Robust Positive Interval Observers For Uncertain Positive Systems *

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Abstract: This paper presents an approach for the estimation of uncertain linear positive systems, based on interval observers. The provided conditions for the existence of robust interval positive observers are expressed in terms of LP. Also, it is shown that the design of tight interval positive observers can be done by minimizing an adequate bound on the steady state of the interval error. An illustrative example is given.

1. INTRODUCTION

This paper deals with the estimation of uncertain positive systems, that is systems for which the trajectory is always nonnegative, whenever the initial conditions are nonnegative. Based on exact measurements, generally, one can estimate the unknown states by using any classical approach. However, when the measurements are not exact the estimation problem becomes nontrivial. This problem can be tackled by the so-called interval observers (bounding observers) that consist of a couple of estimators which provide guaranteed lower and upper bounds on the estimated states, see Rapaport et al. [2003], Durieu et al. [2001], Rami et al. [2007b], Gouzé et al. [2000], Jaulin et al. [2001]. Interval observer has been introduced in Gouzé et al. [2000] for positive systems or cooperative systems. Other set-valued observers has been considered in the literature (see for example Alamo et al. [2005], Raissi et al. [2005]). Recently, different types of robust interval observers have been introduced in Moisan et al. [2009], Rami et al. [2008] for general uncertain systems and Shu et al. [2008] for uncertain positive system. In these previous works, the design of robust interval observers is based on the theory of positive systems (see Luenberger [1979], Farina et al. [2000], Kaczorek [2002], Leenheer et al. [2001], Rami et al. [2007a]). The basic idea behind the design of interval observers is to enforce the nonnegativity of the error of estimation, which leads to guaranteed bounds on the estimated states by taking into account priori bounds on initial conditions and uncertain parameters of the systems (see for instance Rami et al. [2008], Shu et al. [2008], Moisan et al. [2009]).

In this paper, we show how one can use constrained Luenberger-type interval positive observers in order to estimate the states of uncertain positive systems (possibly with no exact measurements). In addition, for the construction of tight robust interval positive observer, we propose an approach based on the minimization of an adequate bound on the steady state of the interval error.

The proposed methodology is based on an LP approach introduced by Rami et al. [2005].

The paper is organized as follows. In the next section, we present some preliminary results. Section 3 gives the main result about the existence of positive interval observer in terms of LP, and provides an approach for obtaining tight robust positive interval observers. Section 4 presents an illustrative example. Section 5 gives some conclusions.

Notations For a real matrix $M$ or a vector, $M > 0$ means that its components are positive, and $M \geq 0$ means that its components are nonnegative. $R_{n \times m}^{+}$ denotes the set of real $n \times m$ matrices with nonnegative entries. $M^{T}$ denotes the transpose of $M$. vec$(M)$ denotes the vector formed by the columns of $M$. diag$(v)$ denotes a diagonal matrix with diagonal components formed by the elements of a vector $v$. $1_{n}$ is the vector of size $n$ with all elements equal to 1. $0_{n \times p}$ is the zero matrix of dimension $n \times p$. The identity matrix of size $n$ is denoted by $I_{n}$.

2. PRELIMINARY

This section provides some definitions and basic results for positive linear systems.

Consider the following linear system

$$
\dot{x}(t) = Ax(t), \quad x(0) = x_{0}.
$$

Definition 1. System (1) is said to be positive if its trajectories are nonnegative for all nonnegative initial conditions.

Definition 2. A real matrix $M$ is called a Metzler matrix if its off-diagonal elements are nonnegative: $m_{ij} \geq 0, i \neq j$.

Remark 3. A matrix $M$ is a Metzler matrix if and only if there exists $\beta \in R_{n}$ such that $M + \beta I \geq 0$.

Definition 4. A real matrix $M$ is called a Hurwitz matrix if all its eigenvalues have a real strictly negative part.

The following result can be deduced from Luenberger [1979], Farina et al. [2000], Rami [2007].

