Structure of Discrete Systems with Switched Delay

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Abstract: We clarify, for discrete time systems, two notions of a system with time variant delay, and explore their structural properties. The ultimate purpose is to shed some light on the potential ill-posedness associated with problems with rapidly increasing delay (delay increment greater than one system update step). The main approach is to consider the time delay system using an extension of the state space. For the case of periodically varying delays, this problem becomes a special case of the lifting for periodic systems. Structural problems, such as stability and reachability, can then be investigated using time-invariant theory. For instance, the property that the quotient space of reachable periodic delay systems modulo state space similarity is a smooth manifold, is inherited.

1. INTRODUCTION: BEHAVIORAL APPROACH

We start with a brief overview of the behavioral approach towards discrete time systems as expounded in Polderman and Willems [1998]. Let $\mathbb{T}$ denote the time set. For discrete systems $\mathbb{T} = \mathbb{Z}$. We let $\mathbb{W}$ be the set of values of trajectories, say for some $n \geq 1$, $\mathbb{W} = \mathbb{R}^n$. Let $\mathbb{W}^T$ denote the set of all maps from $\mathbb{T}$ to $\mathbb{W}$. A dynamical system $\Sigma$ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$, where $\mathbb{B}$ is called the behavior and is an appropriately restricted subset of $\mathbb{W}^T$. We define the evaluation functional $\sigma_k$ by $\sigma_k(w) = w_k$. The shift operator $\mathbf{T}$ is defined by $\sigma_k(\mathbf{T}w) = \sigma_{k+1}w$. Hence $\mathbf{T}n$ corresponds with an $n$-step shift to the left.

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$ is said to be linear if $\mathbb{W}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$, and the behavior $\mathbb{B}$ is a linear subspace of $\mathbb{W}^T$. The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$ is said to be shift invariant if $w \in \mathbb{B}$ implies $\mathbf{T}^\tau w \in \mathbb{B}$ for all $\tau \in \mathbb{T}$.

In full generality, the external signals describing a discrete phenomenon are an infinite $\mathbb{R}^n$ valued sequence, that may be partitioned into an infinite number of finite dimensional components.

A sequence in $\mathbb{W}^T$ is said to be $\ell_2$-locally bounded if $\sum_{i=m}^p \|w_i\|_2^2 < \infty$ for all finite $m < p$. The space of all locally bounded sequences in the $\ell_2$ norm is denoted $\ell_2^{\infty}$. This space is a separable Hilbert space. Any bounded operator defined everywhere on a separable Hilbert space $\mathcal{H}_1$ and mapping to a separable Hilbert space $\mathcal{H}_2$ admits a matrix representation which uniquely defines this operator. Hence in the most generality, any discrete time behavior can be represented by a infinite matrix. A behavioral description restricts the trajectories that are allowed by the system. Thus, a global (timeless) viewpoint is the description of the behavior as the nullspace of some infinite matrix. For causal systems, this matrix is upper triangular (using the convention that scrolling the vector down the signal vector means going back in time). For a time invariant system, the matrix is (block)-Toeplitz. Such a global matrix approach is taken in Dewilde and van der Veen [1998] and other work referenced therein.

A behavior is called autonomous if for all $w^{(1)}, w^{(2)} \in \mathbb{B}$ $w_k^{(1)} = w_k^{(2)}$ for $k \leq 0$ implies $w_k^{(1)} = w_k^{(2)}$ for all $k$. For an autonomous system, the future is entirely determined by its past.

The notion of controllability is an important concept in the behavioral theory, as in all of system theory. Let $\mathbb{B}$ be the behavior of a linear time invariant system. This system is called controllable if for any two trajectories $w^{(1)}$ and $w^{(2)}$ in $\mathbb{B}$, there exists a $\tau \geq 0$ and a trajectory $w \in \mathbb{B}$ such that

\[ \sigma_k(w) = \begin{cases} \sigma_k(w^{(1)}) & k \leq 0 \\ \sigma_k(\mathbf{T}^{-\tau}w^{(2)}) & k \geq n \end{cases} \]

i.e., one can switch from one trajectory to the other, with perhaps a delay of $\tau$ steps. Note that an autonomous system cannot get of a trajectory once you are on it. Hence an autonomous system is not controllable.

