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# Robust Experiment Design through Randomisation with Chance Constraints

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**Abstract:** It is well known that robust optimal experiment design is an extremely computationally expensive problem. The design is generally solved by discretisation of the design space resulting in a discrete semi-infinite convex programming problem. To ease the computational burden it is possible to solve the design problem using the scenario approach to robust convex optimisation. In this paper we examine the application of a recently proposed idea of 'variable robustness' to the experiment design problem. This approach provides insight into the problem in terms of the effect of reducing the number of scenarios in a manner that has a suitable trade-off between performance and guarantees. A numerical example is used to examine the applicability to robust experiment design.

Keywords: Optimal Experiment Design, System Identification, Robustness, Convex Optimization

## 1. INTRODUCTION

Experiment design essentially involves the manipulation of the experimental conditions such that maximal information is secured from the experiment. The goal to maximise the amount of information obtained from a system has motivated substantial research on experiment design during the last century. Early research in the statistics literature includes Cox [1958], Fedorov [1972], Wald [1943], Whittle [1973], Wynn [1972], and, in the engineering literature, Gagliardi [1967], Goodwin et al. [1973], Goodwin and Payne [1977], Hildebrand and Gevers [2003b], Levadi [1966], Mehra [1974], Zarrop [1979]. Some more recent surveys on experiment design are included in Gevers [2005], Hjalmarsson [2005], Pronzato [2008] where many additional references can be found.

From an engineering perspective it is well known [Goodwin and Payne, 1977, Gevers, 2005, Hjalmarsson, 2005] that the accuracy of models obtained from system identification is largely dependent upon the experimental conditions and hence requires a consolidated design. In general, the focus in the engineering literature has been on experiment design for dynamic system identification.

A critical issue for experiment design in dynamical systems is that the model is, typically, nonlinearly parameterised. This means that the Fisher information matrix [Goodwin and Payne, 1977], which is generally used as the basis for experiment design, depends, inter alia, on the true system parameters, i.e. the very thing that the experiment is aimed at finding.

Preliminary work in the engineering literature on robust experiment design includes substantial work on iterative design [Gevers, 2005, Hjalmarsson, 2005], and an insightful sub-optimal min-max solution for a one parameter problem in Walter and Pronzato [1997]. Also, a number of recent engineering publications refer to the idea of min-max optimal experiment design [Gevers and Bombois, 2006, Mårtensson and Hjalmarsson, 2006, Rojas et al., 2007].

A min-max robust design criterion is the basis of the approach described in the current paper. Specifically, we assume that we have available a-priori information that the parameters can take any value in a compact set $\Theta$. We also constrain the allowable set of input signals. A typical constraint [Goodwin and Payne, 1977, Zarrop, 1979, Walter and Pronzato, 1997] used in experiment design is one placed on the input energy. The purpose of min-max robust experiment design is to optimise the input spectrum for the worst case performance of the identification procedure (typically measured as a scalar function of the information matrix of the model parameters).

The min-max optimisation problem can be considered as a special case of a robust convex program [Ben-Tal and Nemirovski, 1998]. In this case a linear objective function is minimised subject to a number of convex constraints, one for each instance of the uncertainty.

In robust experiment design it is usual to describe the uncertainty as a continuous set. Presenting this as a robust convex optimisation problem would give rise to an infinite number of constraints. This leads to a semi-infinite optimisation problem that is known to be difficult to solve and possibly NP-Hard [Ben-Tal and Nemirovski, 1998].

An approach that has been recently developed in Calafiore and Campi [2005, 2006] to deal with semi-infinite convex programming at a general level is known as the 'scenario approach'. The advantage of this method is that solvability can be obtained through random sampling of constraints provided that a probabilistic relaxation of the worst case robust paradigm is accepted. The probabilistic relaxation consists in being content with robustness against the large majority of situations rather than against all situations. In the scenario approach the number of situations is under the control of the designer and can be made arbitrarily close to the set of 'all' situations. It has

been recently shown [Welsh and Rojas, 2009] that utilising the scenario approach that the min-max experiment design problem can be approximated quite closely with considerable gains made in the reduction of computation time. It should be noted that the scenario approach does not impose any conditions upon the dependence on the true parameter, providing the nominal problem is convex.

