Multisensor set-membership state estimation of nonlinear models with potentially failing measurements

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Abstract: A hierarchical state bounding estimation method is presented for a nonlinear dynamic system where different sensors offer several potentially faulty measurements of the same state vector, each of which is subject to unknown but bounded disturbances and is equipped with a local processor. The proposed algorithm works at two levels: at each sampling time, each local processor computes the state estimate and its outer-bounding ellipsoid according to the local measurements given by the corresponding sensor. These ellipsoids are transmitted simultaneously from all local processors to the fusion center which synthesizes them to compute the global state estimate bounding ellipsoid, rejecting, if necessary, the faulty measurements through a judicious choice of some weighting parameters. Then it feeds these data back to all the local processors. This feedback allows the local processors to adjust their results by taking into account the measurements of all the other sensors.

1. INTRODUCTION

Consider a multisensor environment where several sensors observe the same dynamic system and where each sensor is attached to a local processor. Under the assumption of white measurement noises, the distributed Kalman filtering presented in Chong et al. (1990), Zhu et al. (2001) and other references within can be applicable. The algorithm proposed in Song et al. (2007) applies to systems where the measurement noises are cross-correlated with given covariance matrices.

In this paper, we suppose that the system and measurement noises have not any known statistical property, they are supposed to belong to sets with ellipsoidal characterizations. The Kalman filter is no longer applicable in such a case but a set-membership estimation method presented in Becis-Aubry et al. (2008) is. The present work has its beginnings in Becis-Aubry (2010) and now is extended for systems with nonlinear dynamics and with potentially aberrant measurements.

Notations and preliminary facts :

i. The symbols := and =: mean “is equal by definition”.

ii. $\|\cdot\|_2$ is the 2-norm : for any vector $x$, $\|x\|^2 := x^T x$ and for a matrix $A$, $\|A\| := \sup_{\|x\|=1} \|Ax\|$.

iii. $E(c,P) := \{ x \in \mathbb{R}^n \mid (x-c)^T P^{-1} (x - c) \leq 1 \}$ is an ellipsoid in $\mathbb{R}^n$ ($s \in \mathbb{N}$), where $c \in \mathbb{R}^n$ is its center and $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite (SPD) matrix that defines its shape, size and orientation in the $\mathbb{R}^n$ space.

iv. $\mathbf{0}$ denotes a zero vector of appropriate dimension.

v. Block-diag$(X_1, X_2, \ldots, X_N)$ is a block-diagonal (or diagonal) matrix with $X_1, X_2, \ldots, X_N$ its diagonal blocks (or elements).

vi. Let $M$ be a square matrix, $\text{tr}(M)$ is its trace (the sum of its eigenvalues) and $\det(M)$ is its determinant (the product of its eigenvalues).

vii. The segments $[c, b]$ et $[a, b]$ in $\mathbb{R}^n$ are defined as follows:

$$ [a, b] := \{ c \in \mathbb{R}^n \mid c = (1 - \theta)a + \theta b, \theta \in [0, 1] \}, $$

$$ [a, b] := \{ c \in \mathbb{R}^n \mid c = (1 - \theta)a + \theta b, \theta \in [0, 1] \}. $$

2. PROBLEM FORMULATION : TWO-STAGES STATE-BOUNDING PROCESS

2.1 The predicted ellipsoid

The unknown state vector to be estimated, $x_k \in \mathbb{R}^n$, evolves according to the following discrete-time model:

$$ x_k = \varphi(x_{k-1}, u_{k-1}) + w_{k-1}, \quad k \in \mathbb{N}^*, $$

where $u_k \in \mathbb{R}^m$ is a known input vector; $w_k \in \mathbb{R}^n$ is an unknown additive bounded noise vector, it may include modeling inaccuracies like discretization errors and can be viewed as unknown but bounded input, the only property it has is $w_k \in \mathcal{E}(0, W_k) := \{ x \in \mathbb{R}^n \mid x^T W_k^{-1} x \leq 1 \}$, where $W_k$ is a known SPD matrix characterizing the shape, size and the orientation of the ellipsoid containing all possible values of this noise vector; and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $(x, u) \mapsto \varphi(x, u)$. Let $\varphi_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto \varphi_k(x) := \varphi(x, u_k)$ be a bijective $C^1$ function. It is assumed that its Jacobian matrix is bounded for any bounded $x$. It is also assumed that

$$ (x_0^* - \hat{x}_0)^T P_0^{-1} (x_0^* - \hat{x}_0) \leq \sigma_0^2 \iff x_0^* \in \mathcal{E}(\hat{x}_0, \sigma_0^2 P_0) $$

where $\hat{x}_0$ is the estimate of the initial state vector $x_0^*$ and where $P_0$ (a SPD matrix) and $\sigma_0$ (an arbitrary positive scalar) are chosen as large as the confidence in $\hat{x}_0$ is poor.
The following lemma expresses a bounding ellipsoid for \( \varphi_k(x) \), where \( x \in E(\hat{x}_k, \sigma^2_k P_k) \).

