A Maximum Principle for Constrained
Infinite Horizon Dynamic Control Systems

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Abstract: This article presents and discusses a maximum principle for infinite horizon constrained optimal control problems with a cost functional depending on the state at the final time. The main feature of these optimality conditions is that, under reasonably weak assumptions, the multiplier is shown to satisfy a novel transversality condition at infinite time. It is also shown that these conditions can also be obtained for impulsive control problems whose dynamics are given by measure driven differential equations.

Keywords: Optimal Control, Maximum Principle, Impulse control

1. INTRODUCTION

This article concerns the derivation of necessary conditions of optimality for infinite horizon control problems whose cost functional depend on the state at the final time which is subject to constraints. Moreover, the optimization is conducted over trajectories which converge asymptotically to an equilibrium point in the given compact set $C_{\infty}$. This includes the case in which it is not possible to steer the state to the equilibrium point in finite time. This feature distinguishes the problem addressed in this article from the usual finite time control problem.

The basic problem can be stated as follows:

\[(P) \text{ Minimize } h(\xi) \]
\[\text{subject to } \dot{x}(t) = f(t, x(t), u(t)), \mathcal{L} - a.e. \]  
\[(x(0), \xi) \in C_0 \times C_{\infty}, \xi = \lim_{t\to\infty} x(t) \]  
\[u \in \mathcal{U} \]

where $h: \mathbb{R}^n \to \mathbb{R}, f: [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, $C_0$ and $C_{\infty}$ are compact sets, $\mathcal{U}$ is the set of Borel measurable functions $u: [0, \infty) \to \mathbb{R}^m$ with $u(t) \in \Omega$ where $\Omega$ is a given compact set.

The optimization is carried out over all feasible control processes that converge asymptotically to equilibria in an infinite time horizon. A point $\xi \in \mathbb{R}^n$ is an equilibrium as $t \to \infty$ if $\exists$ a feasible control process $(x(\cdot), u)$ such that

$$\lim_{t \to \infty} x(t) = \xi,$$

and $0 \in \lim_{t \to \infty} \text{Int } f(t, x(t), \Omega)$.

This problem is cast in the context of nonsmooth analysis (see Clarke et al. [1998]) due to both the assumptions on its data and the approach used to derive the optimality conditions.

The key feature is the novel notion of transversality condition at infinite time - directional inclusion at infinity - which is obtained by assuming hypotheses which are significantly weaker than those usually required by competing results currently available in the literature. This condition enables the adjoint variable to “propagate” at least a partial effect of the cost function and state constraint penalization from the final time to any finite time. This result represents a compromise between the additional wealth of information provided by the transversality condition and the extent of the applicability range of the optimality conditions. An additional feature is the nondegeneracy of the conditions resulting from an endpoint controllability assumption required to prove our main result. The approach to derive these conditions essentially consist in considering a family of finite horizon optimal control problems approaching the original one and, then, in showing that the desired result is obtained as the limit of a properly extracted subsequence.

There are a number of necessary conditions of optimality that, over the years, have been obtained for infinite horizon control problems. Back in 1974, in Halkin [1974], a problem with an integral cost functional was considered, having an appropriate solution concept been given and the derived optimality conditions do not exhibit any transversality conditions. In Michel [1982], it was shown that, under a certain controllability assumption, the Hamiltonian tends to zero as time goes to infinity. Inspired in stability theory, Smirnov [1996] provides necessary and sufficient conditions of optimality for infinite horizon control problems with a transversality condition by imposing a regularity assumption formulated in terms of Lyapanov exponents to be satisfied by the adjoint variable. In Seierstad [1999], a nonsmooth maximum principle encompassing final time transversality conditions was derived for nonsmooth optimal control problems with final state dependent cost conditions.
functional as well as final time state constraints both with a linear structure. In Weber [2006], an infinite horizon discounted optimal control problem is considered and a maximum principle with a transversality condition is derived under assumptions on the data of the problem which imply that the adjoint variable remains bounded.