In this paper we examine the application of a recently proposed variable robustness algorithm [Campi and Garatti, 2010a], which is based on the scenario approach to optimisation, to robust experiment design. Essentially this algorithm provides a mechanism for trading guarantees of robustness for performance. Basically this algorithm consists of a 'chance constrained' optimisation [Ben-Tal and Nemirovsk, 2002, Nemirovski and Shapiro, 2006] problem. In this problem there is a chance that a particular bound on the cost is violated, however, this chance is constrained by another variable. In this sense probability can be used to quantify the chance that a certain performance specification is not met. The variable robustness algorithm can be considered as a sample based approximation of the 'chance constrained' optimisation problem, with a discrete distribution, where it is possible to determine the violation versus performance tradeoff. Variable robustness inherently uses randomisation to determine the region to remove from the optimisation problem, to improve the cost, by the elimination of active constraints.

The robust experiment design problem involves a min-max approach as outlined above and described further in section 2. In this case, when using the variable robustness algorithm the max requirement will be relaxed in a probabilistic sense. Using a simple one parameter example, the tradeoff between guarantees and performance is examined.

The layout of the remainder of the paper is as follows: Section 2 describes the basic setup of the robust experiment design problem. The scenario approach for solving robust convex programs is explained in Section 3. Section 4 describes the scenario approach with chance constraints and how it is utilised in the variable robustness algorithm for experiment design. Numerical examples illustrating this approach to robust experiment design are presented in Section 5. Finally, Section 6 provides the conclusions.

## 2. ROBUST EXPERIMENT DESIGN

### *2.1 The Information Matrix*

A well known and intuitive way to compare different experiments is to choose a measure related to the expected accuracy of the parameter estimator of the model to be obtained from the experimental data. However, the accuracy of the parameter estimator is a function of both the experimental conditions and the form of the estimator. Since we would prefer to have an 'estimator-independent' measure, we may assume that the estimator used is statistically efficient in the sense that the parameter covariance matrix achieves the Cramér-Rao lower bound [Goodwin and Payne, 1977], i.e.

$$\operatorname{cov} \hat{\theta} = M^{-1},$$

where $M$ is the Fisher's information matrix [Casella and Berger, 2002, Silvey, 1970]. Note that estimators are denoted by a superscript '$\widehat{\ }$' and implicitly depend on the data length, $N$. Therefore, the first step is to determine an expression for $M$.

To be specific, consider a single-input single-output (SISO) linear continuous time system, with input $u(t)$ and output $y(t)$, of the form

$$y(t) = G(p)u(t) + H(p)w(t)$$

where $G$ and $H$ are stable rational transfer functions, $p$ is the time derivative operator, $H$ is minimum phase with $H(\infty) = 1$, and $w(t)$ is zero mean Gaussian white noise of intensity $\sigma^2$. We assume that the system is operating in open loop, hence $u(t)$ and $w(t)$ are independent. We let $\theta := [\rho^T \ \eta^T \ \sigma^2]^T$ where $\rho$ denotes the parameters in $G$ and $\eta$ denotes the parameters in $H$. Therefore, we assume that $G$, $H$ and $\sigma^2$ are independently parameterised.

