Optimal decentralized servo control for systems with diagonal decoupling matrix

Sadaaki Kunimatsu ∗ Kosuke Tateishi ∗ Mitsuaki Ishitobi ∗ Takao Fujii **

*Kumamoto University, JAPAN
**Fukui University of Technology, JAPAN

Abstract: In this paper, we propose an optimal servo design method of decentralized control systems from the inverse linear quadratic problem viewpoint by using the structure of controllers obtained by the ILQ (Inverse Linear Quadratic) design method. Particularly, we propose a decentralized ILQ servo design method for systems with relative degree no more than 2 and diagonal decoupling matrix. We show that the closed loop system can always be stabilized in the framework of the decentralized ILQ design method. Moreover, we show that the optimal decentralized servo controller with the decoupled input/output characteristics is obtained. Finally, we illustrate the effectiveness of our proposed method by simulation.

1. INTRODUCTION

Recently, there is a great interest in large-scale systems since many control problems in the industrial world are related to complex interactive systems. However, as controlled systems become larger scale and more complex, it is not necessarily suitable to use centralized control scheme for systems with one control machine from the viewpoint of economy, efficiency and so on. Then, the system is divided into several subsystems to overcome these disadvantages, and much attention is paid on decentralized control [10, 11].

There are many researches on decentralized control problems [5, 7, 10, 11]. To the best of our knowledge, however, there are few design methods such that optimal decentralized servo control is provided with considering transient response. It is necessary from the practical viewpoint that the input/output characteristics is as decoupled as possible so that the influence from one of the subsystems to the others is reduced as much as possible.

Our purpose in this paper is not only to stabilize systems but also to shape the input/output characteristics so as to have LQ optimality by using the optimal decentralized servo control.

2. PRELIMINARIES

Let us consider the following \( m \)-input, \( m \)-output and \( n \)-th order linear time-invariant system with stable zeros, which is controllable and observable,

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

where \( A, B, C =: \text{col}_{1 \leq i \leq m} \{ c_r \} \) are given constant matrices with appropriate dimensions and rank \( B = m \). The notation \( \text{col}_{1 \leq i \leq m} \{ F_i \} \) is defined by \( \text{col}_{1 \leq i \leq m} \{ F_i \} := \begin{bmatrix} F_{11}^T, F_{12}^T, \cdots, F_{1n}^T \end{bmatrix} \). \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^m \) is the output, \( u \in \mathbb{R}^m \) is the input. The system (1) is composed of \( N \)-subsystems. All the subsystems are single-input, single-output and \( n_i \)-th order such that \( \sum_{i=1}^N n_i = n \). Note that

\[
J = \int_0^\infty \left( x_e^T Q x_e + u_e^T R u_e \right) dt \quad (4)
\]
Lemma 1. [2, 4] The state feedback $u_e = -K_e x_e$ minimizes the performance index (4) with appropriate weight matrices $Q > 0$, $R > 0$ only if there exists some diagonal matrix $\Sigma > 0$, some real nonsingular matrix $V$ and real matrices $K_0^F$, $K_0^C$ satisfying the following equations:

$$K_e = V^{-1} \Sigma \left[ K_0^F - K_0^C \right]$$  \hspace{1cm} (5a) $$K_0^F B = I$$  \hspace{1cm} (5b)

where $K_0^F$ and $K_0^C$ are partitioned according to the sizes of $x$ and $x_e$, respectively.

Lemma 2. [3, 9] The matrix $K_e$ satisfying (5) is optimal for $J$ of (4) with some $Q > 0$ and $R = V^T \Sigma^{-1} V$ if and only if there exist $Y > 0$ and diagonal $\Sigma > 0$ satisfying the following LMI (Linear Matrix Inequality):

$$\begin{bmatrix} Y A_K + A_K^T Y & Y B_V + (K_V A_K)^T \\ B_V^T Y + K_V A_K & K_V B_V + (K_V B_V)^T - \Sigma \end{bmatrix} < 0$$


The gains $K_0^F$ and $K_0^C$ are given below. Assume that the decoupling matrix $D_i$ defined by (6) is nonsingular.

$$D_i := \text{col} \left\{ c_i A_i^{d_i - 1} B \right\}$$  \hspace{1cm} (6)

where $d_i := \min\{c_i A_i^{d_i - 1} B \neq 0\}$ is the relative degree of the $i$-th row. By using the ILQ design method, we can specify a desired input/output characteristics $G_y^c(s)$ from the reference input $r$ to the output $y$ asymptotically.