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Lemma 5. For any Metzler matrix $M$, we have that $e^{tM}$ is nonnegative for all $t \geq 0$. Moreover, the following statements are equivalent:

i) $M$ is Hurwitz.

ii) $\exists \lambda > 0$ such that $\lambda M < 0$.

iii) $\exists D$ such that $MD + DM^T < 0, D = D^T > 0$.

Remark 6. If $A$ is Metzler, then the solution of the affine system

$$
\dot{z}(t) = Ax(t) + b(t)
$$

is nonnegative for all $t \geq 0$. This can be seen by using the fact that $e^{tA}$ is nonnegative for all $t \geq 0$ (see Lemma 5) and the expression of its solution $x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}b(s)ds$.

Lemma 7. Consider the following system

$$
\dot{e}(t) = Ne(t) + f(t), \quad e(0) = e_0 \in \mathbb{R}^n,
$$

where $N$ is Metzler, if there exists a constant nonnegative vector $g \in \mathbb{R}^n_+$ such that $f(t) \leq g$, then the solution of the system (2) is bounded from above $e(t) \leq z(t)$, where $z(t)$ is the solution of the differential equation $\dot{z} = Nz + g, z(0) = e_0$. Moreover, if $N$ is Hurwitz then $z(t)$ converges to the equilibrium $e_{eq} = -N^{-1}g$ and so that $e(t)$ converges towards the box

$$
B(0, e_{eq}) := \{z \in \mathbb{R}^n \mid 0 \leq z \leq e_{eq}\}.
$$

Proof. see for example Gonzo et al. [2000].

3. POSITIVE INTERVAL OBSERVERS

In this section, we treat the estimation problem of the following linear positive system which is subject to interval uncertainties.

$$
\begin{align*}
\dot{x}(t) &= Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n_+ \\
y(t) &= Cx(t),
\end{align*}
$$

where $x \in \mathbb{R}^n_+$ is the state vector and $y \in \mathbb{R}^p$ is the output formed by the measurements. The initial conditions $x(0)$ are assumed to be unknown and bounded $x_0 \leq x(0) \leq \overline{x}$.

The matrix $A \in \mathbb{R}^{n \times n}$ which is Metzler and the matrix $C \in \mathbb{R}^{p \times n}$ are assumed to be unknown and bounded by known constant bounding matrices $\bar{A}, \underline{A}, \bar{C}$ and $\underline{C}$, that is

$$
\underline{A} \leq A \leq \bar{A}, \quad \underline{C} \leq C \leq \bar{C}.
$$

Our aim is to obtain a confident interval observer of system (3), that provides guaranteed bounds on the estimated states, in function of given bounds on the uncertainties that affect the dynamics of system (3). The proposed positive interval observers is defined as follows

$$
\begin{align*}
\hat{y} &= (\bar{A} - \bar{L}C)\hat{y} + L\bar{y}, \quad \hat{y}(0) = \overline{y} \\
\hat{z} &= (\underline{A} - \underline{L}C)\hat{z} + L\underline{y}, \quad \hat{z}(0) = \underline{y} \\
0 \leq \hat{y}(t) &\leq x(t) \leq \overline{y}(t),
\end{align*}
$$

where the gain $L$ must fulfill the additional constraints

$$
L \geq 0, \quad \bar{L}C \geq 0, \quad \underline{A} - \underline{L}C \text{ is Metzler}. \tag{5}
$$

By using the above conditions (5) on the gain $L$ of the couple of observers (4), we can show that if $L \geq 0$ and $\underline{A} - \underline{L}C$ is Metzler, then

$$
\underline{y}(t) \leq y(t) \leq \overline{y}(t).
$$

Effectively, it suffices to use the fact the lower and upper errors

$$
\begin{align*}
\hat{x} &= (\bar{A} - \bar{L}C)\hat{x} + (\bar{A} - A + L(C - \bar{C}))x, \tag{6} \\
\underline{x} &= (\underline{A} - \underline{L}C)\underline{x} + (A - \underline{A} + L(C - \underline{C}))x, \tag{7}
\end{align*}
$$