Assume that the input $u(t)$ has a zero order hold mechanism, with sampling period $h$, and that we sample the output $y(t)$ with the same sampling period $h$. Then for estimation purposes we will have $N$ samples $\{u(kh), y(kh)\}_{k=1}^{N}$. Fisher's information matrix $M$ is given by [Goodwin and Payne, 1977]

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

where $M_1$ is the part of the information matrix related to $\rho$, and $M_2$ is independent of the input. Assuming $N$ is large, it is more convenient to work with the scaled average information matrix for the parameters $\rho$ [Goodwin and Payne, 1977, Walter and Pronzato, 1997],

$$\begin{aligned} \overline{M}(\theta, \Phi_u) &:= \lim_{N\to\infty} \frac{1}{Nh} M_1 \sigma^2 \\ &= \int_0^\infty \widetilde{M}(\theta, \omega) \Phi_u(\omega) d\omega. \end{aligned} \tag{1}$$

where

$$\widetilde{M}(\theta, \omega) := \operatorname{Re}\left\{ \frac{\partial G(j\omega)}{\partial \rho} |H(j\omega)|^{-2} \left[\frac{\partial G(j\omega)}{\partial \rho}\right]^H \right\}, \tag{2}$$

$G$ and $H$ are continuous time transfer functions (assumed independently parameterised) and $\Phi_u$ is the continuous time input spectral density.

### *2.2 Criteria for Nominal Experiment Design*

Since $\overline{M}$ is a matrix, we need a scalar measure of $\overline{M}$ for the purpose of experiment design. In the nominal case, typically treated in the engineering literature (i.e. when a fixed prior estimate of $\theta$ is used), several measures of the 'size' of $\overline{M}$ have been proposed which measure the 'goodness' of the experiment. Some examples include,

(i) $D$ - optimality [Goodwin and Payne, 1977]

$$J_d(\theta, \Phi_u) := [\det \overline{M}(\theta, \Phi_u)]^{-1}. \tag{3}$$

(ii) Experiment design for robust control [Hildebrand and Gevers, 2003a,b, Hjalmarsson, 2005].

$$J_{rc}(\theta, \Phi_u) := \sup_\omega g(\theta, \omega)^H \overline{M}^{-1} g(\theta, \omega) \tag{4}$$

where $g$ is a frequency dependent vector related to the $\nu$-gap [Hildebrand and Gevers, 2003a,b].

Many other criteria have been described in the statistics literature, such as $A$-optimality ($\operatorname{tr} \overline{M}(\theta, \Phi_u)^{-1}$), $L$-optimality ($\operatorname{tr} W\overline{M}(\theta, \Phi_u)^{-1}$, for some $W \geq 0$) and $E$-optimality $\left(\lambda_{\max}(\overline{M}(\theta, \Phi_u)^{-1})\right)$; see Kiefer [1974]. On the other hand, in the engineering literature, Bombois et al. [2006] proposed a criterion that specifies the required accuracy to achieve a given level of robust control performance.

A common feature of all these nominal experiment design approaches is that they are aimed at choosing $\Phi_u$ to minimise a function of the type such as in (3) and (4). Most criteria are convex in $\Phi_u$, so in the sequel, we will consider that the chosen criterion has this property.

### *2.3 Min-Max Robust Design*

A min-max robust design criterion is the basis of our experiment design technique. Specifically, we assume that a-priori information is available indicating that the parameters can take any value in a compact set $\Theta$. We also constrain the allowable set of input signals. Typically in experiment design, a constraint is imposed on input energy [Goodwin and Payne, 1977, Walter and Pronzato, 1997, Zarrop, 1979]. Here we define the constraint as

$$\mathscr{S}(\mathbb{R}_0^+) := \left\{ \Phi_u : \mathbb{R} \to \mathbb{R}_0^+ : \quad \Phi_u \text{ is even and } \int_{-\infty}^{\infty} \Phi_u(\omega) d\omega = 1 \right\}.$$

The min-max robust optimal input spectral density, $\Phi_u^{opt}$, is then chosen as

$$\Phi_u^{opt} = \arg \min_{\Phi_u \in \mathscr{S}(\mathbb{R}_0^+)} \sup_{\theta \in \Theta} J(\theta, \overline{M}(\theta, \Phi_u)) \tag{5}$$

where $J$ is an appropriate scalar measure of $\overline{M}$. We assume that $\Phi_u^{opt}$ exists and is unique; see Rojas et al. [2007]. Notice also that we allow $J$ to depend explicitly on $\theta$.