Now we add some new definitions:
Let $\mathbb{B}$ be the set of all sequences. Let $\Pi$ denote the future-time projection operator, mapping $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$, such that $\Pi(w_k)^{\infty}_{k=-\infty} = (w_k)^{\infty}_{k=0}$, i.e., the past gets forgotten under application of $\Pi$. Introduce also $\mathbf{R}$, be the parity or reflection operator, defined by: $\mathbf{R}(w_k)^{\infty}_{k=-\infty} = (w_{-k})^{\infty}_{k=-\infty}$.

2. DISCRETE DELAY SYSTEM

In this paper we are interested in a simple interconnected system (fig. 1) described locally (in time) by

\[ x_{k+1} = Ax_k + z_k + bu_k \]
\[ z_k = Bx_{k-n} \]

Let us first consider the case where the delay, $n$, is fixed.
We assume that $A$ and $B$ are in $\mathbb{R}^{m \times m}$, and that at time $k$, the variables $x_k$ and $u_k$ are available (full observability).

We first make a preliminary observation. If $\text{rank}(B) < m$, not all components of the $m$-vector $x_k$ need to be memorized. Indeed, a simple similarity transformation (in $GL_m(\mathbb{R})$) may recast the system as

$$
\xi_{k+1} = A_1 \xi_k + A_2 \eta_k + B_1 \eta_{k-n} + b_1 u_k \quad (3)
$$

$$
\eta_{k+1} = A_3 \xi_k + A_4 \eta_k + B_2 \eta_{k-n} + b_2 u_k \quad (4)
$$

where now $\dim \xi_k = m - \text{rank}B$, $\dim \eta_k = \text{rank}B$, and $[B_1^T, B_2^T] \in \mathbb{R}^{\text{rank}(B) \times m}$. It is obvious from these equations that in order to evolve the system forward in time, from $k$ onward, we will need accessibility to the variables $\xi_k$, and $\eta_{k-n}, \eta_{k-n+1}, \ldots, \eta_k$. This means: $(m - \text{rank}B) + (n + 1) \text{rank}B = m + n \text{rank}B$ real variables suffice to predict the future behavior, if also the future inputs, $u_k, u_{k+1}, \ldots$, are known. We call this sufficient information a “state” of the system at time $k$, as this is indeed a sufficient statistic replacing full knowledge of the past behavior.

![Fig. 1. Representation of delay system](image)

Figure 1 gives a realization of such a delay system for the case when $B$ has full rank. The register on the top is the memory device, and operates by right shifting the content from each cell (here an $m$-vector) to the one on its right at each clock pulse. The data in the rightmost cell is subsequently lost, and the new item generated by the adder is loaded in the leftmost cell.

3. TIME-VARIANT DELAY

Two causal interpretations of such a system are possible if the delay depends on time $k$. Thus $n = n(k)$. For notational purposes, we assume that the minimal delay is at least 1. The other case can easily be handled as well.

3.1 Causal Models

In the first interpretation, we imagine that each memory cell connects to a copy of the $B$-matrix, but a switching device chooses which one to connect to the adder. Only one connection is made at each step. This is shown in Figure 2. Here $n_{\text{max}}$ is the maximal delay that is anticipated, but this could be unbounded in principle. In this form, the buffer has a fixed size. Hence, the state space dimension is constant, but this may not be a minimal state space. Effectively, the system is a special case of a

![Fig. 2. Interpretation with fixed register](image)

distributed time delay system, but with time variant delay distribution. A system with time variant delay embedded in a large dimensional system as above was called the lossless causalization in Verriest [2009a, 2010]. The fact that the buffer length depends on future memory sizes makes this system noncausal in a certain sense. In many cases this required (buffer size) information may be known a priori, but there are situations, such as with state dependent buffer size (see Michiels and Verriest [2011]), where this cannot be anticipated.

The other interpretation considers at each time $k$ a system as in figure 1 but with $n = n(k)$, thus with $k$-dependent buffer size (see Figure 3). At each step, the input for the $B$-matrix is taken from the rightmost register in the buffer. In particular, this implies that if $n_{k+1} \geq n_k + 1$, no information is available to store in the added cells on the right, except for the leftmost of these cells, where thus $x_{k-n} = x_{(k+1)-(n+1)}$ will be stored. On the other hand, if $n_{k+1} < n_k$, the buffer gets shortened, and the information that was stored in the deleted cells is now irretrievably lost. No anticipation of the future delays is needed here. For this reason, this system causalization was called the forgetful causalization in Verriest [2009a, 2010]. Of course both causalization models will differ in their outcome, and it makes no sense to determine which is right or wrong. We just point out that the physics behind both models is different, and therefore which one is appropriate depends on the phenomenon to be studied. We caution that a blind use of a mathematical model may lead to inconsistencies.