In a second part of this article, this result for conventional systems is extended to infinite horizon optimal impulsive problems with dynamics specified by measure driven differential equations. This step involves two stages. First, an infinite horizon extension of the concept of proper solution adopted in Pereira and Silva [2000] - which, in turn, is a development of the results in Silva and Vinter [1996] and Silva and Vinter [1997] - is provided. The fundamental issue in this concept is how to ensure a consistent definition of the state trajectory on the set of points in which it exhibits discontinuities. In particular, one key issue for a trajectory to be well defined is the existence of a path joining the jump endpoints that satisfies the singular dynamics. Second, a family of conventional (e.g., without control measures) finite horizon optimal control problems approximating the given impulsive control problem is considered. The concepts and methods used to obtain the transversality conditions in the first part of this article are combined with those pertaining to impulsive control in order to extend the optimality conditions for the impulsive context. These developments can be regarded as an extension of the ones discussed in Pereira and Silva [2006].

This paper is organized as follows. In the next section, we provide some preliminary concepts that play a key role in the main contribution of this article. Then, in section 3, we discuss the problem in detail together with assumptions, state the maximum principle for conventional optimal control problems, illustrate it with a simple example, and provide an outline of its proof. In section 4, we state the optimal impulsive control problem in infinite horizon and the associated solution concept. In the ensuing section, section 5, we present the necessary conditions of optimality for this problem, together with a very brief outline of its proof. Finally, some considerations are given as conclusions of this research effort.

2. THE EXTENDED \( \mathbb{R}^N \)

In the set of necessary conditions for infinite horizon control problems, the condition the co-estate must satisfy as \( t \to \infty \) is the key issue being addressed in this paper. In order to capture the behavior of the co-estate variable as \( t \to \infty \), we need to use a notion for the extended n-dimensional space. We adopt the cosmic closure of \( \mathbb{R}^n \), denoted by \( \text{csm}\mathbb{R}^n \), that is well discussed in Rockafellar and Wets [1998].

A direction for us is a ray, i.e., a closed half-line emanating from the origin. We think of the rays as abstract direction points which lie beyond \( \mathbb{R}^n \) and form what is called the horizon of \( \mathbb{R}^n \), denoted by \( \text{hzn}\mathbb{R}^n \). Sometimes it is convenient to represent a direction point by \( \text{dir}x \), where \( x \) is any nonzero vector in the ray representing the direction point in question. The zero vector is considered to have no direction.

The cosmic space \( \text{csm}\mathbb{R}^n \) is understood as the union of the \( \mathbb{R}^n \) with its horizon \( \text{hzn}\mathbb{R}^n \). The notation \( \lambda_k \not	o 0 \) means that \( \lambda_k \to 0 \) with \( \lambda_k > 0 \).

A sequence of points \( x_k \in \mathbb{R}^n \) converges to a direction point \( \mathbf{x} \in \text{hzn}\mathbb{R}^n \), written \( x_k \to \text{dir} \mathbf{x} \) \((x_k \not	o 0)\) if \( \lambda_k x_k \to \mathbf{x} \) for some choice of \( \lambda_k \not	o 0 \). Similarly, a sequence of direction points \( \mathbf{d} x_k \in \text{hzn}\mathbb{R}^n \) converges to a direction point \( \mathbf{d} \mathbf{x} \in \text{hzn}\mathbb{R}^n \), written \( \mathbf{d} x_k \to \text{dir} \mathbf{d} \mathbf{x} \) \((\mathbf{d} x_k \not	o 0)\) if \( \lambda_k \mathbf{d} x_k \to \mathbf{x} \) for some choice of \( \lambda_k \not	o 0 \).

With this definition it becomes clear that the cosmic \( \mathbb{R}^n \) is a compact space. For a set \( C \subseteq \mathbb{R}^n \), the cosmic closure of \( C \) is given by \( \text{csm} C := \text{cl}C \cup \text{hzn}C \) where \( \text{cl}C \) is the usual closure of \( C \) in \( \mathbb{R}^n \) while the \( \text{hzn}C \) is the collection of all direction point limits. Given a cone \( K \) in \( \mathbb{R}^n \), denote the set of direction points represented by the rays of \( K \) by \( \text{dir} K \).

For a given set \( C \subseteq \mathbb{R}^n \), the horizon cone representing the direction set \( \text{hzn} C \), is defined by \( C^\infty := \{x : \exists x \in C, \lambda \lambda_k \not	o 0, \lambda_k x_k \to x\} \). Note that if \( C \) is bounded, then \( C^\infty = \{0\} \). With this notation we have that \( \text{hzn} C = \text{dir} C^\infty \) and \( \text{csm} C = \text{cl} C \cup \text{dir} C^\infty \).