### *2.4 Discrete Approximation to the Optimal Input*

Note that (5) is an infinite dimensional optimisation problem. In order to solve this problem we must approximate (1) by discretisation of the design space. To this end, we first restrict the positive support of $\Phi_u$ to a compact interval, say $K := [\underline{\omega}, \overline{\omega}] \subset \mathbb{R}_0^+$, hence $\Phi_u \in \mathscr{S}(K)$. Next we approximate the integral in equation (1) by a Riemann sum. Specifically, we choose a grid of $d+1$ points $\omega_m \in [\underline{\omega}, \overline{\omega}]$ for $m = 0, \ldots, d$ such that $\omega_0 = \underline{\omega}$, $\omega_d = \overline{\omega}$. Then

$$\begin{aligned} \overline{M}(\theta, \Phi_u) &:= \int_{\underline{\omega}}^{\overline{\omega}} \widetilde{M}(\theta, \omega) \Phi_u(\omega) d\omega \\ &\approx \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n) \Phi_u(\omega_n)(\omega_{n+1} - \omega_n) \\ &= \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n) E_n \end{aligned} \tag{6}$$

where $E_n := \Phi_u(\omega_n)(\omega_{n+1} - \omega_n)$. We can now state the following discrete semi-infinite convex programming approximation to (5):

$$\begin{aligned} \min_{t \in \mathbb{R},\ E \in \mathbb{R}^d} \ & t \\ \text{s.t.} \quad & J\left(\theta, \sum_{n=0}^{d-1} \widetilde{M}(\theta, \omega_n) E_n\right) \le t, \qquad \theta \in \Theta \\ & \sum_{n=0}^{d-1} E_n = 1 \\ & E_n \ge 0, \qquad n = 0, \ldots, d-1. \end{aligned} \tag{7}$$

where 's.t.' denotes 'subject to'.

## 3. THE SCENARIO APPROACH

In this section we provide a brief overview of the Scenario Approach for solving a robust convex problem. Basically the scenario approach assumes a probabilistic description of uncertainty, that is, the uncertainty is characterised through a set $\Delta$ describing the set of admissible situations, and a probability distribution $P_r$ over $\Delta$.

In Section 2.4, it is shown that the min-max optimisation problem, when converted to a robust convex optimisation program yields an unwieldy number of constraints, c.f. (7). The underlying benefit of the scenario approach is that it involves selecting a small number of these constraints to include in the optimisation problem. Therefore by extracting, at random, $N$ instances or 'scenarios' of the uncertainty parameter $\delta$ according to some probability $P_r$ we consider only the corresponding constraints in the scenario optimisation problem.

Consider the following general Robust Convex Program:

$$RCP: \begin{array}{ll} \min\limits_{\gamma \in \mathbb{R}^d} & c^T \gamma \\ \text{s.t.} & f_\delta(\gamma) \le 0, \quad \delta \in \Delta. \end{array} \tag{8}$$

where $f_\delta : \mathbb{R}^d \to \mathbb{R}$ is convex for every $\delta \in \Delta$. This robust convex program can be formulated in a scenario-based approximation.

The scenario based optimisation problem [Calafiore and Campi, 2006] can be stated as: Extract N independent identically distributed samples $\delta^{(1)}, \ldots, \delta^{(N)}$, according to the probability $P_r$ and solve the scenario convex program:

$$SCP_N: \begin{array}{ll} \min\limits_{\gamma \in \mathbb{R}^d} & c^T \gamma \\ \text{s.t.} & f_{\delta^{(i)}}(\gamma) \le 0, \quad i = 1, \ldots, N. \end{array} \tag{9}$$

It can be seen from (9) that it is a standard finite dimensional convex optimisation problem with a finite number of constraints. Therefore the computational cost, provided $N$ is not large, will be significantly smaller than the cost associated with the min-max optimisation problem.