3.2 State Space and Trajectories

The main difference in these interpretations is that for the first, the memory size stays constant, but not everything stored in memory may be used at any time. This version
of the time-varying delay systems may be lifted to a large dimensional time-varying system, but perhaps at the expense of giving a nonminimal model.

In the second representation, the memory size is a function of \(k\). However, this creates a problem with the conventional notion of a state space. A state space should itself have a time-invariant structure, otherwise there cannot be a notion of trajectory, and surely not of stability. This problem can be resolved by casting the model in the class of multi-mode, multi-dimensional (\(M^3D\)) systems.

In Verriest [2009b] we studied the continuous version of this problem.

Characterize each physically different buffer size as a mode of the system. The state space of such a \(M^3D\) system is given by a discrete bundle, with the set of buffer lengths \(n_j\) as base space. If \(B\) has full rank \((m)\), the fibre over \(j\) is a vector space of dimension \((n_j + 1)m\) where \(n_j\) is the delay in phase \(j\). In general, the dimension of the \(j\)-th fibre is \(d_j = n_j\) rank \(B + m\). In addition, between switches, the state transition from one buffer to the next is simply given by projection (to the lower dimensional fiber) or embedding (in the higher dimensional fiber), where empty cells are loaded with the zero state (the equilibrium). We note that the bundle structure is necessary as we want a stationary structure for the state space. If that were not the case we could not reasonably talk about trajectories in the state space.

4. PERIODIC TIME DELAY SYSTEM

For a periodically varying delay, the previous model simplifies considerably. Let the system have delay sequence \(n(k); k = 1, 2, \ldots, N\) which repeats periodically. Let \(n_{\max}\) be their maximum. With the first interpretation of lossless causalization, the system can be lifted to a system of dimension \((n_{\max} + 1)m\).

\[
\chi_{k+1} = \begin{bmatrix} A & \beta^{(k)}_{1} & \cdots & \beta^{(k)}_{n_{\max}} \end{bmatrix} \chi_k + \begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_k. \tag{5}
\]

where only one of the \(\beta^{(k)}_{j}\) in the top block row equals \(B\) at any time, the other entries being zero. Denote this by

\[
\chi_{k+1} = A[k] \chi_k + b[k] u_k, \tag{6}
\]

where the subscript \([k]\) denotes \([k] = ((k - 1) \text{mod} N) + 1\). This determines an \(N\)-period of system of fixed state space dimension. Its stability is completely determined by the monodromy matrix \(\hat{F} = \hat{A}_{[N]} \hat{A}_{[N-1]} \cdots \hat{A}_{[1]}\), and its reachability properties by the monodromy systems \((\hat{F}, \hat{G}_1)\) with \(\hat{F}_i = \hat{A}_{[i-1]} \hat{A}_{[i-2]} \cdots \hat{A}_{[1]} \hat{A}_{[N]} \cdots \hat{A}_{[i]}\), and \(\hat{G}_i = [b^d, A_{[i-1]} b^d, A_{[i-2]} b^d, \ldots, A_{[i-2]} A_{[i-1]} A_{[i-2]} b^d, \ldots, A_{[i-1]} A_{[i-1]} b^d]\).

The monodromy system \((\hat{F}_i, \hat{G}_i)\) is the system that describes how all inputs over one past period contribute to \(\chi_t\). If the \(i\)-th monodromy system is reachable, then the full memory state \(\chi_{i}\) can be made arbitrary, and thus as a special case, can be zeroed. The time-invariant nature of the monodromy systems implies that if this is possible at all, it will require at most \(Nm(n_{\max} + 1)\) steps. However it may still be possible to control to zero, even if the monodromy system is not reachable: Reachability implies controllability (accessibility of the zero state), but is not implied by it in the discrete time case [Kailath, 1980, p. 100].

