m_z unmeasurable outputs z, m measurements y, and q ≥ m disturbances w represented by

$$\begin{aligned} x(k+1) &= Ax(k)+Gw(k) \quad , \quad x(0)=0 \\ z(k) &= C_z x(k) \\ y(k) &= \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(k) + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w(k) \\ &= C_y x(k) + Dw(k) \end{aligned} \tag{1}$$

where C_y is supposed to have full row rank. The output y is subdivided such that y_1 contains the m-κ disturbed measurements and y_2 the κ perfect ones with $0 \le \kappa \le m$. It is assumed that the pair (C_y, A) is detectable.

Further consider a reduced order state observer of order n-κ for the system (1) (Luenberger, 1971), namely

$$\xi(k+1) = F\xi(k) + [H_1 \quad H_2] \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \tag{2}$$

In an undisturbed steady state $\xi(k) = Tx(k)$ holds when $TA - FT = [H_1 \quad H_2]C_y$. Using the undisturbed measurements $y_2(k)$ together with the observer state $\xi(k)$ the state estimate results as

$$\hat{x}(k) = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2(k) \\ \xi(k) \end{bmatrix} = [\Psi_2 \quad \Theta] \begin{bmatrix} y_2(k) \\ \xi(k) \end{bmatrix} \tag{3}$$

such that an *a priori* estimate for the unmeasurable output z(k) is given by $\hat{z}(k) = C_z \hat{x}(k)$.

As a consequence of (3), the relations

$$\begin{bmatrix} C_2 \\ T \end{bmatrix} [\Psi_2 \quad \Theta] = \begin{bmatrix} C_2\Psi_2 & C_2\Theta \\ T\Psi_2 & T\Theta \end{bmatrix} = \begin{bmatrix} I_\kappa & 0 \\ 0 & I_{n-\kappa} \end{bmatrix} \tag{4}$$

and

$$\Psi_2 C_2 + \Theta T = I_n \tag{5}$$

hold. With L_1 such that $TL_1 = H_1$ an alternative representation (see (Gelb, 1974)] of the observer equation (2) is

$$\begin{aligned} \xi(k+1) = T(A - L_1C_1)\Theta\xi(k) + \\ + T[L_1 \ , (A - L_1C_1)\Psi_2] \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \end{aligned} \tag{6}$$

If $L_1 = A\lambda_1$ this equation can be written as

$$\begin{aligned} \xi(k+1) = TA(I - \lambda_1C_1)\Theta\xi(k) + \\ + TA[\lambda_1 \ , (I - \lambda_1C_1)\Psi_2] \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \end{aligned}$$

and if one is interested in also using the current (disturbed) measurements $y_1(k)$ to reconstruct z(k), the *a posteriori* estimate $\hat{z}^+(k) = C_z\hat{x}^+(k)$ with $\hat{x}^+(k) = \hat{x}(k) + \lambda_1(y_1(k) - C_1\hat{x}(k))$, can be employed (Anderson and Moore, 1979; Hippe and Wurmthaler, 1990). This estimate gives a reduced estimation error covariance for the Kalman filter in a stochastic setting and it can also give a reduced infimal value γ for the H_∞ filter (see below).

3. PROBLEM FORMULATION

Given m measurements y find an H_∞ filter for the system (1) that generates an estimate $\hat{z}(k)$ for the unmeasurable m_z linear combinations z(k) of the state x(k) in the H_∞ norm minimization sense. With $l_2[0,\infty)$ denoting the set of real square summable functions on the interval $[0,\infty)$ define the (worst case) performance measure

$$J = \sup_{\substack{w \in l_2[0,\infty) \\ w \neq 0}} \frac{\|z - \hat{z}\|_2}{\|w\|_2} = \|T_{\varepsilon w}\|_\infty \tag{7}$$

when using the *a priori* estimate $\hat{z}(k)$, and in the case of an *a posteriori* estimate $\hat{z}^+(k)$ use

$$J^+ = \sup_{\substack{w \in l_2[0,\infty) \\ w \neq 0}} \frac{\|z - \hat{z}^+\|_2}{\|w\|_2} = \|T^+_{\varepsilon w}\|_\infty \tag{8}$$

In (7) and (8) $T_{\varepsilon w}$ and $T^+_{\varepsilon w}$ denote the filter transfer function matrices from the disturbance w to the estimation errors $\varepsilon = z - \hat{z}$ and $\varepsilon^+ = z - \hat{z}^+$, respectively. Now consider the following (sub-optimal) singular H_∞ filtering problems

1) *A priori filtering*: For a given limit $\gamma > 0$ find a stable filter, if it exists, yielding the *a priori* estimate $\hat{z}(k)$ such that $J \le \gamma$.