This is the context that allows us to define the concept of directional inclusion which will enable us to make precise boundary conditions involving variables which may either become unbounded-valued or persist in a certain set as time goes to infinity.

Let \( y \colon [0, \infty) \to \mathbb{R}^n \) be a continuous function. Let \( P(y) := P_L(y) \cup \text{dir} P^\infty(y) \), also alluded to as the set of persistency points of \( y \), where

- \( P_L(y) := \{\xi \in \mathbb{R}^n : \exists t_i \to \infty, \lim_{i \to \infty} y(t_i) = \xi\} \)
- \( \text{dir} P^\infty(y) := \{\xi \in \mathbb{R}^n : \exists t_i \to \infty, \lambda_i \not	o 0, \lim_{i \to \infty} \lambda_i y(t_i) = \xi\} \).

Given a function \( y \colon [0, \infty) \to \mathbb{R}^n \) and a set \( C \subseteq \mathbb{R}^n \) we say that \( y \) satisfies the weak directional inclusion in \( C \) at \( \infty \) if \( P(y) \cap \text{csm} C \not= \emptyset \). This relation can be referred to in short notation by \( y \in C^\infty \).

3. NECESSARY CONDITIONS OF OPTIMALITY FOR CONVENTIONAL PROBLEMS

3.1 Assumptions on the data of \( (P) \)

Consider problem \( (P) \) whose data satisfies a somewhat strengthened version of the usual standing nonsmooth assumptions. That is,

- S1 \( h \) is continuously differentiable.
- S2 \( f \) measurably Lipschitz, i.e., \( f \) is Lebesgue times Borel measurable in \((t, u), \forall u \in \mathbb{R}^n\), and Lipschitz continuous in \( x, \forall (t, u) \in [0, \infty) \times \Omega \).
- S3 The endpoint state constraint sets \( C_0 \), and \( C_\infty \) and the control constraint set \( \Omega \) are closed.

Although assumption S1 can be weakened, we keep it in order to facilitate some developments discussed later in the article. In order to show the optimality conditions, we also need the following additional assumptions. In what follows, let \( \xi^* := \lim_{t \to \infty} x^*(t) \).
H1 The limit \( f(t, x(t), \Omega) \) exists in a sense of Hausdorff and is denoted by \( F_\infty(\xi) \) where \( \xi := \lim_{t \to \infty} x(t) \).

H2 There exists \( \delta > 0 \) such that \( \forall x \in \xi^* + \delta B, 0 \in \text{Int} \lim_{t \to \infty} f(t, x(t), \Omega) \).

H3 \( \exists v_0 \in \mathbb{R}^n \) satisfying \( v_0 \in \text{Int} f(0, x^*(0), \Omega) \) and such that either \( x^*(0) \in \text{Int} C_0 \), or \( (\xi_0, v_0) < 0, \forall \xi_0 \in N_{C_0}(x^*(0)) \).

Some comments are in order. While \( H1 \) reflects a sort of persistence of the velocity set at the limiting value of the state variable, \( H2 \) implies controllability in a neighborhood of the optimal reference trajectory as time goes to \( \infty \). \( H3 \) is an initial point controllability condition with respect to the initial state constraint.

3.2 Necessary Conditions of Optimality

Our necessary conditions of optimality for \( (P) \) are stated in the form of a maximum principle and they involve the pseudo-Hamiltonian or Pontryagin function which is defined as

\[
H(t, x, u, p) = p^T f(t, x, u).
\]

The adjoint variable \( p : [0, \infty) \to \mathbb{R}^n \) satisfies a boundary condition at \( t = \infty \). This is stated as the existence of a non-empty set of its persistency points, \( P(p) \), on the convex closure of the right hand set of the usual transversality condition. Moreover \( p \) is a subgradient of the value function \( V \) along the optimal trajectory, being \( V(t, z) := \text{Min} \{ h(\xi) : \text{all admissible } (x, u) \text{ with } x(t) = z \} \).

In particular, if \( p \) converges asymptotically to some point \( \bar{p} \), then \( P(p) = \{ \bar{p} \} \). If \( p \) approaches a limit cycle \( C_L \) at infinite time, then \( P(p) = C_L \). The pattern of realization of the limiting approach towards a given infinitely often visited set of points might not be periodic. In what follows, \( N_{C}(c) \) denotes the normal cone to the set \( C \) at point \( c \) and by \( \partial_x f(x) \) the generalized gradient of the function \( f \), both in the sense of Clarke, Clarke et al. [1998].