With the forgetful causalization, the individual mode systems have fibre dimension \((n_k + 1)m\). Denote by \(\chi_k\) the entire contents of the shift register, by concatenating the state in each cell. Thus

\[
\chi_k = [x_k^T, x_{k-1}^T, \ldots, x_{k-n_{\max}}^T]. \tag{7}
\]

Represent the system in phase \(k\) by

\[
\chi_{k+1} = \begin{bmatrix} A & 0 & \cdots & 0 & B \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} \chi_k + \begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_k \tag{8}
\]

which we shall denote as

\[
\chi_{k+1} = A[k] \chi_k^+ + b[k] u_k, \tag{9}
\]

where the subscript \([k]\) again denotes \([k] = ((k - 1) \text{mod} N) + 1\). In addition, if the delay switches from \(n(k)\) at step \(k\) to \(n(k + 1)\) at step \(k + 1\), this transition involves a mode change, which induces a change in the dimension of the delay system representation. This requires the additional transition

\[
\chi_{k+1}^+ = \begin{bmatrix} I \\ 0 \end{bmatrix} \chi_k, \text{ if } n_k < n_{k+1} \tag{10}
\]

\[
\chi_{k+1}^+ = \begin{bmatrix} I \\ 0 \end{bmatrix} \chi_k^-, \text{ if } n_k > n_{k+1}. \tag{11}
\]

These matrices are respectively a projection and an embedding. Denote this transition matrix from \(k \rightarrow k + 1\) by \(S_{k+1,k}\). Combining, we set \(A_k = S_{k+1,k} A[k]\) as the total transition from the \(m(n_{k+1} + 1)\)-dimensional state at \(k\) to the \(m(n_{k+1} + 1)\)-dimensional state at \(k + 1\), and likewise, let \(S_{k+1,k} b[k] = b_k\) be the effect of the input \(u_k\) to \(\chi_{k+1}\). In general, the \(A_k\) are no longer square matrices.

Every \(N\)-periodic system is therefore completely described by a periodic sequence of matrices

\[
(A_k, b[k])_N = ((A_1, B_1), (A_2, B_2), \ldots, (A_N, B_N)) \tag{12}
\]

Again noting that \(\text{dim}\ \chi_k = m(n_{k+1} + 1)\), we see that the \(j\)-th phase matrix \(A_j\) is in \(\mathbb{R}^{m(n_{j+1} + 1) \times m(n_{j+1})}\). The system \((A_k, B_k)_N\) is now purely periodic, and the structure of such systems was described in Helmke and Verriest [2011].

We associate with every \(N\)-periodic system \((A_k, B_k)_N\) a time-invariant linear system as follows. Let \((A_e, B_e)\) denote the extended system defined as follows:

\[
A_e := \begin{bmatrix} 0 & \cdots & 0 & A_N \\ A_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{N-1} & 0 \end{bmatrix} \tag{13}
\]
This extended system has dimensions
\[ n_c = m \sum_{k=1}^{N} (n_k + 1) ; \quad m_c = N \]

If we use the time-varying state space transformations and input transformations as in Park and Verriest [1989]
\[ (x_k) \mapsto (z_k) , \quad (u_k) \mapsto (v_k) \]
then the system with periodically varying delay is equivalent to the extended time-invariant dynamics
\[ z_{k+1} = A_c z_k + B_c v_k , \quad k \in \mathbb{Z}. \]

We shall now relate the reachability properties of the periodic system to the extended system. But first we need to define what these concepts mean for systems with variable dimensions. The analysis of observability proceeds by duality.

**Definitions**

(i) The periodic system is **reachable at phase** \( j \) if the state \( x_j \) can be made arbitrary by applying suitable inputs prior to \( j \), regardless the initial state.

(ii) The periodic system is **reachable**, if it is reachable at all phases \( j = 1, \ldots, N \).

In the literature a stronger notion of reachability is defined for systems of constant dimension \( n \), sometimes referred to as uniform reachability. This notion is equivalent to reachability in \( n \) steps. In [Bittanti and Colaneri, 2009, p. 115] a characterization of uniform reachability is given via the existence of a periodic rational canonical form.

In Helmke and Verriest [2011], we proved the extension:

**Theorem 1.** The periodic system is reachable if and only if the extended system is reachable.

Various invariant theoretic characterizations of reachability are given in Helmke and Verriest [2011], and it was shown that the quotient space of reachable periodic systems modulo state space equivalence is a smooth manifold. An embedding into a Grassman manifold can be given, extending the well-known Kalman embedding of reachable pairs. Hence all this carries over for discrete periodic delay systems.