2) *A posteriori filtering*: For a given limit $\gamma > 0$ find a stable filter, if it exists, yielding the *a posteriori* estimate $\hat{z}^+(k)$ such that $J^+ \le \gamma$.

4. THE REDUCED ORDER *A PRIORI* H_∞ FILTER

First the case of the *a priori* estimate $\hat{z}(k)$ is considered. When using the estimate (3) to get $\hat{z}(k) = C_z \hat{x}(k)$, the estimation error transfer function matrix $T_{\varepsilon w}$ is given by

$$T_{\varepsilon w}(z) = C_z (zI - \Theta T(A - L_1 C_1))^{-1} \Theta T(G - L_1 D_1) \quad (9)$$

or since $(zI - \Theta T(A - L_1C_1))^{-1}\Theta T =$

$$= \Theta T\, (zI - (A - L_1 C_1)\Theta T\,)^{-1}$$

this is equivalent to

$$T_{\varepsilon w}(z) = C_z \Theta T(zI - (A - L_1 C_1)\Theta T)^{-1} (G - L_1 D_1) \quad (10)$$

Now consider the transfer function matrix

$$F(z) = \overline{C}(zI - \overline{A})^{-1}\overline{B} + \overline{D} \quad (11)$$

of a linear discrete time system with state space realization $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$.

A bound on the H_∞ norm of F as defined by (11) is provided by the following *Discrete-time Bounded Real Lemma*, which is a dual result of the lemma given in (de Souza and Xie, 1992).

Lemma 1: *The following statements are equivalent*:

(a) $\overline{A}$ *is a stable matrix and*

$$\left\| \overline{C}(zI - \overline{A})^{-1}\overline{B} + \overline{D} \right\|_\infty \le \gamma\,;$$

(b) $(\overline{A}, \overline{B})$ *has no uncontrollable modes on the unit circle, and there exists a solution* $\overline{P} = \overline{P}^T \ge 0$ *to the algebraic Riccati equation (ARE)*

$$\overline{P} = \overline{A}\overline{P}\overline{A}^T + \overline{B}\overline{B}^T - (\overline{A}\overline{P}\overline{C}^T + \overline{B}\overline{D}^T) \quad (12)$$
$$(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)^{-1}(\overline{C}\overline{P}\overline{A}^T + \overline{D}\overline{B}^T)$$

such that $-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T < 0$ *and* $\overline{P}$ *being a stabilizing solution, that is* $\overline{A} - (\overline{A}\overline{P}\overline{C}^T + \overline{B}\overline{D}^T)$ $(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)^{-1}\overline{C}$ *has all its eigenvalues inside the closed unit disc.*

In the following Lemma 1 will be applied to the error transfer function matrix $T_{\varepsilon w}$ to obtain the gain matrices characterizing the reduced order a priori H_∞ filter.

Theorem 1: *Consider the system* (1) *with* κ *perfect measurements* y_2. *Then the singular* H_∞ *filtering problem giving an a priori estimate for the unmeasurable output z is solved by the reduced order filter of order n -*κ *(if it exists)*

$$\xi(k+1) = T(A - L_1C_1)\Theta\xi(k) + $$
$$+ T[L_1\,,\quad (A - L_1C_1)\Psi_2]\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \quad (13)$$

$$\hat{z}(k) = C_z \Psi_2 y_2(k) + C_z \Theta \xi(k) \quad (14)$$

where the filter gain matrices result via

$$L = [\underset{n,m_z}{L_z}\,,\quad \underset{n,n-\kappa}{L_1}\,,\quad \underset{n,\kappa}{L_2}] =$$
$$= (A\overline{P}C^T + S_f)\,(R_f + C\overline{P}C^T)^{-1} \quad (15)$$

and

$$\Psi_2 = (A - L_z C_z - L_1 C_1)^{-1}\, L_2 \quad (16)$$

with the abbreviations

$$C = \begin{bmatrix} C_z \\ C_1 \\ C_2 \end{bmatrix};\quad R_f = \begin{bmatrix} -\gamma^2 I_{m_z} & 0 & 0 \\ 0 & D_1 D_1^T & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad (17)$$

$$S_f = \begin{bmatrix} 0 & G D_1^T & 0 \end{bmatrix}$$

and $\overline{P} = \overline{P}^T \ge 0$ *is a stabilizing solution of the ARE*

$$\overline{P} = A\overline{P}A^T + GG^T - \quad (18)$$
$$- (A\overline{P}C^T + S_f)(R_f + C\overline{P}C^T)^{-1}(C\overline{P}A^T + S_f^T)$$

such that $R_f + C\overline{P}C^T < 0$, *and* $((A - L_1C_1)\Theta T,\ G - L_1D_1)$ *has no uncontrollable modes on the unit circle.*