Lemma 3. [3, 9] The asymptotic transfer function $G_y^c(s)$ of $G_y^c(s)$ as $\Sigma \rightarrow \infty$ can be specified as (7) in terms of some arbitrary stable $d_i$-th order monic polynomial $\phi_i(s) := s^{d_i} + \phi_{i+1} s^{d_i-1} + \cdots + \phi_0$.

$$G_y^c(s) = C(sI - A_K)^{-1} B K_0^C = \text{diag} \left\{ \frac{\phi_i(0)}{\phi_i(s)} \right\}$$  \hspace{1cm} (7)

Then, $K_0^F$ and $K_0^C$ are constructed by (8).

$$K_0^F = D_c^{-1} N_y, \quad N_y := \text{col} \left\{ c_i \psi_i(A) \right\}$$  \hspace{1cm} (8a) $$K_0^C = D_c^{-1} \Phi_0, \quad \Phi_0 := \text{diag} \left\{ \phi_i \right\}$$  \hspace{1cm} (8b)

where $\psi_i(A)$ is a polynomial matrix determined by $\psi_i(s) := (\phi_i(s) - \phi_i(0))/s$.

Note that $K_0^F$ always satisfies $K_0^F B = I$ in Lemma 3. Then any $\Sigma$ satisfying $\Sigma > \Sigma$ guarantees LQ optimality[3, 9].

4. CASE OF SYSTEM FOR WHICH DECENTRALIZED CONTROLLER CAN BE OBTAINED

In the following, we consider a decentralized control problem in which the subsystems use the information about themselves alone. Namely, we consider the case where $K_e$ of (5) consists of a diagonal nonsingular matrix $V$, a block diagonal matrix $K_0^F$ and a diagonal $K_0^C$.

In this section, we state conditions on the system (1) such that $K_e$ of (8) can be obtained as a decentralized controller.

4.1 Case with all the subsystems having relative degree 1

When the relative degrees of all the subsystems are 1, the decoupling matrix $D_i$ becomes $D_i = CB$. If $D_i$ is nonsingular, then $K_0^F$ and $K_0^C$ of (8) satisfy $K_0^F = D_c^{-1} C$ and $K_0^C = D_c^{-1} \Phi_0$, respectively. Thus, the controller is obviously decentralized.

4.2 Case with all the subsystems having relative degree no more than 2

When the relative degrees of all the subsystems are no more than 2, the following theorem is obtained as a condition on systems to achieve decentralized controllers.

Theorem 4. We assume that the decoupling matrix $D_i$ is nonsingular, and $d_i \leq 2$ for any $i$ is satisfied. Under this assumption, $K_e$ is decentralized if and only if each subsystem $S_i$ with $d_i = 2$ satisfies the following equation.

$$c_i A_i b_j = 0 \ \forall j \neq i$$  \hspace{1cm} (9)

Proof. (Necessity) If $K_0^F$ is diagonal, $D_c^{-1}$ is diagonal since $\Phi_0$ is a diagonal nonsingular matrix. This implies that $D_i$ is diagonal. By the definition of $D_i$, $S_i$ with $d_i = 2$ satisfies $c_i A_i b_j = 0 \ \forall j \neq i$. Furthermore, if $K_0^F$ is block diagonal, $S_i$ with $d_i = 2$ satisfies $c_i A_i b_j = 0 \ \forall j \neq i$ since $D_c^{-1}$ is diagonal. This condition is included in the necessary condition of $c_i A_i b_j = 0 \ \forall j \neq i$ so that $K_0^C$ is block diagonal. Hence, if $K_e$ is decentralized, it is necessary that $S_i$ with $d_i = 2$ satisfies $c_i A_i b_j = 0 \ \forall j \neq i$.

(Sufficiency) We have $D_i = d_{i=1} \leq 2 \{ c_i A_i^{d_i - 1} b_i \}$, $N_y = b_{i=1} \leq 2 \{ c_i A_i + \phi_1 \}^{d_i - 1}$ by direct calculation. Therefore, it is obvious that $K_0^F$ and $K_0^C$ are both decentralized.

5. CASE OF SYSTEM FOR WHICH DECENTRALIZED CONTROLLER CANNOT BE OBTAINED

5.1 Construction method of decentralized controller

When we construct a decentralized controller $u_e = -K_e x_e$ by using Lemma 1, which is derived by applying the inverse problem of the optimal control[2], we obtain the following lemma.