### 5. A SIMPLE EXAMPLE

Consider the scalar system \( x_{k+1} = a x_k + b x_{k-n(k)} + u_k \). We call \( x_k \) a **partial state**. For the fixed register system (lossless causality), the extended system is given in reachability canonical form, with

\[
A_c = \begin{bmatrix}
a & 0 & \cdots & 0 \\
1 & & & 0 \\
& \ddots & & \vdots \\
& & 1 & 0 \\
\end{bmatrix}.
\]

where the first row contains a \( b \) somewhere, and \( b^c = [1, 0, \ldots, 0]^T \). Denote the extended state in \( \mathbb{R}^d \) by \( \chi \), then from

\[
\chi_0 = A_c^{n-1} \cdots A_c^{d} \chi_{-d} +
\]

\[
[b_c^{(n-1)} A_c^{n-2} b_c \cdots A_c^{d} b_c^{d} ]
\]

the reachability matrix is upper triangular. Hence one can always find a sequence so that \( \chi_0 \) can be made arbitrary. Consequently, this representation is fully reachable, no matter what the sequence of delays is.

Now we consider the same scalar delay system but with the **time varying buffer** interpretation. To show the proof of concept, we consider for simplicity the case of two delay modes. Let \( n_a \) be the short delay, maintained for \( k_a \) consecutive steps, and \( n_b \) the larger delay, which is maintained for \( k_b \) consecutive steps. Let the extended systems respectively be \( (A_c^a, b_c^a) \) and \( (A_c^b, b_c^b) \). This leads to the **partial state description** (we take \( d_a = 1 \) and \( d_b = 2 \), \( k_a = k_b = 2 \), and \( u_k \equiv 0 \) for notational simplicity)

\[
x_1 = a x_0 + b x_{-1}, \quad d(0) = 1 \\
x_2 = a x_1 + b x_0, \quad d(1) = 1 \\
x_3 = a x_2 + b x_{-1}, \quad d(2) = 2 \\
x_4 = a x_3 + b x_1, \quad d(3) = 2 \\
x_5 = a x_4 + b x_2, \quad d(4) = 1
\]

where we also indicated the delay time \( d(k) \). The full state **local description** is given by the local models

\[
k = 0, \quad \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \end{bmatrix}
\]

\[
k = 1, \quad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}
\]

\[
k = 2, \quad \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} a & 0 & b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}
\]

\[
k = 3, \quad \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}
\]

\[
k = 4, \quad \begin{bmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}
\]

Something is missing from the local models, and that is how the state is ‘communicated’ for the next update. The state sequence is

\[
\chi_0 \rightarrow \chi_1 \rightarrow \chi_2 \rightarrow \chi_3 \rightarrow \chi_4 \rightarrow \chi_5
\]

Thus the states on the left hand side are actually \( \chi_{(k+1)-} \), whereas the states on the right hand side is \( \chi_k \). The additional map \( \chi_k \rightarrow \chi_{k+1} \) is needed.

\[
\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}_+ = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}_- ; \quad \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}_+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}_- + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}_- \]

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where the subscript \([-1]\) denotes a delay of 1, so that effectively \([x_2, x_1]^\top\)\([-1\]+n = \([x_1, x_0]^\top\). Then also
\[
\begin{bmatrix}
x_3 \\
x_2 \\
x_1 +
\end{bmatrix}
= \begin{bmatrix}
x_3 \\
x_2 \\
x_1 +
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
x_4 \\
x_3 \\
x_2 
\end{bmatrix}.
\]
This leads to the \textit{stitched models}
\[
k = 0, \quad \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}_0 = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \end{bmatrix}_+
\]
\[
k = 1, \quad \begin{bmatrix} x_2 \\ x_1 \\ x_0 \end{bmatrix}_1 = \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}_+
\]
\[
k = 2, \quad \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}_1 = \begin{bmatrix} a & 0 & b \\ 1 & 0 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}_+
\]
\[
k = 3, \quad \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}_1 = \begin{bmatrix} a & 0 & b \\ 1 & 0 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}_+
\]
\[
k = 4, \quad \begin{bmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}_1 = \begin{bmatrix} a & 0 & b \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}_+. \]
Symbolically, this models a birth and death like evolution, shown in Figure 4. ‘Births’ occur at \(k = 0, k = 4,\)
\[-1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots \]