Next, we state the main result of this article.

**Theorem 1.** Let \((x^*, u^*)\) be a solution to \((P)\). Then, there exists a multiplier \((p, \lambda_0)\), with \( \lambda_0 \geq 0 \), satisfying:

- \( \lambda_0 + \|p\| \neq 0 \) (nontriviality).
- \( \exists p(0) \in N_{C_0}(x^*(0)) \) for which there is a solution to

\[
-\dot{p}(t) = \partial_x H(t, x^*(t), u^*(t), p(t)), \text{L-a.e.,}
\]

s. t. \( -p(t) \in \partial_t V(t, x^*(t), u^*(t), \xi_0), \text{L-a.e. on } [0, \infty), \) and

\[
P(-p) \cap \text{csm} (\lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset.
\]

- \( u^*(t) \) maximizes in \( \Omega \) the map \( v \mapsto H(t, x^*(t), v, p(t), \lambda_0) \).

Remark that \( P(-p) \cap \text{csm} (\lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*)) \) can be interpreted as \( \exists \xi \in \lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*) \) for which

- either \( \xi \in P_L(-p) \), if \( p \) is bounded,
- or \( \frac{\xi}{|\xi|} \in \text{dir}\mathbb{P}(p) \), otherwise.

The information provided by this concept is certainly weaker than that given by the boundary condition of the adjoint variable in the finite time interval context. In general there are many functions \( p \) that persist in an absolute or a directional sense towards a point of \( \lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*) \) at infinite time. However, this information is still useful in delimiting the number of multipliers which satisfy the maximum condition.

3.3 Outline of the proof

The proof is based on extracting the limit of a subsequence of multipliers associated with the corresponding sequence of solutions to a family of auxiliary finite horizon optimal control problems converging to \((P)\). Take \((T_k), T_k \to \infty\) and consider the following auxiliary problem.

\[
(P_{T_k}) \text{ Minimize } V(T_k, x(T_k))
\]

subject to \( x = f(t, x, u), \text{L-a.e. in } [0, T_k], \)

\[
x(0) \in C_0, u \in U([0, T_k]),
\]

where \( V(t, z) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) is defined by

\[
V(t, z) := \min_{u(t) \in U} \{ h(\xi) : x(t, x, u), \text{L-a.e. in } [t, \infty), \}
\]

\[
x(t) = z, \lim_{s \to \infty} x(s) = \xi \in C_\infty \}
\]

Notice that, by the principle of optimality, the solution to \((P_{T_k})\), denoted by \((x_k^*, u_k^*, p_k)^*\) coincides modulo Borel-a.e. w.r.t. the control with \((x^*, u^*)\) on \([0, T_k]\). Before pursuing, we would like to point out the following properties of \( V \) which follow from Vinter [2000].

**Proposition.**

(i) \( V \) is measurable in \( t \) and Lipschitz continuous on \( z \) for any finite time.

(ii) \( \lim_{t \to \infty} V(t, z) = \{ h(z) \text{ if } z \in C_\infty \}
\]

(iii) The map \( \lim_{t \to \infty} V(t, x^*(t)) \) is \( h(\xi^*) \) uniformly.

The various items of this proposition are straightforward extensions, under the hypotheses assumed in this article, of the corresponding finite horizon counterparts. It should be remarked that item (i) draws heavily from the controllability at infinity in order to ensure the Lipschitz dependence instead of the usual lower semicontinuity usually proved for finite horizon problems whose state variable is constrained at the final time.

Next, we apply the Maximum Principle to \((P_{T_k})\), and we obtain a multiplier \( p_k \) satisfying

\[
-p_k(t) \in \partial_x H_k(t, x^*(t), u^*(t), p_k(t)), [0, T_k]-\text{a.e.}
\]

\[
p_k(0) \in N_{C_0}(x^*(0)), -p_k(T_k) \in \partial_x V(T_k, x^*(T_k)),
\]

\[
u^*(t) \text{ maximizes in } \Omega \text{ the map } v \mapsto H(t, x^*(t), v, p_k(t)), [0, T_k]-\text{a.e.}
\]

Now, let us compute an estimate of \( \partial_x V(T_k, x^*(T_k)) \).