are nonnegative for all initial conditions $x_0 = \bar{x}_0 - x_0 \geq 0$ and $\underline{x}_0 = x_0 - \underline{x}_0 \geq 0$. In addition, note that the nonnegativity of the lower estimate $\underline{y}(t) \geq 0$ is guaranteed by the condition $L \underline{C} \geq 0$. To prove this claim, we can use Remark 6 and it suffices to use the fact that $\bar{A} - \bar{L}C$ is Metzler combined with $(\bar{A} - A + L(C - \bar{C}))x \geq 0$ and $(A - \underline{A} + L(C - \underline{C}))x \geq 0$. Also, we can show that the lower estimate

$$
\underline{y} = (\underline{A} - \underline{L}C)\underline{y} + L\underline{C}x, \tag{8}
$$

is nonnegative since $\bar{A} - \bar{L}C$ is Metzler and $LC$ is nonnegative matrix.

Now, we state our main result.

Theorem 8. Assume that the trajectory of system (3) is bounded

$$
x(t) \leq \bar{x}, \forall t \geq 0,
$$

then, there exists a positive interval observer of system (3) of the form (4) with bounded error, if the following LP problem in the variable $\beta \in \mathbb{R}_+, \lambda \in \mathbb{R}^n$ and $Z \in \mathbb{R}^{p \times n}$ is feasible

$$
\begin{align*}
\underline{A}^T\lambda - \underline{C}^TZ\lambda &\leq 0, \\
\lambda &> 0, \\
Z &\geq 0, \\
\bar{C}^TZ &\geq 0, \\
\bar{A}^T\text{diag}(\lambda) - \bar{C}^TZ + \beta I &\geq 0.
\end{align*}
$$

Moreover, the gain matrix $L$ of the interval observer (4) can be computed as follows

$$
L = \text{diag}(\lambda)^{-1}Z^T,
$$

where the vector $\lambda$ and the matrix $Z$ are any feasible solution to the above LP problem (9).

In addition, the interval error $e = \overline{y} - \underline{y}$ is bounded and converges towards the box

$$
B(0, v) := \{z \in \mathbb{R}^n \mid 0 \leq z \leq v\}, \tag{10}
$$

where $v = -\lambda^{-1}(\bar{A} - A + L(C - \bar{C}))\overline{y}$.

Proof. Assume that condition (9) holds and define $L = \text{diag}(\lambda)^{-1}Z^T$ then by simple calculation, conditions (9) is equivalent to

$$
\begin{align*}
(\underline{A}^T - \underline{C}^TL^T)\lambda &\leq 0, \\
\lambda &> 0, \\
L^T\text{diag}(\lambda) &\geq 0, \\
\bar{C}^TL^T\text{diag}(\lambda) &\geq 0, \\
(\underline{A}^T - \underline{C}^T)L^T\text{diag}(\lambda) + \beta I &\geq 0.
\end{align*}
$$

The condition $\underline{A}^T - \underline{C}^TL^T\text{diag}(\lambda) + \beta I \geq 0$ is nothing else than $\underline{A}^T - \underline{C}^T$ $L^T$ is Metzler or $\underline{A} - \underline{L}C$ is Metzler. The
non negativity of $L$ and $LC$ is obviously equivalent to the condition $LT \text{diag}(\lambda)$ and $CT LT \text{diag}(\lambda)$ are nonnegative. By previous considerations, we know that the conditions $L \geq 0$, $LC \geq 0$, $\Delta - LC$ is Metzler define a positive interval observer of the form (4).

In order to complete the proof, we use the fact that $\Delta - LC$ is Hurwitz (see Lemma 5 and use $\Delta - LC$ is Metzler together with the condition $(\Delta - LC)^T L^T \lambda < 0, \lambda > 0$).