Fig. 4. Representation of the discrete delay system

and ‘deaths’ at \(k=3, 8, \) etc. Perhaps this is made more clear by invoking the \textit{continuation} of the model. This is the reverse of discretization, and associates with for a discrete time model, \(x_{k+1} = \Phi x_k,\) a continuous time model \(\dot{x} = A x\) such that, if \(\xi(0) = x_0,\) then \(\xi(k\Delta) = x_k.\)

See Verriest [1993]. The horizontal line on he top of the figure represents the continuous evolution of the second order system, the down sloped arrow represents the trivial dynamics \(\dot{\xi}(t) = 0\) between \(t = 0\) and \(t = \Delta\) (and again between \(t = 4\Delta\) and \(t = 5\Delta,\) etc. This simply memorizes \(x_0\) and \(x_1, x_2, \ldots\)) Then the continuous evolution proceeds between \(t = \Delta,\) and \(t = 3\Delta.\) After \(t = 3\Delta,\) the third state variable is no longer needed (the death), and the update of the second order system resumes. Continuation is based on the existence of a matrix logarithm, which can only be defined if the matrix \(\Phi\) is nonsingular. Moreover, it is shown in [Verriest, 1993, Theorem 1] that a real matrix \(\Phi\) does not have a real matrix logarithm if \(\Phi\) has a negative real eigenvalue which corresponds to an odd number of equal Jordan blocks. A minimal discrete \(n\)-th order system with nonsingular \(\Phi\) (thus not a dead-beat system) can be represented by a sampled version of \((N+n)\)-dimensional continuous time system where \(N = \sum_{i=0}^{\nu_s} \nu_i,\) with \(\alpha\) the number of different elementary divisors \((s - \lambda_i)^{\nu_i}\) of \(\Phi\) that vanish on the negative real axis and occur a odd number of times. (See Theorem 2 in Verriest [1993]). In this case the real minimal continuous time system has order between \(n\) and \(2n.\) Even with such extensions, the general form of figure 4 remains. It is interesting to note that this continuized system is not a delay system, but an \(M^3D-\)system where the \textit{invariant} state-space has the structure of a discrete bundle (Verriest [2009b]).

First of all, it should be clear that if the “small” state \(x_\ell\) can be made zero, then the entire system will be brought to rest. The corresponding (small) monodromy system for the above example is \((F_s, G_s)\)
\[
F_s = \begin{bmatrix} 1 & a^2 + b & a^3b + ab^2 + b^2 \\
a^3b + ab^2 + b^2 & a^2b + ab \end{bmatrix}
\]
and
\[
G_s = \begin{bmatrix} 1 & a^2 + b & a^3 + ab \\
a^3 + ab & a^2 \end{bmatrix}.
\]

By collocating the steps in each mode, the evolution in this system may be described as an extended period 2 system, thus resulting in an extended multi-input system of smaller dimension. It readily follows from the small monodromy system that in general: If \(k_\ell \geq n_\ell,\) the system is reachable. If \(k_\ell \leq n_\ell - 1,\) then \([b_\ell, \ldots, A_{\ell}b_\ell] = \text{diag}[1, 1, \ldots, 1, 0, \ldots, 0]\) of rank \(k_\ell + 1\) and size \(n_\ell \times n_\ell.\) Consequently, the following cases can occur:

i) If \(n_\ell < k_\ell + 1,\) the reachability matrix of the monodromy system has full rank, and the switched delay system is reachable.

ii) If \(n_\ell > k_\ell + 1,\) then only if \(n_\ell > n_\ell,\) will the system be reachable.

6. REFLECTO-DIFFERENCE EQUATION

The above discussion focused on finite delay. In this section, we consider the behavior
\[
\mathcal{B} = \{w | Tw = Aw + Brw\},
\]
as an example of one with unbounded delay.