**Lemma 1.** Since \( \varphi_k \) is \( C^1 \), if \( \varphi_k \) is locally Lipschitz on \( \mathcal{E}(\hat{x}_k, \sigma^2_k P_k) \) and there exists a bounded matrix \( \Phi_k \in \mathbb{R}^{nxn}, \Phi_k := \Phi_k(\xi^*) \) such that

\[
\xi^* = \arg \max_{\xi \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k)} \| \Phi_k(\xi) \| ,
\]

(2)

where \( \Phi_k(\xi) \) is the Jacobian matrix of \( \varphi_k(x) \) at \( \xi \):

\[
\Phi_k(\xi) := \frac{d \varphi_k}{dx}(\xi) = \frac{d \varphi_k(x)}{dx}|_{x=\xi},
\]

(3)

and the following proposition is true for all \( x \in \mathbb{R}^n \)

\[
x \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k) \Rightarrow \varphi_k(x) \in \mathcal{E}(\varphi_k(\hat{x}_k), \sigma^2_k \Phi_k P_k \Phi_k^T).
\]

\[\blacksquare\]

**Proof.** Let \( \phi : [0,1] \rightarrow \mathbb{R}^n, \tau \rightarrow \phi(\tau) = \varphi_k(\hat{x}_k + \tau (x - \hat{x}_k)) \); when \( \tau \) covers the interval \([0,1], \) the vector \( \hat{x}_k + \tau (x - \hat{x}_k) \) sweeps the segment \([x, \hat{x}_k] \). Let us apply the mean value theorem to \( \phi : \theta \in [0,1], \phi(1) - \phi(0) = \int_0^1 \phi'(\tau) d\tau = \phi'(\theta) \).

Letting \( \xi := \hat{x}_k + \tau (x - \hat{x}_k) \); since the ellipsoid is convex, \( x \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k) \) implies \( \| \varphi_k(x) - \varphi_k(\hat{x}_k) \| \leq \| \Phi_k(\xi) \| \| x - \hat{x}_k \| \leq \| \Phi_k(\xi) \| \| x - \hat{x}_k \| \). Letting \( \delta_k := y_k - F_k \hat{x}_k \), the above inequality leads to

\[
-x \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k) \Rightarrow \varphi_k(x) \in \varphi_k(\hat{x}_k) + \Phi_k(\xi)(x - \hat{x}_k).
\]

(4)

The condition (2) means that \( \Phi_k(\xi) \) is invertible and using (4):

\[
x \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k) \Rightarrow \varphi_k(x) \in \varphi_k(\hat{x}_k) + \Phi_k(\xi)(x - \hat{x}_k),
\]

\[
\| \varphi_k(x) - \varphi_k(\hat{x}_k) \| \leq \| \Phi_k(\xi) \| \| x - \hat{x}_k \| \leq \| \Phi_k(\xi) \| \| x - \hat{x}_k \| \]

\[
\Rightarrow \| \varphi_k(x) - \varphi_k(\hat{x}_k) \| \leq \| \Phi_k(\xi) \| \| x - \hat{x}_k \| \]

\[
\Rightarrow \varphi_k(x) \in \mathcal{E}(\varphi_k(\hat{x}_k), \sigma^2_k \Phi_k P_k \Phi_k^T).
\]

\[\blacksquare\]

**Remark 1.** The matrix \( \Phi_k \) should be computed by resolving at each time step \( k \) the optimization problem (2). Now, setting

\[
\Delta_k := \Phi_k - \Phi_k(\hat{x}_k),
\]

\[
\Delta_k(x_k - \hat{x}_k) \text { is the linearization error vector of } \varphi_k(x_k) \text { at } \hat{x}_k \text { and if } \varphi_k \text { is twice differentiable, it represents the sum of the second and higher order terms of the development of } \varphi_k(x_k) \text { around } \hat{x}_k \text { and hence } \Delta_k \text { can be written as } \Delta_k = H(x, \xi^*) \text { such that } (x^*, \xi^*) := \arg \max_{x, \xi \in \mathcal{E}(\hat{x}_k, \sigma^2_k P_k)} H(x, \xi),
\]

where

\[
H(x, \xi) := \frac{1}{2} \begin{pmatrix} (x - \hat{x}_k)^T H_{k,1}(\xi) \\ (x - \hat{x}_k)^T H_{k,2}(\xi) \\ \vdots \\ (x - \hat{x}_k)^T H_{k,n}(\xi) \end{pmatrix}
\]

and where \( H_{k,j}(\xi) \) is the \( n \times n \) Hermitian matrix of the \( j \)-th component \( \varphi_k(x) \) \( (j = 1, 2, \ldots, n) \), \( H_{k,j}(\xi) := \left( \frac{\partial \varphi_{k,j}}{\partial x} \right)^T \), at \( x = \xi \). A practical method to resolve either optimization problem described above is not yet determined actually (we are still working at). In the mean time, we are just trying to bound the matrix \( \Delta_k \) (see Remark 4).