Lemma 5. The state feedback $u_e = -K_e x_e$ minimizes the performance index (4) with appropriate weight matrices $Q > 0$, $R > 0$ only if there exists some diagonal matrix $\Sigma > 0$, real block diagonal matrix $K_0^F$ and real diagonal matrix $K_0^C$ satisfying the following equations:

$$\dot{K}_e = \Sigma \left[ K_0^F - K_0^C \right]$$  \hspace{1cm} (10a) $$K_0^F B = I$$  \hspace{1cm} (10b)

When the diagonal elements of $K_0^F$ and $K_0^C$ are given by $K_0^F[i]$ and $K_0^C[i]$, respectively, $K_0^F$ and $K_0^C$ are represented as follows:

$$\dot{K}_0^F[i] = b_{i=1} \leq N \left\{ K_0^F[i] \right\}, \quad \dot{K}_0^C[i] = \text{diag} \left\{ K_0^C[i] \right\}$$  \hspace{1cm} (11)

If the optimal servo system for the system (1) is designed by using ILQ design method, we can obtain $K_0^F$, $K_0^C$ by choosing a desired $\phi_i(s)$. In this case, since $K_0^F$, $K_0^C$ are not block diagonal in general, it cannot be guaranteed that the controller is decentralized. According to the size of the subsystem $S_i$, $K_0^F(i, j)$ and $K_0^C(i, j)$ denote the $(i, j)$ block of $K_0^F$ and $K_0^C$, respectively. We define $K_0^F[i] = K_0^F(i, i)$.
and $\hat{K}_0^C[i] = K_0^C(i,i)$ as a pair of decentralized controllers. Now, by noting that $B = b$-diag$_{1 \leq i \leq N}\{b_i\}$, we see $\hat{K}_F^0 B = \hat{K}_C^0 B = I$. Therefore, since Lemma 5 that is the necessary condition of LQ optimality is satisfied, the next theorem is obtained.

**Theorem 6.** The matrix $\hat{K}_e$ satisfying (10) is optimal for $J$ of (4) with some $Q > 0$ and $R := \Sigma^{-1}$ if and only if there exist $Y > 0$ and diagonal $\Sigma > 0$ satisfying the following LMI:

$$
\begin{bmatrix}
Y A_K + A_K^T Y & Y B + (\hat{K} A_K) \Sigma \\
B^T Y + K A_K & \hat{K} B + (\hat{K} B) \Sigma - \Sigma
\end{bmatrix} < 0
$$

where $A_K := A - B \hat{K}_e$, $\hat{K} = \hat{K}_F^0 A + \hat{K}_C^0 C$.

**Corollary 1.** There exist $Y > 0$ and diagonal $\Sigma > 0$ satisfying (12) if and only if $A_K$ is stable.

Corollary 1 implies that an optimal decentralized servo system can be constructed when $\hat{K}_F^0$ and $\hat{K}_C^0$ are designed such that $A_K$ is stable.

**5.2 Input/Output characteristics**

A main feature of ILQ design method is that the desired input/output characteristics $\hat{G}_{yr}(s)$ converges to the asymptotic transfer function $\tilde{G}_{yr}(s)$ uniformly with $\sigma(\Sigma) \to \infty$. In the case of the decentralized controller, this property holds similarly.

**Theorem 7.** If $A_K$ including the decentralized controller $\hat{K}_e$ is stable, the desired input/output characteristics $\hat{G}_{yr}(s)$ converges to the following asymptotic transfer function uniformly with $\sigma(\Sigma) \to \infty$.

$$
\hat{G}_{yr}(s) = C(sI - A_K)^{-1} B \hat{K}_C^0
$$

That is, $\lim_{\sigma(\Sigma) \to \infty} \| \hat{G}_{yr}(s) - \tilde{G}_{yr}(s) \|_{\infty} = 0$.

Although the asymptotic decoupling of the input/output characteristics is guaranteed by using the ILQ controller $K_e$, this is not guaranteed by using the decentralized controller $\hat{K}_e$ in general. Therefore, it is necessary that $\hat{K}_F^0$, $\hat{K}_C^0$ are designed so that $\hat{G}_{yr}(s)$ have a desired characteristics. Note that since $\tilde{G}_{yr}(0) = C(-A_K)^{-1} B \hat{K}_C^0 = I$, each output response of steady states is decoupled.