By considering only a finite subset of constraints, which are chosen in a random manner, we would like the scenario-based optimisation program $SCP_N$ to provide a solution $\gamma^{opt}$ which, with high probability, say $1 - \beta$, satisfies all the constraints in $\Delta$, except for a fraction with a small probability, say $\epsilon$ (with respect to the probability measure $P_r$). Here $\beta$ is denoted as the 'confidence parameter' and $\epsilon$ is the 'violation parameter'. These variables are user choices which determine the minimum number of scenarios $N$ to be randomly selected.

Several bounds on the minimum number of scenarios required have been derived in the literature, see Alamo et al. [2007, 2008], Calafiore and Campi [2006], Campi and Garatti [2007]. To date, the tightest bound has been established in Campi and Garatti [2007], according to which $N$ has to satisfy

$$\sum_{i=0}^{d-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \le \beta. \tag{10}$$

This relationship establishes an implicit dependence of $N$ on $d$, and comes from the following proposition, first established in Calafiore and Campi [2005]:

*Theorem 1.* Consider the convex program:

$$\mathcal{P}: \begin{array}{ll} \min\limits_{x \in \mathbb{R}^d} & c^T x \\ \text{s.t.} & x \in \mathcal{X}_i, \quad i = 1, \ldots, m, \end{array}$$

where $c \in \mathbb{R}^d$, and $\mathcal{X}_i$, $i = 1, \ldots, m$, are closed convex sets in $\mathbb{R}^d$. Also define for every $k = 1, \ldots, m$,

$$\mathcal{P}_k : \min_{x \in \mathbb{R}^d} c^T x \quad \text{s.t.} \quad x \in \mathcal{X}_i, \quad i = 1, \ldots, k-1, k+1, \ldots, m.$$

Let $x^{opt}$, $x_k^{opt}$ be any optimal solutions of $\mathcal{P}$ and $\mathcal{P}_k$, $k = 1, \ldots, m$, respectively. We say that $\mathcal{X}_k$ is a *support constraint* for $\mathcal{P}$ if $c^T x_k^{opt} < c^T x^{opt}$. Then, the number of support constraints for $\mathcal{P}$ is at most $d$.

**Proof.** See Calafiore and Campi [2005].

According to (10), the bound on $N$ is an increasing function of $d$. On the other hand, for robust experiment design, the size of $d$ is related to the discretisation described in Section 2.4. This appears to give rise to a huge curse of dimensionality, since in order to obtain a reasonable degree of approximation, the required number of scenarios might be too large for a practical implementation. However, in most practical cases, it is possible to replace $d$ in (10) by a much smaller number, asymptotically independent of the degree of approximation made in Section 2.4.

## 4. SCENARIO-BASED CHANCE CONSTRAINED OPTIMISATION AND VARIABLE ROBUSTNESS

The basic idea for the scenario based chance constrained optimisation is to remove a number of sampled constraints in order to improve the cost. Obviously, this occurs at the expense of guarantees of the robustness. It was shown [Campi and Garatti, 2010a] that any algorithm that removes active constraints can be utilised in the optimisation procedure. Typical constraint removal algorithms include: the updating the solution by the removal of all active constraints at a given iteration, as well as the greedy algorithm which removes only the active constraint that results in the largest reduction in the cost.

From Campi and Garatti [2010b] we have the following definition for the constraint removal algorithm.

*Definition 1.* Let $k < \mathcal{A}$. An algorithm $\mathcal{A}$ for constraints removal is any rule by which $k$ constraints out of a set of $N$ constraints are selected and removed. The output of $\mathcal{A}$ is the set $\mathcal{A}\{\delta^{(i)}, \ldots, \delta^{(N)}\} = \{I_1, \ldots, I_k\}$ of the indexes of the k removed constraints.