### 2.2 The corrected ellipsoid

The system (1) is observed by \( N \) sensors. Each one delivers a measurable system output vector, \( y_k \in \mathbb{R}^t \),

\[
y_k = F_k x_k + v_k, \quad i \in \{1, 2, \ldots, N\},
\]

(6)

where \( F_k \in \mathbb{R}^{p \times n}, \) \( i \in \{1, 2, \ldots, N\}, \) is the output matrix of full row rank and \( v_k \in \mathbb{R}^p \) is the unobservable measurement noise vector of the \( i \)-th sensor. If this noise vector \( v_k \) is acceptable, it satisfies \( v_k \in \mathcal{E}(0, V_k) \) \( v_k^T V^{-1}_k v_k \leq 1, \forall k \in \mathbb{N} \), where \( V_k \) is a known SPD matrix characterizing the shape of the ellipsoid containing all acceptable values of this noise vector. Otherwise, \( v_k \) is considered as a fault. The i-th measurement equation (6) with the noise bound defines another bounding set for state vector \( x_k \in \mathcal{E}_k := \{ x \in \mathbb{R}^n | (y_k - F_k x)^T V^{-1}_k (y_k - F_k x) \leq 1 \} \).

This is a degenerate ellipsoid which projection on the \( \mathbb{R}^n \) space via the matrix \( F_k \) is the ellipsoid \( \mathcal{E}(y_k, V_k) \).
∀ωki ∈ I R+, xk ∈ Ski ∩ Ek/k−1i ⊆ E(ˆxki, σ2
kiPki) := Eki.

Remark 2. The parameter 0 < ρi < 1 can be chosen to minimize, at each step k, either the squared volume of Eki/k−1, i.e., σ2
ki−1, or the squared length of its axes, i.e., σ2
ki−1 tr Pki/k−1. For the latter case, there is an explicit solution ρi = \sqrt{\text{tr}\ Wki−1 (\text{tr}\ Φki Pki/k−1, Φki−1)}.

Maksarov and Norton (1996). As for the parameter ωki (see Becis-Aubry et al. (2008) for details), it is given by the following lemma

Proposition 1. The value of ωki that solves

\max \{ \omega ki ∈ I R+, \ yki (xk−xki (xk−xki)T Pki−1 (xk−xki) ) \}

is \(\omega^*_ki = \begin{cases} 0 & \text{if } \delta^2 ki Vki−1 \delta^2 ki \leq 1 \text{ where } \delta^2 ki, \text{ otherwise} \end{cases}\)

Remark 3. The latter equation of the \textit{Proposition 1} is equivalent to \(\| yki − Fki xki \| Vki−1^2 = 1 \Leftrightarrow (I − \omega ki Fki Fki−1 Vki−1) \delta^2 ki \| Vki−1^2 = 1\).

3.2 The global estimation at the fusion agent without feedback

At the central processor, the prediction stage is performed without the need for measurements and during the correction stage, the quantities \(\hat{x}ki, Fki, \text{ and } \sigma^2 ki\) defining the global bounding ellipsoid for the state vector are first expressed using the overall output data.

Corollary 1. If \(xk−1 ∈ E(\hat{x}k−1, \sigma^2 k−1 Fk−1) =: Ek−1\) obeying to (1) for any \(\omega ki ∈ E(0, Wki−1)\) and \(\hat{x}ki = \varphi_k(\hat{x}k−1), Fki = \Phi_k Pki−1\Phi_k−1 + Wki−1,\) such that \(0 < \rho ki−1 < 1, \sigma^2 ki−1 := \sigma^2 ki−1,\)

then \(\forall\rho k−1 ∈[0, 1], xk ∈ E(\hat{x}k/k−1, \sigma^2 k/k−1 Fk/k−1) =: Ek/k−1,\), letting \(p := \sum_{i=1}^N p_i, yki, vki ∈ I R^p\) and \(Fki ∈ I R^{p×n}\) be defined as follows