Finally, by using the fact that $\Delta - LC$ is Hurwitz we show that the interval error is bounded and converges toward the box $B(0, v)$. The total error $e = \theta - \beta$ satisfies

$$
\dot{e}(t) = (\Delta - LC) e(t) + (\Delta - A + L(C - L) e(t), \tag{11}
$$

and it is nonnegative $e(t) \geq 0, \forall t \geq 0$. Since $\theta(t) = x(t) \leq \text{diag}(\lambda)$ and $\Delta - LC$ is Metzler, then by using Lemma 7 we conclude that the interval error is bounded from above by the solution of the differential equation

$$
\dot{z} = (\Delta - LC)^{-1} z + (\Delta - A + L(C - L)) z(t),
$$

which admits the stable equilibrium

$$
(\Delta - LC)^{-1} (\Delta - A + L(C - L)) v,
$$

since $\Delta - LC$ is Hurwitz. We conclude that the interval error $e(t)$ is bounded and converges towards the box $B(0, v)$.

**Remark 9.** It is easy to re-express the LP problem (9) as a standard LP problem by using Kronecker product and vec operations as follows

$$
\begin{align*}
\begin{bmatrix}
\bar{A}^T & 0_n & -1^T \otimes C^T \\
-I_n & 0_n & 0_q & -I_q \\
0_{m \times n} & 0_m & -I_m \otimes C^T \\
-M & \text{vec}(I_n) & I_n \otimes C^T
\end{bmatrix} w &< 0, \\
& (12)
\end{align*}
$$

where $M = \sum_{i=1}^{n} v_i v_i^T \otimes A^T v_i$, $v_i$ is the canonical vector of $\mathbb{R}^n$, $m = n \times n$, $q = n \times p$ and the new vector variable $w$ is defined as

$$
w := \begin{bmatrix} \lambda \\ \beta \\ \text{vec}(Z) \end{bmatrix}.
$$

**Remark 10.** By using Lemma 5 the above LP problem (9) is equivalent to the following LMI problem in the variables $\beta \in \mathbb{R}^+, d \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n \times n}$:

$$
\begin{align*}
\begin{bmatrix} \bar{A}^T \text{diag}(d) + \text{diag}(d) \bar{A} - C^T K - K^T C & < 0, \\
\text{diag}(d) & > 0, \\
C^T K & > 0, \\
K & > 0,
\end{bmatrix}
\end{align*}
$$

$$
\begin{align*}
\Delta^T \text{diag}(d) & - \text{diag}(d) \lambda + \beta I > 0.
\end{align*}
$$

The gain matrix $L$ of the interval observer (4) can be computed as follows

$$
L = \text{diag}(d)^{-1} K^T,
$$

where the matrices $\text{diag}(d)$ and $K$ are any feasible solution to the above LMI (13).

We can use the LP optimization approach in Rami et al. [2008] to design tight interval observers. This can be done by minimizing the size of the box $B(0, v)$ or the sum of its contours which is proportional to $l_1$ norm of the vector $v = -(\bar{A} - LC)^{-1} (\bar{A} - A + L(C - L)) \bar{x}$.

This is possible by using the following result.

**Theorem 11.** A tight interval observer of the form (4) can be computed as solving the following LP problem in the variables $\lambda \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$ and $Z \in \mathbb{R}^{n \times n}$

$$
\min \{ (\bar{A} - A) \lambda + [(C - L) \lambda]^T Z \}
$$

subject to:

$$
\begin{align*}
&\bar{A}^T \lambda - C^T Z 1_n = -1_n, \\
&Z > 0, \\
&CT Z > 0, \\
&\Delta^T \lambda - CT Z + \beta I > 0.
\end{align*}
$$

Moreover, the gain matrix $L$ of the interval observer (4) can be computed as follows $L = \text{diag}(\lambda)^{-1} Z$, where the vector $\lambda$ and the matrix $Z$ are any feasible solution to the above LP problem (14).