**6. DECENTRALIZED ILQ DESIGN METHOD FOR SYSTEMS WITH RELATIVE DEGREE NO MORE THAN 2**

From this section, we assume that each subsystem of the system (1) has relative degree $d_i$ satisfying $d_i \leq 2$.

**6.1 System with diagonal decoupling matrix**

Due to the complexity in analyzing the case when neither $K_F^0$ nor $K_C^0$ is decentralized, we consider the case where $K_C^0$ is decentralized. From the proof of Theorem 4, it is clear that $K_C^0$ is decentralized if and only if $D_c$ is diagonal, that is, $S_i$ with $d_i = 2$ for $\gamma_i$ is $c_i A_j b_j = 0$ for $\gamma_j \neq i$.

Consequently, we consider the system which has the diagonal decoupling matrix $D_c$ and satisfies $d_i \leq 2$ for $\gamma_i$. The decoupling matrix $D_c$ is given by

$$
D_c = \text{diag} \{d_i\}
$$

where $d_i := c_i A^{-1} b_i$. $\hat{K}_F^0$ and $\tilde{K}_C^0$ of (10) are also given as follows:

$$
\hat{K}_F^0 = D_c^{-1} \text{diag}\{c_i(A_{i,i} + \phi_i I)^{-1}\}
$$

$$
\tilde{K}_C^0 = D_c^{-1} \Phi_0 = \hat{K}_C^0
$$

Since (15a) satisfies $\hat{K}_F^0 B = I$, we can apply Theorem 6. Therefore, from Corollary 1, it is an important problem whether there always exists $K$ such that $A_K$ is stable or not.

Since $\hat{K} = \hat{K}_F^0 A + \hat{K}_C^0 C$, $A_K$ is described as follows by using $A_K$.

$$
A_K = A_K + B K_D
$$

$$
K_D = (\hat{K}_F^0 - \hat{K}_0^0) A = D_c^{-1} M
$$

$$
M := \text{col}\{e_{c_i}(A_{i,d_i} - A_{i,d_i})A\}
$$

where $\hat{A} = b$-diag$_{1 \leq i \leq N}\{A_i\}$. We can see that $K_D$ is independent of the design parameters and is dependent only on the system matrices $(A, B, C)$. Therefore, the error $B K_D$ can be obtained before the controller is designed. The framework of robust control is available by regarding $B K_D$ as an uncertainty. Then we obtain Theorem 8.

**Theorem 8.** Assume that the system (1) has a nonsingular diagonal decoupling matrix, and the relative degrees of all the subsystem are no more than 2. Then, there always exists $K$ stabilizing $A - B K$ when $\hat{K} = \hat{K}_F^0 A + \hat{K}_C^0 C$ is obtained from $\hat{K}_F^0$ and $\hat{K}_C^0$ of (15).

Proof. See Appendix A.

**6.2 Shaping of desired input/output characteristics**

In Theorem 8, it was shown that $\hat{K}$ stabilizing $A_K$ always exists. However, shaping of a desired input/output characteristics, which is the main feature of ILQ design method, is not necessarily achieved by using the decentralized controller which is different form the original one. In this section, we propose a method to obtain the decentralized controller by which the input/output characteristics approaches the desired one.

When $K_e$ designed by the original ILQ design method is used, the desired input/output characteristics can be specified in the form of asymptotical decoupling as $\Sigma \to \infty$. On the other hand, when the decentralized controller $\hat{K}_e$ is used, the asymptotic desired input/output characteristics as $\Sigma \to \infty$ given by (13) is not guaranteed to have a diagonal structure, in other words, the decoupling of the transient response is not guaranteed, in which the decoupling of the steady response is guaranteed. Thus, we first specify the decoupling $\hat{G}_{yr}(s)$ by (7), and then by (8) we obtain $K_F^0$ and $K_C^0$ achieving it. Next we obtain $\hat{K}_F^0$ and $\hat{K}_C^0$ minimizing $\| (\hat{G}_{yr}(s) - \tilde{G}_{yr}(s)) \|_{\infty}$ for $\hat{G}_{yr}(s)$ obtained by the decentralized controller. By this optimization, it is expected that the input/output characteristics by the decentralized controller approaches the desired one. The above procedure is summarized as the following problem.