\(yki := \begin{pmatrix} yki1 \\ yki2 \\ \vdots \\ ykin \end{pmatrix}, \quad vki := \begin{pmatrix} vki1 \\ vki2 \\ \vdots \\ vkin \end{pmatrix}, \quad Fki := \begin{pmatrix} Fki1 \\ Fki2 \\ \vdots \\ Fkin \end{pmatrix},\)

such that \(yki ∈ E(0, Vki), i ∈ \{1, 2, \ldots, N\}\) and letting \(\hat{x}ki = \hat{x}k−1 + Fki Fki−1 \delta ki\)

\(Pki−1 = Fki−1 Fk−1,σ^2 ki−1 = \sum_{i=1}^N \alpha_i ωki,\)

\(\delta ki := \sqrt{\text{tr}\ Wki−1 (\text{tr}\ Φki Pki−1, Φki−1)} + I_p−1\delta ki\)

\(\delta ki := Fki−1 Fki−1,\text{ and }\Omega_k := \text{Block-diag}\{\omega ki \| Vki−1\}, i ∈ \{1, 2, \ldots, N\} ,\)

\(\sum_{i=1}^N \alpha_i = 1, xk ∈ Ski ∩ Ek/k−1i ⊆ E(\hat{x}ki, \sigma^2 ki Pki) := Eki;\)

where \(Eki = \{x ∈ I R^n | (yki − Fki x)^T Vki−1 (yki − Fki x) ≤ 1\} .\)

Proof. Can be easily deduced from Lemma 2 by noticing that for any \((\alpha_1, \alpha_2, \ldots, \alpha_N) ∈ [0, 1]^N,\)

such that \(\sum_{i=1}^N \alpha_i = 1, \alpha_i vki T Vki−1 vki ≤ 1 \Leftrightarrow vki T (\text{Block-diag}\{Vki \| \omega \}), i ∈ \{1, 2, \ldots, N\}\). \(\square\)

Remark 4. The prediction stage of (9) of the estimation algorithm deals separately with the linearization error bound related to the matrix \(\Delta k\), defined by (5) in \textit{Remark 1}, and additive bounded noise ellipsoid of shape matrix \(Wk\). There are two other possibilities to deal with these errors affecting the state dynamics:

- It is possible to consider the linearization error \(\psi_k\) as an additive error, like \(\omega ki\), belonging to an ellipsoid. In fact, using (4), \(\varphi_k(xk−1) − \varphi_k(\hat{x}k−1) = \Phi_k(\hat{x}k−1)(xk−1 − \hat{x}k−1) + \psi_k,\) where \(\psi_k := (\Phi_k(\hat{x}k−1) − \Phi_k(xk−1))(xk−1 − \hat{x}k−1).\)

It is obvious that \(\psi_k ∈ E(0, \sigma^2 k−1 Fk−1 \Delta k).\) Thus, in the \(P_{k−1}\) expression of (9), \(\hat{x}k−1\) can be replaced by \(\Phi_k(\hat{x}k−1)\) and \(Wk\) by \(\Delta k\), the shape matrix of the Minkowski sum of the ellipsoids \(E(0, \sigma^2 k−1 Fk−1 \Delta k)\) and \(E(0, Wk).\)

- It is also possible to assume that the input noise, instead of being additive, is affecting the nonlinear dynamic of the state vector. Therefore, the matrix \(\Delta k\) should be replaced by \(\Delta k ≥ \Delta k\) which takes into consideration this input error in addition to the linearization error. In this case, \(Wk\) could be set to be 0 in (9).

How to bound \(\Delta k\) : Noticing that the only image of the linearization error we have (in the measurement space) is the a priori output error (the innovation vector \(\delta ki\) given by (10d) to within about a measurement noise vector \(F_k(\varphi_k(xk−1) − \varphi_k(\hat{x}k−1)) = \delta ki − \psi_k\), it is possible to state that there exists an increasing function \(f : I R_+ → I R_+\) such that \(\Delta k−1 f^2(\| \delta ki \| Vki−1)fn\) is a SPD matrix.

The local processors are performing their local estimates in an open-loop manner according to Lemma 2 and the parameters \(\omega ki, (i ∈ \{1, 2, \ldots, N\})\) are still computed separately according to Proposition 1.

During the correction stage of the central processor, the global ellipsoid \(E(\hat{x}ki, \sigma^2 ki Pki)\) is computed by using the local ellipsoids \(E(\hat{x}ki, \sigma^2 ki Pki)\) obtained at the local processors and transmitted to the central one. The latter does not receive other information than the attributes of the local ellipsoids from local agents (neither measurements \(yki\) nor noises characterizations \(Vki\)). In this purpose, the
quantities \( \hat{x}_k, P_k \) and \( \sigma^2_k \) defining the global bounding ellipsoid for the state vector are expressed as functions of \( \hat{x}_{k|i}, P_{k|i} \) and \( \sigma^2_{k|i} \), and \( \hat{x}_{k/k-1}, P_{k/k-1} \), and \( \sigma^2_{k/k-1} \).