With this definition the chance constrained scenario optimisation problem Campi and Garatti [2010b] can be stated as:

$$SCP_{N,k}^{\mathcal{A}} : \begin{array}{ll} \min\limits_{\gamma \in \mathbb{R}^d} & c^T \gamma \\ \text{s.t.} & f_{\delta^{(i)}}(\gamma) \leq 0, \quad i \in \{1, \ldots, N\} \\ & \qquad -\mathcal{A}\{\delta^{(i)}, \ldots, \delta^{(N)}\}. \end{array} \tag{11}$$

It has been shown in Campi and Garatti [2010b] that a feasible solution can be found for the chance constrained scenario optimisation problem with a given probability, $1 - \beta$, provided that the number of scenarios, $N$, and discarded constraints, $k$, fulfill the following condition:

$$\binom{k+d-1}{k} \sum_{i=0}^{k+d-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta. \tag{12}$$

Taking the above approach to solve the optimisation problem, it is possible to construct a performance-violation plot [Campi and Garatti, 2010a] for the optimal solution. This allows the user to make decisions regarding the required guarantees and the performance that can be obtained. The violation parameter can be obtained [Campi and Garatti, 2010a] by solving the following equation for $\epsilon_k$,

$$\binom{k+d}{k} \sum_{i=0}^{k+d} \binom{N}{i} \epsilon_k^i (1-\epsilon_k)^{N-i} = \frac{\beta}{\bar{k}+1}, \tag{13}$$

where $\bar{k}$ is the number of discarded scenarios.

## 5. NUMERICAL EXAMPLES

In this section, two examples are presented. Both examples involve the robust experiment design for a one parameter first order system [Rojas et al., 2007]. Note that all the optimisation problems in this section were solved using semidefinite programming with the LMI parser YALMIP [Löfberg, 2004] and the solver SeDuMi.

Consider a model given by $H(s) = 1$ and

$$G(s) = \frac{1}{s/\theta + 1},$$

where it is assumed that $\theta \in [0.1, 10]$. For this model structure, the 'single frequency' normalised information matrix is given by

$$\widetilde{M}(\theta, \omega) = \frac{\omega^2/\theta^4}{(\omega^2/\theta^2 + 1)^2}.$$

Consider a criterion of the form

$$J(\theta, \overline{M}(\theta, \Phi_u)) = \frac{1}{\theta^2 \overline{M}(\theta, \Phi_u)}.$$

The reason for multiplying $\overline{M}$ by $\theta^2$ is that $\overline{M}^{-1}$ is a variance measure and thus $[\theta^2 \overline{M}]^{-1}$ gives relative (mean square) errors.

As shown in Rojas et al. [2007], this robust experiment design problem can be solved by discretising the interval for $\theta$, and rewriting the problem as a linear program. This approach is similar to the one described in Section 3, except for the fact that in Rojas et al. [2007] a deterministic (in fact, uniform) sampling of the constraints has been used. In both examples we consider an interval $[0.1, 10]$ for the support of $\Phi_u$ (which, according to Rojas et al. [2007], actually contains the optimal spectrum),

### *5.1 Example 1*

The first example we consider is where the optimisation parameters are chosen as $d = 30$, $\epsilon = 0.01$, and $\beta = 10^{-10}$, the bound (12) shows that $N$ should be at least $8950$. In all cases the distribution $P_r$ is uniform on $\ln \theta$. The percentage of discarded constraints was chosen to be 5% which equates to the removal of 447 active constraints (or active scenarios).

Figure 1 shows the cost associated with the robust experiment design for three cases: 1). Scenario approach with all 8950 scenarios, 2). Removal of 337 active constraints (scenarios) by a non-greedy algorithm, and 3). the greedy approach to the removal of 337 active constraints. Note that the horizontal lines of the same type (solid, dot, dash) represent the optimal cost for the corresponding solutions for the scenario approach with respect to the above mentioned cases.