As previously, we can re-express the LP problem (14) in the standard form by using Kronecker product and vec operations as follows

$$
\begin{align*}
\min \{ (\bar{A} - A) \lambda + [(C - L) \lambda]^T \}
\end{align*}
$$

subject to:

$$
\begin{align*}
&\bar{A}^T 0_n - 1^T \otimes C^T w = -1_n, \\
&-I_n 0_{n \times q} 0_q w < 0, \\
&0_{q \times n} 0_q - I_q w < 0, \\
&0_{m \times n} 0_m - I_m \otimes C^T w < 0, \\
&-M - \text{vec}(I_n) I_n \otimes C^T w < 0,
\end{align*}
$$

where $M = \sum_{i=1}^{n} k_i k_i^T \Delta^T k_i$, $k_i$ is the canonical vector of $\mathbb{R}^n$, $m = n \times n$, $q = n \times p$ and the new vector variable $w$ is defined as

$$
w := \begin{bmatrix} \lambda \\ \beta \\ \text{vec}(Z) \end{bmatrix}.
$$

**4. ILLUSTRATIVE EXAMPLE**

In order to illustrate the proposed interval estimation approach, we consider a mammillary model from Distefano et al. [1975] described by the following system

$$
\begin{align*}
\dot{x}(t) = \begin{bmatrix} -k_{21} - k_{31} & k_{12} & k_{13} \\
k_{31} & -k_{02} - k_{12} & 0 \\
k_{31} & 0 & -k_{03} - k_{13}
\end{bmatrix} x(t), \\
y(t) = [c \ 0 \ 0] x(t),
\end{align*}
$$

where the components $k_{ij} > 0$ represent the flow rates. It is assumed that the system’s parameters are uncertain and are within the following ranges: $k_{21} = 0.1 \pm 0.042 \times \delta$, $k_{12} = 2.0 \pm 0.06 \times \delta$, $k_{31} = 1.5 \pm 0.05 \times \delta$, $k_{02} = 0.15 \pm 0.05 \times \delta$, $k_{03} = 1.8 \pm 0.1 \times \delta$, $k_{03} = 1.23 \pm 0.13 \times \delta$ and $\delta = 1.0 \pm 0.1 \times \delta$, where $0 \leq \delta \leq 1$.

Thus, one can obtain the following priori bounds on system (16)

$$
\Delta = \begin{bmatrix} -0.158 & 2.06 & 1.55 \\
0.142 & -3.64 & 0 \\
0.2 & 0 & -2.55
\end{bmatrix},
$$

By previous considerations, we know that the conditions $L \geq 0$, $LC \geq 0$, $\Delta - LC$ is Metzler define a positive interval observer of the form (4).
\[
A = \begin{bmatrix}
-0.342 & 1.94 & 1.45 \\
0.058 & -3.96 & 0 \\
0.1 & 0 & -2.91
\end{bmatrix},
\]
\[
C = \begin{bmatrix} 1.1 & 0 & 0 \end{bmatrix}
\]
and \[
C = \begin{bmatrix} 0.9 & 0 & 0 \end{bmatrix}.
\]
We have considered two kind of interval observers, the first one is based on Theorem 8 and its gain \(L\) is calculated by using the standard form of LP (see Remark 9). Then, we have obtained the following gain \(L = [1.9358 \ 0.0482 \ 0.0836]^T\). The second interval observer should be optimal, for this we have used Theorem 11 and solved the optimization problem (15). The obtained optimal gain is \(L_{opt} = [12.0211 \ 0.0491 \ 0.0620]^T\). For these two interval observers, the simulation results are depicted in Figure 1, Figure 2 and Figure 3.

By considering Figure 1 (for \(x_1\)), we can see that the optimal interval observer have better performance and provides a tight bounds on the estimated states. Also, notice that the upper observer with the optimal gain \(L_{opt}\) provides better estimates for the states \(x_2\) and \(x_3\) (see Figure 2 and 3).

5. CONCLUSION

In this paper, we have considered the design of positive interval observers for uncertain positive systems. It is shown that the existence of robust interval positive observers can be expressed in terms of LP conditions and as consequence one can easily compute them. In addition, we have shown that tight interval positive observers can be derived by minimizing an adequate objective function. The results are illustrated by an example form biology.

REFERENCES


![Fig. 1. Evolution of the first component](image-url)

Fig. 1. Evolution of the first component \(x_1\) and upper and lower bound without and with optimization.
Fig. 2. Evolution of the second component $x_2$ and upper and lower bound without and with optimization

Fig. 3. Evolution of the third component $x_3$ and upper and lower bound without and with optimization