After solving $T\Psi_2 = 0$ *with* T *having full row rank the matrix* Θ *is obtained from*

$$\begin{bmatrix} \Psi_2 & \Theta \end{bmatrix} = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1}. \quad (19)$$

Proof: For the proof of Theorem 1 one uses Lemma 1. Introducing the abbreviation

$$L_z = (\overline{A}\overline{P}\overline{C}^T + \overline{B}\overline{D}^T)(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)^{-1} \quad (20)$$

in (12) and adding the vanishing term

$$L_z(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)L_z^T - (\overline{A}\overline{P}\overline{C}^T + \overline{B}\overline{D}^T)L_z^T +$$
$$+ L_z(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)L_z^T - L_z(\overline{C}\overline{P}\overline{A}^T + \overline{D}\overline{B}^T)$$

to the right hand side of the ARE (12) gives

$$\overline{P} = \overline{A}\overline{P}\overline{A}^T + \overline{B}\overline{B}^T - (\overline{A}\overline{P}\overline{C}^T + \overline{B}\overline{D}^T)L_z^T - \quad (21)$$
$$- L_z(\overline{C}\overline{P}\overline{A}^T + \overline{D}\overline{B}^T) + L_z(-\gamma^2 I + \overline{D}\overline{D}^T + \overline{C}\overline{P}\overline{C}^T)L_z^T$$

Now substituting (compare (10) and (11) and observe $\Theta T = I - \Psi_2 C_2$ (see (5)))

$$\bar{A} = (A - L_1 C_1)(I - \Psi_2 C_2)\,; \quad \bar{B} = G - L_1 D_1\,;$$

$$\bar{C} = C_z (I - \Psi_2 C_2)\,; \quad \bar{D} = 0$$

in (21) the resulting expression can be ordered such that the fictitious feedback matrix

$$L_2 = (A - L_z C_z - L_1 C_1)\Psi_2 \tag{22}$$

can be introduced, giving

$$\bar{P} = A\bar{P}A^T + GG^T - \gamma^2 L_z L_z^T + L_1 D_1 D_1^T L_1^T +$$

$$+ [L_z C_z + L_1 C_1 + L_2 C_2]\bar{P}[L_z C_z + L_1 C_1 + L_2 C_2]^T -$$

$$- [L_z C_z + L_1 C_1 + L_2 C_2]\bar{P}A^T - L_1 D_1 G^T -$$

$$- A\bar{P}[L_z C_z + L_1 C_1 + L_2 C_2]^T - G D_1^T L_1^T$$

This can be reassembled as

$$\begin{aligned}\bar{P} &= A\bar{P}A^T + GG^T - (A\bar{P}C^T + S_f)(R_f + C\bar{P}C^T)^{-1}(C\bar{P}A^T + S_f^T) + \\ &+ \{L - (A\bar{P}C^T + S_f)(R_f + C\bar{P}C^T)^{-1}\}(R_f + C\bar{P}C^T) \\ &\quad \{L^T - (R_f + C\bar{P}C^T)^{-1}(C\bar{P}A^T + S_f^T)\}\end{aligned} \tag{23}$$

with L, C, R_f and S_f as defined in (15) and (17). This shows that when choosing L according to (15), the ARE (12) is satisfied if $\bar{P}$ results from the ARE (18). Furthermore the solution is a stabilizing solution in the sense of part b) of Lemma 1, since

$$\bar{A} - (\bar{A}\bar{P}\bar{C}^T + \bar{B}\bar{D}^T)(-\gamma^2 I + \bar{D}\bar{D}^T + \bar{C}\bar{P}\bar{C}^T)^{-1}\bar{C} =$$

$$= \bar{A} - L_z \bar{C} =$$

$$= (A - L_1 C_1 - L_z C_z)(I - \Psi_2 C_2) = A - LC$$

(see (20), (15) and (16)).