Lemma 3. If \( \hat{x}_{k|i}, P_{k|i} \) and \( \sigma^2_{k|i} \) satisfy (8) and if \( \hat{x}_{k-1}, P_{k-1} \) and \( \sigma^2_{k-1} \) satisfy (10), then

\[
\begin{align*}
\hat{x}_k &= P_k \left( P^{-1}_{k/k-1} \hat{x}_{k/k-1} + \sum_{i=1}^{N} \alpha_i \left( P^{-1}_{k|i} \hat{x}_{k|i} - P^{-1}_{k/k-1} \right) \right), \\
P_k &= \left( P^{-1}_{k/k-1} + \sum_{i=1}^{N} \alpha_i \left( P_{k|i} - P_{k/k-1} \right) \right) \cdot P_{k/k-1}, \\
\sigma^2_k &= \sigma^2_{k/k-1} + \sum_{i=1}^{N} \alpha_i \left( \sigma^2_{k|i} - \sigma^2_{k/k-1} - \left( \hat{x}_{k|i} - \hat{x}_{k/k-1} \right)^T \sigma_{k/k-1} \right) \\
&= \sum_{i=1}^{N} \alpha_i \left( \sigma^2_{k|i} - \sigma^2_{k/k-1} - \left( \hat{x}_{k|i} - \hat{x}_{k/k-1} \right)^T \sigma_{k/k-1} \right) \\
&= \sum_{i=1}^{N} \alpha_i \left( \sigma^2_{k|i} - \sigma^2_{k/k-1} - \left( \hat{x}_{k|i} - \hat{x}_{k/k-1} \right)^T \sigma_{k/k-1} \right). 
\end{align*}
\]

This fusion algorithm is not finalized yet since it needs the attributes of both predicted and corrected local ellipsoids to be transmitted to the central processor.

Proof. Substituting (10d) in (10a), the latter is rewritten as

\[
P^{-1}_k \hat{x}_k = \left( P^{-1}_{k/k-1} - F^T \Omega_k F_k \right) \hat{x}_{k/k-1} + F^T \Omega_k y_k
\]

and in light of (8), we have

\[
P^{-1}_k \hat{x}_k - P^{-1}_{k/k-1} \hat{x}_{k/k-1} = \sum_{i=1}^{N} \alpha_i \left( P^{-1}_{k|i} \hat{x}_{k|i} - P^{-1}_{k/k-1} \hat{x}_{k/k-1} \right).
\]

In the same way, (10b) can be rewritten as

\[
P_k - P_{k/k-1} = \sum_{i=1}^{N} \alpha_i \left( P_{k|i} - P_{k/k-1} \right).
\]

Noting that \( \sigma^2_{k/k-1} = \sum_{i=1}^{N} \omega_k \alpha_i \left( 1 - \delta_{k|i} V_{k|i}^{-1} \right) \) and that \( \sigma^2_{k/k-1} = \omega_k \left( 1 - \delta_{k|i} V_{k|i}^{-1} \right) \) and \( \sigma^2_{k/k-1} = \omega_k \left( 1 - \delta_{k|i} V_{k|i}^{-1} \right) \) and \( \sigma^2_{k/k-1} = \omega_k \left( 1 - \delta_{k|i} V_{k|i}^{-1} \right) \), we have

\[
\sigma^2_k - \sigma^2_{k/k-1} = \left( \hat{x}_k - \hat{x}_{k/k-1} \right)^T P^{-1}_k \left( \hat{x}_k - \hat{x}_{k/k-1} \right) = \sum_{i=1}^{N} \alpha_i \left( \sigma^2_{k|i} - \sigma^2_{k/k-1} - \left( \hat{x}_{k|i} - \hat{x}_{k/k-1} \right)^T \sigma_{k/k-1} \right). 
\]

3.3 The choice of the weighting parameters \( \alpha_i \) for fault tolerant approach

Definition 1. The \( i^{th} \) \( (i \in \{1, 2, \ldots, N\}) \) measurement vector \( y_{k|i} \) is considered as faulty (or aberrant), at the time step \( k \), if the noise vector \( v_{k|i} \) is not acceptable, that is, \( v_{k|i} \notin \mathcal{E}(0, V_{k|i}) \).

Proposition 2. If \( x_k \in \mathcal{E}_{k/k-1} \) and if \( v_{k|i} \in \mathcal{E}(0, V_{k|i}) \) then \( \mathcal{S}_k \cap \mathcal{E}_{k/k-1} \neq \emptyset \).

Thus to check whether a given measurement is faulty, it suffices to check whether the intersection of the sets \( \mathcal{S}_k \) and \( \mathcal{E}_{k/k-1} \) (or the intersection of their projections in \( \mathbb{R}^p \) space via the matrix \( F_k \)) is empty.