**Proposition.** Under the assumptions \( H1 - H3 \) and \( S1 - S3 \), we have that \( \partial_x V(T_k, x^*(T_k)) \) contains the set

\[
\{ \bar{p}_k \in \mathbb{R}^n \exists (\bar{p}, \bar{\lambda}) \text{ satisfying : (i) } \|\bar{p}(t)\| + \bar{\lambda} \neq 0, \bar{\lambda} \geq 0 \}
\]

\[
(ii) -\bar{p}(t) \in \partial_x H_k(t, x^*(t), u^*(t), p_k(t)), [T_k, \infty)-\text{a.e.}
\]

\[
(iii) \bar{p}(T_k) = \bar{p}_k
\]

\[
(iv) P(-p) \cap \text{csm} (\lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset
\]

\[
u^*(t) \text{ maximizes in } \Omega [T_k, \infty)-\text{a.e.,}
\]

the map \( v \mapsto H(t, x^*(t), v, p(t), \lambda_0) \}

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Let \( x \in AC([0,\infty); \mathbb{R}^{n}) \) be such that \( x(T_{k}) = z, \dot{x} = f(t, x, u) \) a.e. and \( \lim_{t \to \infty} x(t) = \xi \) asymptotically. Then, by using the fact that \( h \) is assumed to be \( C_{1} \), we have
\[
h(\xi) = h(z) + \int_{T_{k}}^{\infty} \nabla h(x(t))f(t, x(t), u(t))dt.
\]
We also need an additional auxiliary variable \( y \) satisfying \( \dot{y} = 0 \) with \( y(T_{k}) \in C_{\infty} \) and also \( \lim_{t \to \infty} (y(t) - x(t)) = 0 \).

Note that, since \( \tilde{C} := \{(x,y) : x = y\} \), we have that \( N_{\tilde{C}_{\infty}} = (p_{x}, p_{y}) : p_{x} = -p_{y} \). Now, notice that \( V(T_{k}, z) \) is the minimum cost of the following auxiliary optimal control problem

Minimize \( \int_{T_{k}}^{\infty} \nabla h(x(t))f(t, x(t), u(t))dt \)
subject to \( \dot{x} = f(t, x, u), \dot{y} = 0, \) \([T_{k}, \infty)\)-a.e.
\[
\lim_{t \to \infty} (x(t), y(t)) \in \tilde{C}_{\infty},
(x(T_{k}), y(T_{k})) \in \{z\} \times C_{\infty}.
\]

Observe that the generalised gradient of \( V \) with respect \( o \) at time \( T_{k} \) at \( x^{*}(T_{k}) \) is given by the set of values of the (symmetric of the) adjoint variable at time \( T_{k} \). Remark also that the cost functional of this problem does not depend on state at infinite time. The final endpoint constraint does not cause any difficulty since it is affine in the state variable and always active.

By applying the maximum principle to this auxiliary problem, and, then, by expressing the obtained conditions in terms of the data of the original problem, it is straightforward to derive the intended characterization of the estimate of \( \partial_{x} V(T_{k}, x^{*}(T_{k})) \). Indeed, we have
\[
H(t, x, y, u, p_{x}, p_{y}, \lambda_{0}) = p_{x}^{T}f(t, x, u) - \lambda_{0}\nabla h(x)f(t, x, u)
\]
and, thus
- \( -\tilde{p}_{x}(t) = \partial_{x}H(t, x^{*}(t), y^{*}(t), u^{*}(t), p_{x}(t), p_{y}(t), \lambda_{0}). \)
- \( -\tilde{p}_{y}(t) = 0, \) and thus \( p_{y}(t) \equiv p_{y}(T_{k}) \in N_{\tilde{C}_{\infty}}(x^{*}(T_{k})). \)
- \( \exists\{t_{i}\}, t_{i} \uparrow \infty, \exists\{\alpha_{i}\}, \alpha_{i} > 0, \alpha_{i} \to \alpha_{\infty} \geq 0, \) such that \( \lim_{i \to \infty} \alpha_{i}p_{x}(t_{i}) = -\tilde{p}_{x}(T_{k}). \)

H(t, x^{*}(t), y^{*}(t), u^{*}(t), p_{x}(t), p_{y}(t), \lambda_{0}) \geq 0.
\[
H(t, x^{*}(t), y^{*}(t), u^{*}(t), p_{x}(t), p_{y}(t), \lambda_{0}) \geq 0, \forall u \in \Omega.
\]

Notice that the third item arises naturally from the fact that the adjoint equation relative to \( p_{y} \), involving also \( \lambda_{0} \), can be scaled down by some positive number.