**Problem 1.** Set up $\hat{K}_F^0$ and $\hat{K}_C^0$ achieve the following optimization problem.

$$
\min_{X, K_F^0, K_C^0} \gamma
$$

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subject to $X > 0$

$$\begin{bmatrix}
A_p X + X A_p^T & X C_p^T & B_p \\
C_p X & -\gamma I & 0 \\
B_p^T & 0 & -\gamma I
\end{bmatrix} < 0\quad (18a)$$

where $A_p$, $B_p$, $C_p$ are defined in the following state-space form of $G_{yr}(s) - \hat{G}_{yr}(s)$.

$$\begin{bmatrix}
A_K \\
0 \\
A - B(\hat{K}_F^0 A + \hat{K}_C^0 C) \\
-C
\end{bmatrix} = B K_0^0 \quad (19)$$

Since Problem 1 is BMI (Bilinear Matrix Inequality), it is difficult to obtain the optimal solution of BMI. Then we partition $X$ according to the size of (19) as follows:

$$X = \begin{bmatrix}
X_1 & X_{12} \\
X_{12}^T & X_2
\end{bmatrix} \quad (20)$$

We apply the following iterative algorithm based on LMI to Problem 1.

**Iterative Algorithm**

**Step 1:** Obtain an initial solution pair $\hat{K}_F^0(0)$, $\hat{K}_C^0(0)$ of $K_F$ and $K_C$ stabilizing $A_K$.

**Step 2:** For given $K_F^0(k)$ and $K_C^0(k)$, obtain $X = X(k)$ minimizing $\gamma = \gamma(k)$ under the LMI condition (18) with respect to $X$.

**Step 3:** For partitioned matrices $X_{12}(k), X_2(k)$ of $X(k)$ calculated in Step 2, obtain $K_F^0(k + 1)$ and $K_C^0(k + 1)$ minimizing $\gamma = \gamma(k + 1)$ under the LMI condition (18) with respect to $X_1, K_F^0, K_C^0$.

**Step 4:** Repeat back to Step 2 until $|\gamma(k + 1) - \gamma(k)| < \varepsilon$ is satisfied for some given $\varepsilon > 0$.

From Theorem 8, we note that it is possible to choose $K_F^0$ and $K_C^0$ as an initial solution pair.

**Theorem 9.** When the controller $K_F^0(0)$ and $K_C^0(0)$ stabilizing $A_K$ are given, the above iterative algorithm is always feasible. Its solution pair $K_F^0(k)$ and $K_C^0(k)$ stabilizes $A_K$ and satisfies the following inequality:

$$\gamma := \inf_{K_F^0, K_C^0} \|G_{yr}(s) - \hat{G}_{yr}(s)\|_\infty < 0$$

These $K_F^0(k)$ and $K_C^0(k)$ also satisfy Theorem 6 and Corollary 1. Hence, the optimal servo system can be achieved with some appropriate $\Sigma > 0$.

**7. NUMERICAL EXAMPLE**

We consider the following system consisting of three subsystems:

$$\dot{x}_i = A_{ii}x_i + \sum_{j=1,j\neq i}^3 A_{ij}x_j + b_ui, \quad y_i = c_ix_i$$

where

$$A_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A_{13} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

The relative degrees $d_i (i = 1, 2, 3)$ of each subsystem are $d_1 = d_2 = d_3 = 2$, respectively. The invariant zeros of the system are $(-3, -3, -4)$. The decoupling matrix $D_e$ becomes $D_e = CAB = I$, which is diagonal. The desired input/output characteristics $G_{yr}(s)$ of (7) as $\Sigma \to \infty$ is specified by

$$G_{yr}(s) = \text{diag} \left\{ \frac{16}{(s + 4)^2} \right\}$$

First, $K_F^0$ and $K_C^0$ are obtained by (8) as follows:

$$K_F^0(0) = \begin{bmatrix} 0 & 8 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 8 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & 3 & 0 & 0 & 8 & 1 \end{bmatrix}, K_C^0(0) = \text{diag} \left\{ 16 \right\}$$

Next, an initial solution pair of $K_F^0(0)$ and $K_C^0(0)$, which consist of the diagonal blocks of $K_F^0$ and $K_C^0$, is obtained as follows:

$$K_F^0(0) = b\text{-diag}[0 8 1], K_C^0(0) = \text{diag} [16]$$

For $K$ based on this initial solution pair, it can be confirmed that $A - BK$ is stable, and $\|G_{yr}(s) - \hat{G}_{yr}(s)\|_\infty = 0.251$. Finally, the local optimal solution pair of $K_F^0$ and $K_C^0$ obtained by applying our proposed iterative algorithm is as follows:

$$K_F^0 = b\text{-diag}[0 39.2 1], K_C^0 = \text{diag} [60.2, 60.0, 113]$$

from which we have $\|G_{yr}(s) - \hat{G}_{yr}(s)\|_\infty = 0.206$.