The stability of the filter is assured by Lemma 1, which states that $\bar{A} = (A - L_1 C_1)\Theta T$ is stable if condition (b) is satisfied. An application of the similarity transformation

$$\begin{bmatrix} C_2 \\ T \end{bmatrix} \bar{A} \begin{bmatrix} \Psi_2 & \Theta \end{bmatrix} =$$

$$= \begin{bmatrix} C_2(A - L_1 C_1)\Theta T \Psi_2 & C_2(A - L_1 C_1)\Theta T \Theta \\ T(A - L_1 C_1)\Theta T \Psi_2 & T(A - L_1 C_1)\Theta T \Theta \end{bmatrix} =$$

$$= \begin{bmatrix} 0_\kappa & C_2(A - L_1 C_1)\Theta \\ 0 & T(A - L_1 C_1)\Theta \end{bmatrix}$$

(see (19)) and observing the relations (4) shows that κ eigenvalues of $\bar{A}$ are located at z = 0 while the remaining are the stable optimal filter eigenvalues.

5. THE REDUCED ORDER *A POSTERIORI* H_∞ FILTER

Also taking the disturbed measurements $y_1(k)$ to reconstruct the state x(k) at time instant k, the *a posteriori* estimate $\hat{z}^+(k)$ gives an H_∞ norm bound which may be lower than the one that results from the *a priori* estimate $\hat{z}(k)$ (i.e. $J^+ < J$). The equations for the reduced order H_∞ filter are in this case

$$\xi(k+1) = TA(I - \lambda_1 C_1)\Theta\xi(k) + + TA[\lambda_1\,, \quad (I - \lambda_1 C_1)\Psi_2]\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \tag{24}$$

with $\hat{x}(k) = \Psi_2 y_2(k) + \Theta\xi(k)$ and the *a posteriori* estimate (Anderson and Moore, 1979) given by

$$\hat{z}^+(k) = C_z \hat{x}(k) + C_z \lambda_1 (y_1(k) - C_1 \hat{x}(k)) \tag{25}$$

The error transfer function matrix now has the form

$$T_{\varepsilon w}^+(z) = C_z (I - \lambda_1 C_1)(zI - \Theta TA(I - \lambda_1 C_1))^{-1} \Theta T(G - A\lambda_1 D_1) - C_z \lambda_1 D_1 \tag{26}$$

which can be rewritten as

$$T_{\varepsilon w}^+(z) = C_z (I - \lambda_1 C_1)\Theta T(zI - A(I - \lambda_1 C_1)\Theta T)^{-1} (G - A\lambda_1 D_1) - C_z \lambda_1 D_1 \tag{27}$$

Using Lemma 1 the gain matrices for the *a posteriori* H_∞ filter can be obtained from the following results.

Theorem 2: *Consider the system* (1) *with* κ *perfect measurements* y_2. *Then the singular* H_∞ *filtering problem with a posteriori estimate is solved by the reduced order filter of order n -*κ *(if it exists)*

$$\xi(k+1) = TA(I - \lambda_1 C_1)\Theta\xi(k) + + TA[\lambda_1\,, \quad (I - \lambda_1 C_1)\Psi_2]\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} \tag{28}$$

$$\hat{x}(k) = \Psi_2 y_2(k) + \Theta\xi(k) \tag{29}$$

and

$$\hat{z}^+(k) = C_z \hat{x}(k) + C_z \lambda_1 (y_1(k) - C_1 \hat{x}(k)) \tag{30}$$

where the filter gain matrices result via

$$L = [\underset{n,m_z}{L_z}\,, \quad \underset{n,n-\kappa}{L_1}\,, \quad \underset{n,\kappa}{L_2}] = (A\bar{P}C^T + S_f)(R_f + C\bar{P}C^T)^{-1} \tag{31}$$

$$\lambda_1 = (A - L_z C_z)^{-1} L_1 \tag{32}$$

and

$$\Psi_2 = (A - L_z C_z - L_1 C_1)^{-1} L_2 \tag{33}$$

with the abbreviations (17) and $\overline{P} = \overline{P}^T \geq 0$ *is a stabilizing solution of the ARE*

$$\overline{P} = A\overline{P}A^T + GG^T - \qquad (34)$$

$$-(A\overline{P}C^T + S_f)(R_f + C\overline{P}C^T)^{-1}(C\overline{P}A^T + S_f^T)$$

such that $R_f + C\overline{P}C^T < 0$*, and* $(A(I - \lambda_1C_1)\Theta T,\ G - A\lambda_1D_1)$ *has no uncontrollable modes on the unit circle. After solving* $T\Psi_2 = 0$ *with* T *having full row rank the matrix* Θ *is obtained from (19).*