The parameters \( \alpha_i, i \in \{1, 2, \ldots, N\} \) were introduced to define the ellipsoid containing the overall sensor noises \( v_k \), so it would be natural to choose their values such that the size of the ellipsoid \( \mathcal{E}(0, \text{diag}(\frac{V_{k|i}}{\alpha_i})_{i \in \{1, 2, \ldots, N\}}) \) is minimized, which correspond to the values of \( \alpha_i \) that minimize the size of the ellipsoid containing the vector (Minkowski) sum of the the \( N \) ellipsoids \( \mathcal{E}(0, V_{k|i}) \) (The reader is referred to Durieu et al. (2001) where two minimization method—using the trace and the determinant criterions—with unique solutions are proposed). This solution is not retained for two reasons.

Firstly, to compute such parameters, it would be necessary to transmit all the local noises characteristics \( (V_{k|i}) \) to the fusion center and this is contrary to the aim of the designed method.

Secondly, this algorithm is used to reject the aberrant measurements, thus the \( \alpha_i \) should function of the measurement errors.

If, at a given time step \( k \), one (or more) of the \( N \) measurement vectors, \( y_{k|i} \), is subject to an abnormal perturbation which doesn’t lie in the ellipsoid \( \mathcal{E}(0, V_{k|i}) \) containing all acceptable measurement noises, then the \( \alpha_i \) should reduce the effect of this faulty output \( i \) on the state estimate with respect to other outputs. Therefore, the \( \alpha_i \) should be chosen as a normalized decreasing function of some norm of the a priori output error vector \( \delta_{k|i} \) as \( \| \delta_{k|i} \|_{V_{k|i}} \). Let \( g: [1, +\infty[ \rightarrow \mathbb{R}_+^*, r \mapsto g(r) \) be a decreasing function. The parameters \( \alpha_i \) is chosen as

\[
\alpha_i := \alpha_{k|i} := \left\{ \begin{array}{ll}
g \left( \| \delta_{k|i} \|_{V_{k|i}} \right), & \| \delta_{k|i} \|_{V_{k|i}} \geq 1; \\
\frac{1}{N} \sum_{j=1}^{N} g \left( \| \delta_{k|i} \|_{V_{k|i}} \right), & \text{otherwise.} \end{array} \right.
\]

For example, \( \alpha_{k|i} = \left( \| \delta_{k|i} \|_{V_{k|i}}^{-1} \right)^{-1} \) if \( \| \delta_{k|i} \|_{V_{k|i}} \geq 1 \).

Thus, if all the measurements are fault-free, the output errors norms are roughly of the same order for all the \( N \) sensors, and so are the parameters \( \alpha_i, i \in \{1, 2, \ldots, N\} \).

3.4 The global estimation at fusion agent with feedback

If the local estimates are left to evolve autonomously, in open-loop manner, they can not cope with possible faults occurring on their own outputs and therefore can lead to
wrong trajectories that can be handicapping for the central estimation processor. This is why a feedback from the central to the local processors can enhance the estimation procedure.

The fused data of the preceding time step \( \hat{x}_{k-1} \), \( P_{k-1} \) and \( \sigma^2_{k-1} \) are used to compute the predicted ellipsoid by the central processor; then \( \hat{x}_{k/k-1} \), \( P_{k/k-1} \) and \( \sigma^2_{k/k-1} \) are returned to the local sensor agents where they will be used instead of \( \hat{x}_{k/k-1} \), \( P_{k/k-1} \), and \( \sigma^2_{k/k-1} \), in the following way \( \hat{x}_{k/k-1} \leftarrow \hat{x}_{k-1} \), \( P_{k/k-1} \leftarrow \alpha_i P_{k-1} \), \( \sigma^2_{k/k-1} \leftarrow \sigma^2_{k-1} \). Note that these operations are equivalent to \( \hat{E}(\hat{x}_{k/k-1}, \sigma^2_{k/k-1} P_{k/k-1}) \leftarrow \hat{E}(\hat{x}_{k-1}, \sigma^2_{k-1} P_{k-1}) \).