In Figs. 2 and 3, we show the simulation result using $\Sigma = \text{diag}_{1\leq i\leq 3}[100]$. Note that the LQ optimality for the closed loop system with these three controllers are achieved by this $\Sigma$. The solid line shows the output response obtained by the local optimal solution pair of $K_F^0$ and $K_C^0$, the dashed dotted line shows the one by the initial solution pair of $K_F^0(0)$ and $K_C^0(0)$, and the dashed line shows the one by the original centralized solution $K_F^0$ and $K_C^0$. In Fig. 2, all the responses between each input/output pair are shown. From Fig. 2, we see that there are no interferences of transient response when using the optimal solution pair, while there are interferences when using the initial solution pair. In Fig. 3, each output response is shown. From Fig. 3, we see that the output responses obtained by the optimal solution pair, which are a little slower than the one by the original solution, have no oscillation and good performance, while the one by the initial solution pair have oscillation. Therefore, the effectiveness of our proposed method was confirmed as a design method of optimal servo systems by decentralized control.
Fig. 2. All the responses between each input/output pair

Fig. 3. Each output response

8. CONCLUSION

In this paper, we proposed an optimal servo design method of decentralized control systems by using both the inverse linear quadratic problem and the structure of controllers obtained by the ILQ design method. Particularly, we proposed the decentralized ILQ servo design method for systems with diagonal decoupling matrix and relative degree no more than 2. We showed that the closed loop system can always be stabilized in the framework of the decentralized ILQ design method. Moreover, we showed that the optimal decentralized servo controller with the decoupled input/output characteristics is obtained. Finally, we illustrated the effectiveness of our proposed method by simulation.

REFERENCES


Appendix A. PROOF OF THEOREM 8

Proof. Instead of showing that there always exists $\hat{K}$ such that $A - BK = A_K + BD_c^{-1}M$ is stable, it is enough to show that there always exists $K$ such that $\Delta(s) := M(sI - A_K)^{-1}BD_c^{-1}$ satisfies $\|\Delta(s)\|_{\infty} < 1$.

First let $T$ be defined by

$$T = \left[ \begin{array}{cccc} \text{col}_{1 \leq i \leq m} \{T_i\} & & & \\ W & \end{array} \right], \quad T_i = \frac{\text{col}_{i \leq k \leq n} \{c_iA^{k-1}\}}{}$$

where $W$ is a matrix satisfying both $WB = 0$ and $\det(T) \neq 0$, its existence is always guaranteed. Similarity transformation by $T$ yields $A_T$ and $B_T$ [6] as follows:

$$A_T := TAKT^{-1} = \begin{bmatrix} \text{b-diag}_{1 \leq i \leq m} \{\hat{A}_i\} & 0 \\ \hat{A}_{10} & A_0 \end{bmatrix}$$

$$B_T := TBD_c^{-1} = \begin{bmatrix} \text{b-diag}_{1 \leq i \leq m} \{b_i\} & 0 \\ \end{bmatrix}$$

where $\hat{A}_i = -\phi_{i0}$ and $\bar{b}_i = 1$ in the case of $d_i = 1$, $\hat{A}_i = \begin{bmatrix} 0 & 1 \\ -\phi_{i0} & -\phi_{i1} \end{bmatrix}$, $\bar{b}_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the case of $d_i = 2$. $\hat{A}_0$ is also a matrix such that the eigenvalues of $\hat{A}_0$ are the same as the invariant zeros of the system (1). Since $\phi_{i0}$ and $\phi_{i1}$, which are positive due to the stability of $\hat{A}_i$, can be chosen arbitrarily, we can choose $\phi_{i0} = \alpha p_i$ for $d_i = 1$ by using $\gamma_m \geq 1$, $2p > 0$, and $\phi_{i1} = \alpha^2 q_i$, $\phi_{i1} = \alpha (p + q)$ for $d_i = 2$ by using $\gamma_m \geq 1$, $2p > 0$, $2q (\neq p) > 0$. As a new transformation matrix, let $R$ be defined by

$$R = \left[ \begin{array}{c} \text{b-diag}_{1 \leq i \leq m} \{R_i\} & 0 \\ 0 & I \end{array} \right]$$

where $R_i = 1$ in the case of $d_i = 1$, and

$$R_i = \begin{bmatrix} q & 0 \\ p & 1 \end{bmatrix}$$

in the case of $d_i = 2$. Similarity transformation by $R$ yields $A_T$ and $B_T$.