Proof: As in the proof of Theorem 1, one uses Lemma 1, introduces the abbreviation (20) and obtains the equation (21). Here the matrices

$$\overline{A} = A(I - \lambda_1C_1)(I - \Psi_2C_2)\,; \quad \overline{B} = G - A\lambda_1D_1\,;$$

$$\overline{C} = C_z(I - \lambda_1C_1)(I - \Psi_2C_2)\,; \quad \overline{D} = -C_z\lambda_1D_1$$

derived from the representation (27) of $T^+_{\varepsilon w}$ have to be substituted (see also (11)). The resulting expression can be ordered such that the (fictitious) feedback matrices

$$L_1 = (A - L_zC_z)\lambda_1 \qquad (35)$$

and

$$L_2 = (A - L_zC_z - L_1C_1)\Psi_2 \qquad (36)$$

can be separated, giving the same equation (23) as in the case of the *a priori* estimate. The fact that one gets a stabilizing solution in the sense of part b) of Lemma 1, follows exactly as in Theorem 1. Likewise the stability of the optimal filter can be shown as in the proof of Theorem 1.

6. A SIMPLE EXAMPLE

Given a third order system with one disturbed and one (perfect) measurement (i.e. m = 2, κ = 1) having the state space representation

$$x(k+1) = \begin{bmatrix} 1 & -0.25 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & 0 \end{bmatrix} w(k);$$

with $\quad x(0) = 0$

$$z(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$$

$$y_1(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} w(k)$$

$$y_2(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k)$$

Since the measurement y_2 is not disturbed, the reduced order H_∞ filter is of order n–κ = 1.

Considering the *a priori* estimate first, the infimal value of γ is $\gamma_{opt} = \sqrt{35/6} \cong 2.41523$ and the solution of the ARE (18) for $\gamma = \gamma_{opt}$ (i.e. $J = \gamma_{opt}$) result as

$$\overline{P} = \begin{bmatrix} 6 & 1 \\ 1 & 6 \end{bmatrix}$$

giving with (15) and (16)

$$L = [L_z\ L_1\ L_2] = \begin{bmatrix} -0.1\overline{714285} & 1 & -0.05 \\ -0.1714285 & 1 & 0.2 \end{bmatrix} \text{ and}$$

$$\Psi_2 = (A - L_zC_z - L_1C_1)^{-1}L_2 = \begin{bmatrix} 0.1\overline{6} \\ 1 \end{bmatrix}.$$

With $T = [-1 \quad 0.1\overline{6}]$ the condition $T\Psi_2 = 0$ holds, yielding $\Theta = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Consequently the reduced order a priori H_∞ filter is described by

$$\xi(k+1) = \begin{bmatrix} -0.8\overline{3} & 0.25 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$$

$$\hat{z}(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0.1\overline{6} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_2(k) \\ \xi(k) \end{bmatrix}$$

A better performance index $J^+ < J$ for the H_∞ filter can be obtained when using the *a posteriori* estimate. Here the infimal value for γ is $\gamma_{opt} = \sqrt{\frac{10}{11}} \cong 0.95346$ yielding a solution for $\overline{P}$ whose entries tend to infinity. Taking e.g. $\gamma = \sqrt{\frac{15}{14}} \cong 1.0351$ gives the solution $\overline{P} = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$ to the ARE (34), yielding a performance index $J^+ = 0.989$ and the filter parameters are (see (31), (32), and (33))

$$L = [L_z\ L_1\ L_2] = \begin{bmatrix} -4.\overline{6} & 5 & 4.75 \\ -4.6 & 5 & 5 \end{bmatrix}$$

and from this one obtains

$$\lambda_1 = \begin{bmatrix} 0.882353 \\ 0 \end{bmatrix} \text{ and } \Psi_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

With $T = \begin{bmatrix} 1 & -0.5 \end{bmatrix}$ the relation $T\Psi_2 = 0$ holds, giving $\Theta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Consequently the reduced order H_∞ filter using the *a posteriori* estimate is described by

$$\xi(k+1) = 0.058824\xi(k) + $$

$$+ \begin{bmatrix} 0.4411765 & -0.220588 \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$$

$$\hat{z}^+(k) = [1 \quad 1]\left\{\hat{x}(k) + \begin{bmatrix} 0.882353 \\ 0 \end{bmatrix}[y_1(k) - [1 \quad 0]\hat{x}(k)]\right\}$$

with

$$\hat{x}(k) = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_2(k) \\ \xi(k) \end{bmatrix}$$

7. CONCLUSIONS

Based on the Bounded Real Lemma a solution has been derived for the discrete time H_∞ estimation problem for nth order plants in the presence of κ perfect measurements. The resulting H_∞ filter is of order $n-\kappa$ and has a structure identical to that of the reduced order Kalman filter (see e.g. Hippe and Wurmthaler, (1990)). Both *a priori* and *a posteriori* estimates were considered. A simple example demonstrated the design procedure.

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