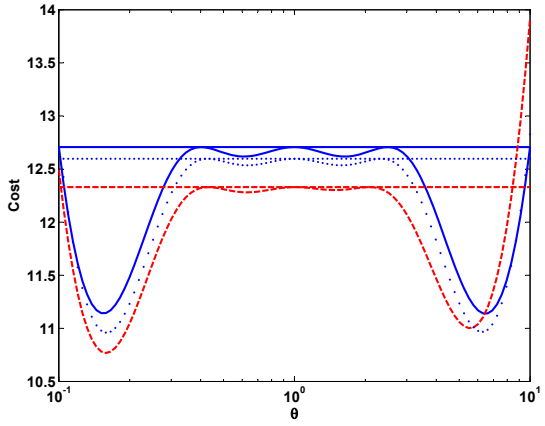


Fig. 1. Cost $J(\theta, \overline{M}(\theta, \Phi_u^{opt}))$ as a function of $\theta$, and the optimal cost, for the three cases in example 1.

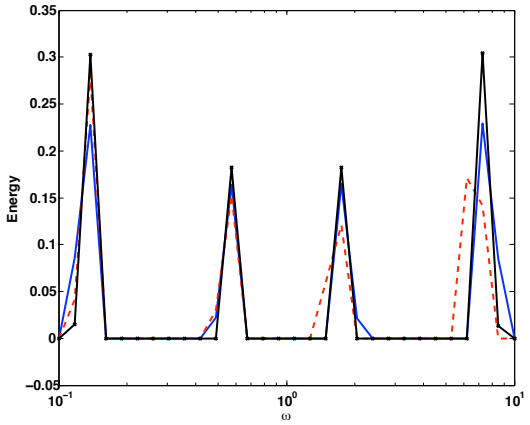


Fig. 2. Input energy, $E$, for the discretised robust optimal input for the three cases of example 1.

From Figure 1 it can be seen that for a given number of constraints (or scenarios) the greedy algorithm obtains the best performance, but, appears to admit the most violations of the cost. However, this is a little misleading as careful examination will show that the greedy algorithm doesn't violate constraints on the lower limit of the interval $[0.1, 10]$. Figure 2 shows the optimal input signal for the three cases above. It can be observed that there appears to be only 4 spectral lines in all cases as predicted in Rojas et al. [2007]. The only difference is essentially in the re-distribution of the input energy (recall that there is a constraint on the total input energy).

From Figure 3 we can observe the tradeoff based on the variable robustness algorithm between performance and violations. It should be noted that the violation plot is an upper bound on the probability that a violation will occur. The star line in Figure 3 represents the cost associated with the non-greedy removal of active constraints (scenarios). It can easily be seen from this figure that the greedy algorithm (dotted line) will, in general, provide the best improvement in performance overall, based on the number of active constraints removed.

### *5.2 Example 2*

In this example we consider the optimisation parameters, $d = 30$, $\epsilon = 0.07$, and $\beta = 10^{-7}$, the bound (12) shows that