Notice also that it would be equivalent to return the fused data \( \hat{x}_{k-1} \), \( P_{k-1} \) and \( \sigma^2_{k-1} \) to local processors (setting \( \hat{x}_{k-1} \leftarrow \hat{x}_{k-1} \), \( P_{k-1} \leftarrow \alpha_i P_{k-1} \) and \( \sigma^2_{k-1} \leftarrow \sigma^2_{k-1} \)) before the prediction step which would be performed by the \( N \) local processors instead of the central one. Obviously this alternative is less efficient:

\[
\begin{align*}
P_{k/k-1}^{-1} &= \sum_{i=1}^{N} \alpha_i P_{k-1}^{-1} - (N-1)P_{k-1}^{-1} \quad \text{(15a)} \\
\hat{x}_k &= \hat{x}_k + \sum_{i=1}^{N} \alpha_i (\hat{v}_{k_i} - \hat{x}_{k/k-1}) \quad \text{(15b)} \\
\sigma^2_{k/k-1} &= \sum_{i=1}^{N} \alpha_i (\sigma^2_{k_i} - (\hat{v}_{k_i} - \hat{x}_{k/k-1})^T P_{k_i}^{-1} (\hat{v}_{k_i} - \hat{x}_{k/k-1}) + (\hat{x}_k - \hat{x}_{k/k-1})^T P_{k/k-1}^{-1} (\hat{x}_k - \hat{x}_{k/k-1}) \quad \text{(15c)}
\end{align*}
\]

These equations are straightforward when replacing respectively, \( \hat{x}_{k/k-1} \), \( P_{k/k-1} \), and \( \sigma^2_{k/k-1} \), by \( \hat{x}_{k/k-1} \), \( \alpha_i P_{k-1} \) and \( \sigma^2_{k-1} \alpha_i \) in (11).

The overall algorithm for the global closed-loop state estimation:

1. \( k \leftarrow 1 \).
2. The predicted ellipsoid \( \hat{E}(\hat{x}_{k/k-1}, \sigma^2_{k/k-1} P_{k/k-1}) \) (which depends only on the system dynamics (\( \Phi_{k-1} \), \( \Phi_{k-1} \), and \( \Delta_k \)), the process noise characteristics \( W_{k-1} \) and \( \hat{E}(\hat{x}_{k-1}, \sigma^2_{k-1} P_{k-1}) \)) is computed by the central processor according to (9) with an appropriate choice of the parameter \( \rho_k \) (see the Remark 2) and transmitted to the local processors setting \( \hat{x}_{k/k-1} \leftarrow \hat{x}_{k/k-1} \), \( P_{k/k-1} \leftarrow \alpha_i P_{k-1} \) and \( \sigma^2_{k/k-1} \leftarrow \sigma^2_{k-1} \alpha_i \), with \( \alpha_i \) given by (13) (see the following section).
3. If \( S_{k_i} \cap \hat{E}_{k/k-1} = \emptyset, \alpha_i = 0 \). If this is true for all \( i \in \{1, 2, \ldots, N\} \), the correction stage for this time step \( k \) is skipped and go to 5. Otherwise, the local processors (for which \( \alpha_i \neq 0 \)) compute the corrected ellipsoids \( \hat{E}(\hat{x}_{k_i}, \sigma^2_{k_i} P_{k_i}) \) using their respective measurements, the output noise characterizations and \( \hat{E}(\hat{x}_{k/k-1}, \sigma^2_{k/k-1} P_{k/k-1}) \) according to (8) and to the Proposition 1.
4. The local processors transmit these ellipsoids to the fusion center which, deduces, using \( \hat{E}(\hat{x}_{k/k-1}, \sigma^2_{k/k-1} P_{k/k-1}) \), the unique global ellipsoid \( \hat{E}(\hat{x}_{k}, \sigma^2_{k} P_{k}) \) containing all possible values of the state vector \( x_k \) according to (15).
5. \( k \leftarrow k + 1 \) and go to 2.

4. EXAMPLE OF A MIXED-CULTURE BIOREACTOR

We consider the dynamics of two cell strains that are competing for a same substrate with an opposite sensitivity to an external growth-inhibiting agent (Hoo and Kantor (1986)). The growth of the species 2 is then inhibited by the addition of an inhibitor while the growth of the species 1 deactivates the inhibitor agent. Let \( x_1 \) be the cell density of the inhibitor resistant cells and \( x_2 \) the the density of the inhibitor sensitive cells. The concentrations of the rate-limited substrate and inhibitor in the fermentation medium are denoted by \( S \) and \( I \). We consider a continuous mixed culture chemostat of constant volume \( V \), total volumetric flowrate \( q \), inlet substrate concentration \( S_n \) and inlet inhibitor concentration \( I_n \). Material balances for the chemostat yield the following equations:

\[
\begin{align*}
\dot{x}_1 &= \mu_1(S)x_1 - D x_1 \\
\dot{x}_2 &= \mu_2(S, I)x_2 - D x_2 \\
\dot{I} &= -p x_1 I - DI + DI_n \\
\dot{S} &= \frac{\mu_1(S)}{Y_1}x_1 - \frac{\mu_2(S, I)}{Y_2}x_2 + DS_n - DS
\end{align*}
\]