$$A_R := RAR^{-1} = \begin{bmatrix} \text{b-diag}_{1 \leq i \leq m} \{\hat{A}_i\} & 0 \\ \hat{A}_{10} & A_0 \end{bmatrix}$$

$$B_R := RB_T = \begin{bmatrix} \text{b-diag}_{1 \leq i \leq m} \{\bar{b}_i\} & 0 \\ \end{bmatrix}$$

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where $\tilde{A}_i = -\alpha p$, $\tilde{b}_i = 1$ for $d_i = 1$, and $\tilde{A}_i = \alpha \times \text{diag}\{-p, -q\}$, $\tilde{b}_i = [1, 1]^T$ for $d_i = 2$ and $\tilde{A}_{10} = \bar{A}_{10} \times b\text{-diag}_{1 \leq i \leq m} \{R_i^{-1}\}$.

As is well known, $\|G_{\Delta}(s)\|_\infty < 1$ is equivalent to the existence of $P > 0$ such that
\[
Z := PA_R + A_R^TP + PB_RB_R^TP + M_R^TMR < 0 \quad (A.1)
\]
where $M_R := MT^{-1}R^{-1}$. Now, we consider the case of $\alpha = 1$. Let $A_{R_1}$ and $B_{R_1}$ be defined by $A_{R_1} := \text{b-diag}_{1 \leq i \leq m}\{\tilde{A}_i\}$ and $B_{R_1} := \text{b-diag}_{1 \leq i \leq m}\{\tilde{b}_i\}$, respectively. Since $A_{R_1}$ is stable, there exists $\bar{P}_1 > 0$ such that
\[
Z_1 := \bar{P}_1A_{R_1} + A_{R_1}^T\bar{P}_1 + \bar{P}_1B_{R_1}B_{R_1}^T\bar{P}_1 < 0
\]
Since the invariant zeros of the system (1) is stable, namely $A_0$ is stable, there exists $\bar{P}_2 > 0$ such that
\[
Z_2 := \bar{P}_2A_0 + A_0^T\bar{P}_2 < 0
\]
We see that for $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $\|M_R\| \leq \gamma_1$ and $\|\bar{A}_{R_1}^TP_2\| \leq \gamma_2$, respectively.

By choosing $P$ of $Z$ in (A.1) as $P = \text{b-diag}\{\alpha \bar{P}_1, \alpha \bar{P}_2\}$ according to the size of $A_R$, we obtain
\[
Z \leq \begin{bmatrix}
\alpha^2Z_1 + \gamma_1^2I & \alpha\bar{A}_{10}^T\bar{P}_2 \\
\alpha\bar{A}_{10}\bar{P}_2 & \alpha\bar{Z}_2 + \gamma_2^2I
\end{bmatrix}
\leq \begin{bmatrix}
(-\alpha^2\varepsilon_1 + \gamma_1^2)I & \alpha\bar{A}_{10}^T\bar{P}_2 \\
\alpha\bar{P}_2\bar{A}_{10} & (-\alpha^2\varepsilon_2 + \gamma_2^2)I
\end{bmatrix} \quad (A.2)
\]
where $\varepsilon_1 := \lambda_{\min}(-Z_1) > 0$ and $\varepsilon_2 := \lambda_{\min}(-Z_2) > 0$.

By applying Schur complement to the right hand side of (A.2), we see that (A.3) is a sufficient condition for $Z < 0$.

\[
\begin{align*}
-\varepsilon_1\alpha^2 + \gamma_1^2 &< 0 \quad (A.3a) \\
-\varepsilon_1\varepsilon_2\alpha^3 + (\gamma_1^2\varepsilon_1 + \gamma_2^2)\alpha^2 + \gamma_1^2\varepsilon_2\alpha - \gamma_1^4 &< 0 \quad (A.3b)
\end{align*}
\]
Since (A.3) is satisfied for some large $\alpha$, it is shown that there always exists $\hat{K}$ such that $A - BK$ is stable.