It corresponds to the system with time varying delay, \(n(k) = 2k,\) for \(k \geq 0, w_{k+1} = A w_k + B r w_{k-2k}P.\) However, for \(k > 0\) it poses additional restriction on the behavior restricted to \([-|k|, |k|].\) How is one to interpret such a relation involving the shift and the parity operator? (The continuous time equivalent is somewhat simpler and analyzed in Verriest [2011].) The idea is to resolve the behavior in an even and an odd sequence, with respect to \(k = 0.\) The even and odd part of a sequence are respectively defined as
\[
Ex = \frac{1}{2}(x + Rx)
\]
\[
Ox = \frac{1}{2}(x - Rx),
\]
and note that
\[
\sigma_k E(TEx) = \frac{1}{2}(\sigma_k (TEx) + \sigma_k (REx)) = \frac{1}{2}(\sigma_k (TEx) + \sigma_k T^{-1}REx)
\]
\[
= \frac{1}{2}(\sigma_k (TEx) + \sigma_k T^{-1}Ex)
\]
\[
= \frac{1}{2}(\sigma_{k+1} (Ex) + \sigma_{k-1} Ex)
\]
\[
= \frac{1}{2}(\sigma_{k+1} + \sigma_{k-1})Ex.
\]
Likewise, we derive
\[
\sigma_k E(TO_x) = \frac{1}{2} (\sigma_{k+1} - \sigma_{k-1}) Ox. \tag{18}
\]
\[
\sigma_k O(TO_x) = \frac{1}{2} (\sigma_{k+1} + \sigma_{k-1}) Ox. \tag{19}
\]
\[
\sigma_k O(TEx) = \frac{1}{2} (\sigma_{k+1} - \sigma_{k-1}) Ex. \tag{20}
\]
The above are needed since we note that $TEx$ is neither odd nor even in general. The even and odd parts of the behavior are then given by
\[
(\sigma_{k+1} + \sigma_{k-1}) Ex + (\sigma_{k+1} - \sigma_{k-1}) Ox = 2(A + B)\sigma_k Ex
\]
\[
(\sigma_{k+1} - \sigma_{k-1}) Ex + (\sigma_{k+1} + \sigma_{k-1}) Ox = 2(A - B)\sigma_k Ox.
\]
The first relation yields, in terms of $u_k$, the pair $(Ex)_{k+1} = (A + B)(Ex)_k - u_k$ and $(Ox)_{k+1} = (A - B)(Ox)_k + u_k$, and for consistency, the substitution in the second relation gives:
\[
[I - (A + B)^2](Ex)_k + 2Au_k = [I - (A - B)^2](Ox)_k.
\]
Likewise, substitution in the first relation yields
\[
[I - (A + B)^2](Ex)_k - 2Bu_k = [I - (A - B)^2](Ox)_k.
\]
Consequently, the sequence $u$ must satisfy $(A + Bu)_{k+1} = 0$. However, this is not sufficient for the existence of a nontrivial solution as the choice $u \equiv 0$ reveals. In fact, these equations may not possess a solution.

Unlike the continuous time case, where solutions can be shown to exist for arbitrary $x_0$ (see Verriest [2011]), the discrete time situation is quite different. For instance, for $A = 0$, it is easily seen that the given behavior should imply $x_{k+1} = Bx_k$ for $k \geq 0$ and $x_{k+1} = Bx_k$ for $k \geq 1$. But this implies $x_{k+1} = B^2x_k$ for all $k$. Thus either all $x_k$ are eigenvectors of $B$, corresponding to an eigenvalue $\pm 1$, or $x_k \equiv 0$ if $B$ does not have such eigenvalues.

If one relaxes the behavior, and only requires satisfaction for nonnegative $k$, then $x_1 = (A + B)x_0$ and noncausal behavior is allowed.

The growing buffer interpretation in the forgetful causalization makes the system behavior equivalent to $x_{k+1} = Ax_k$, since at no time $k > 0$, the value of $x_k$ had been preserved. Hence all trajectories of the system are of the form $x_k = A^{k-1}(A + B)x_0$ for $k > 0$. If $A$ is stable, the system is stable.

If the infinite past is considered as the state space at time zero (lossless causalization), then there is no freedom in the choice of $u$, and the evolution of the system is uniquely defined. Again, if $A$ is Schur-Cohn stable, any $\ell_2$ initial condition will give an $\ell_2$ bounded solution, and thus $x_k \rightarrow 0$, as $k \rightarrow \infty$.

7. CONCLUSIONS

Two approaches (lossless and forgetful causalization) for modeling discrete delay systems with time varying delay were given. Which one should be used depends on the physical nature of the memory or information storage and structure (e.g. foresight in future delay) in the system. For systems with periodic time variation, it was shown that with either system interpretation, the model may be embedded in large dimensional periodic system, and thus the results on the structural properties (parameterization and canonical forms) are inherited from the latter.

We have illustrated that the reflecto-difference equation, unlike its continuous time counterpart, may not have a solution. If however the description is limited to positive time, then solutions exists, and we have contrasted the noncausal behavior with the lossless and forgetful causalization.

REFERENCES