where \( D = \frac{q}{V} \) is the dilution rate, \( Y_1 \) and \( Y_2 \) are the yields of species 1 and 2 per unit substrate. We assume that the species 1 deactivates the inhibitor at a rate proportional to \( p x_1 I \) where \( p \) is the rate constant. \( \mu_1(S) \) and \( \mu_2(S, I) \) are the specific growth rate for species 1 and 2, respectively, given by the Monod expressions:

\[
\mu_1(S) = \frac{\mu_{1_{\text{max}} S}}{K_M + S}, \quad \mu_2(S, I) = \frac{\mu_{2_{\text{max}} S}}{K_I + S \mu_{I_{\text{max}}}}
\]

where we suppose that the Monod saturation constant \( K_M \) is the same for both species. The control variables are the dilution rate \( D \) and the total inhibitor addition rate \( D_I = D I_n \). The outputs are \( y_{k_1} = (x_1 + x_2)^2 + v_{k_1} \) and \( y_{k_2} = S + v_{k_2} \). The bioreactor model is discretized with the Euler method using a fixed step size \( T = 0.1 \) hr. The input variable and the parameters values are fixed to \( D = 0.3 \) hr\(^{-1} \), \( D_I = 0.0027 \) hr\(^{-1} \), \( \mu_{1_{\text{max}}} = 0.4 \) hr\(^{-1} \), \( \mu_{2_{\text{max}}} = 0.5 \) hr\(^{-1} \), \( K_M = 0.05 g/l, K_I = 0.02 g/l, I_1 = 0.2, Y_2 = 0.15, S_n = 2 g/l \) and \( p = 0.5 \) hr\(^{-1} \). The discretization error is represented by the additive process noise \( \nu \in \mathcal{E}(0, W) \), where \( W = (0.5\%)^2 \text{diag}(\text{mean}(x_i))_{i \in \{1, \ldots, 4\}} \). The measurement noises belong to the ellipsoids \( \mathcal{E}(0, V_j) \), \( (j \in \{1, 2\}) \), where \( V_j = (5\%)^2 \text{diag}(\text{mean}^2(v_{ij}))_{i \in \{1, 2\}} \).
The initial conditions $\mathbf{x}_0 = (0.2 \ 0.1 \ 0.01 \ 0.2)^T$ and $\hat{\mathbf{x}}_0 = 0$, $P_0 = 100I_4$ and $\sigma_0^2 = 1$. The matrix $\Phi_k$ is replaced by $\Phi_k(\hat{\mathbf{x}}_k) + 0.005 \left( \alpha_1 \| \delta_{k-1} \|_{V^{-1}} + \alpha_2 \| \delta_{k} \|_{V^{-1}} \right) I_4$. A constant fault is affecting $y_{k_1}$ between the time steps 500 and 510 and an other one is affecting $y_{k_2}$ between the time steps 600 and 620 (the simulation is run during $N=1000$ time steps). The weighting parameters $\alpha_i$, $i \in \{1, 2\}$ is chosen as $\alpha_i = \| \delta_{k_j} \|_{V^{-1}}^{-1} \left( \| \delta_{k_1} \|_{V^{-1}}^2 + \| \delta_{k_2} \|_{V^{-1}}^2 \right)$, $j \neq i$, if $\| \delta_{k_i} \|_{V^{-1}} > 1$ for all $i \in \{1, 2\}$ and $\alpha_i = \| \delta_{k_j} \|_{V^{-1}}^{-1} \left( \| \delta_{k_j} \|_{V^{-1}}^2 + 1 \right)$ if $\| \delta_{k_j} \|_{V^{-1}} \leq 1$, $j \neq i$ and $\alpha_1 = \alpha_2 = 0.5$ otherwise. Fig. 1-2 show satisfying tracking performances of the state estimates. We can notice that they are insensitive to the measurement faults.

The developed algorithm is decomposed into two levels. At the lower or local level, the state estimate and its outer-bounding ellipsoid is computed, at each sampling time, using the measurements given by each sensor separately. The local results are transmitted to the central processor which computes the global estimate with its bounding ellipsoid and feeds this information back to all the local processors which update their data using this global estimate. A particular choice of the parameter $\alpha_{k_i}$, which weights at time step $k$ the contribution of the $i$th sensor measurement to the global state estimate and its bounding ellipsoid, allows to reduce the influence of this measurement when it is subject to abnormal noises.

The optimality for distributed kalman filtering fusion with cross-correlated sensor noises when the measurement is considered as fault-free and unbounded otherwise.

Fig. 4. $x_{4_k} = S (- - - -)$ and $\hat{x}_{4_k} (-)$ of the same state vector, each of them is subject to disturbances which are bounded with known ellipsoidal bounds when the measurement is considered as fault-free and unbounded otherwise.

REFERENCES