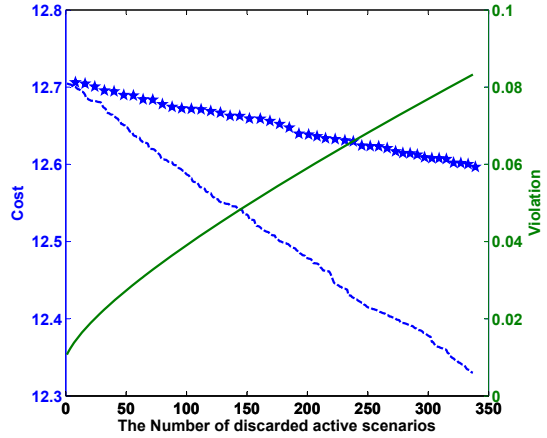


Fig. 3. Performance-Violation plot for example 1

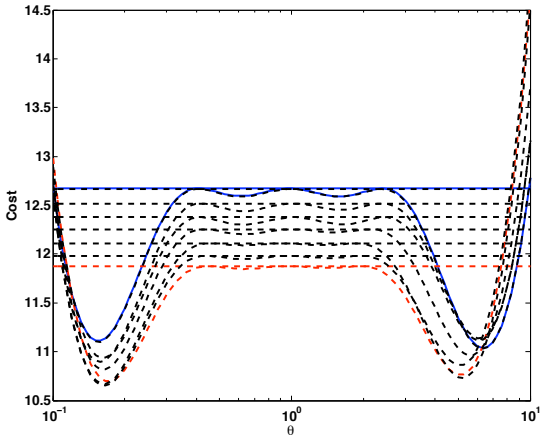


Fig. 4. Cost $J(\theta, \overline{M}(\theta, \Phi_u^{opt}))$ as a function of $\theta$, and the optimal cost for Example 2.

$N$ should be at least 1075. In all cases the distribution $P_r$ is uniform on $\ln \theta$. The percentage of discarded constraints was chosen to be 10% which equates to the removal of 107 active scenarios. Figure 4 shows the progression for this case as active scenarios are removed by the greedy algorithm. As the algorithm progresses it can be seen that the cost is reduced, however the number of violations increase.

Figure 5 shows the tradeoff between performance and violations when we have a smaller number of scenarios, as compared to Example 1, and we discard a higher percentage of these. Figure 6 illustrates the redistribution of input energy when active constraints are removed by the greedy algorithm.

## 6. CONCLUSION

This paper has examined the robust optimal experiment design problem based on a scenario approach to optimisation. In particular it investigated the use of 'variable robustness' as a means of quantifying the performance versus guarantees tradeoff. This is achieved by removing active constraints from the optimisation problem whilst evaluating the associated cost. Plotting this cost versus the number of constraints removed on the same graph as the violation parameter vs constraints removed provides a means to judge the relative tradeoff. Two examples

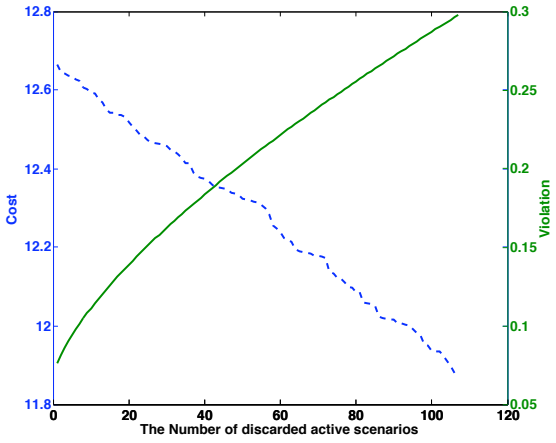


Fig. 5. Performance-Violation plot for example 2

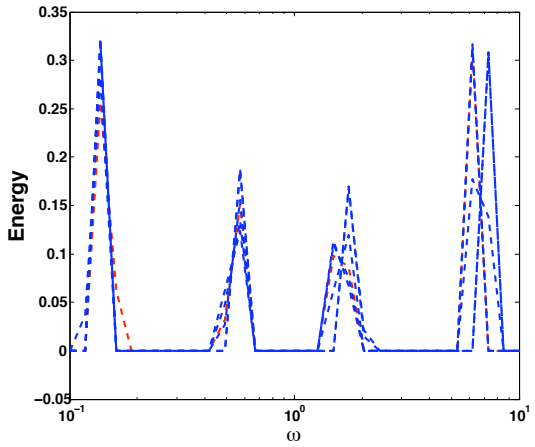


Fig. 6. Input energy, $E$, for example 2, with 20, 40, 60, 80 and 100 active constraints removed.

are provided which provide insight into the applicability of this approach in experiment design.

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