Now, by putting \( \bar{p}(t) = p_{y}(t) - \lambda_{0}\nabla h(x^{*}(t)) \), we have that
\[ -\bar{p}^{T}(t) = \partial_{y}H(t, x^{*}(t), u^{*}(t)) \]
and, by considering sequences \( \{t_{i}\} \) and \( \{\alpha_{i}\} \) with either \( \alpha_{\infty} > 0 \) or \( \alpha_{\infty} = 0 \) we have the stated transversality condition.

To complete the proof of Theorem 1, it is enough to take the limit of the necessary conditions obtained for \((P_{T_{k}})\). By once more recalling the principle of optimality, the properties of \( V \), and using the characterization of the estimate of its generalized gradient derived in the above proposition, we readily obtain the desired conclusions, i.e., the necessary conditions of optimality for \((P)\).

3.4 A simple example

Let us consider the following simple optimal control problem that arises when searching for optimizing economic growth strategies.

\[ h(\xi) \quad \text{subject to} \quad x = u - x, \quad x(0) = 1, \]
\[
\dot{y} = e^{-t} \ln(x - 1/2), \quad y(0) = 0, \]
\[
u(t) \in [0, 1/2], \]
\[
\xi = \lim_{t \to \infty} x(t) \in C_{\infty} = [0, 1/2].
\]

Simple considerations lead to the conclusion that the \((x^{*}, u^{*})\) with \( u^{*}(t) = 1/2, \forall t \), which corresponds the trajectory
\[
\{ x^{*}(t) = (1 + e^{-t})/2, \}
\[
y^{*}(t) = e^{-t}(t + 1 + \ln 2) - 1 - \ln 2 \}
\]

is the solution to \((Ex)\).

By applying the maximum principle, we obtain:
\[
H(t, x, u, x^{*}(t)) = p_{x}(u - x) + p_{y}e^{-t} \ln(x - 1/2),
\]
\[
p_{y} = x^{*} + 2, \quad p_{y}(\infty) \geq 0,
\]
\[
p_{y} = 0, \quad p_{y}(\infty) = -1
\]
\[
p_{x}(t) = e^{-t}(p_{x}(0) + 2) - 2.
\]

We can readily see that these conditions hold if we choose some \( p_{x}(0) > -2 \).

4. THE OPTIMAL IMPULSIVE CONTROL PROBLEM

In this section we show how the necessary conditions stated in the previous section can be extended to the impulsive context. Extensions from results of optimal control theory with absolutely continuous trajectories to systems with trajectories of bounded variation can be found, e.g., in Warga [1962], Rishel [1965], Rockafellar [1976], Bressan and Rampazzo [1994], Motta and Rampazzo [1995], Dijkstra [1996], Pereira and Silva [2000, 2002, 2004], Arutyunov et al. [2005a,b]. There are high profile optimal control applications, e.g., the minimum fuel problem, Rishel [1965], management of resources, Baumeister [2001], impact mechanics Brogliato [1996] (to cite just a few classes) revealing the relevance of the impulsive control paradigm.

Let \((P)\) be the problem

\[
\text{Minimize } h(\xi)
\]

\[
\text{such that } dx(t) \in F(t, x(t))dt + G(t, x(t))\mu(dt)
\]
\[
x(0) \in C_{0}, \quad x(t) \to \xi \in C_{\infty}
\]
\[
\mu \in K.
\]

Here, \( F \) and \( G \) are set-valued maps from \([0, \infty) \times \mathbb{R}^{n}\) to, respectively, \( \mathcal{P}(\mathbb{R}^{n}) \) and \( \mathcal{P}(\mathbb{R}^{m} \times \mathbb{R}^{n}) \), \( h : \mathbb{R}^{n} \times \mathbb{R}^{n} \to \mathbb{R} \) is a given function and the sets \( C_{0} \), and \( C_{\infty} \) are closed subsets of \( \mathbb{R}^{n} \). \( C \subset C^{*}([0, \infty), \mathbb{R}^{m}) \) is a cone of measures with range in a positive, convex, closed, pointed cone \( K \subset \mathbb{R}^{n} \), i.e., for all \( \mu \)-measurable \( A \in [0, \infty) \), \( \mu(A) \in K \). Moreover, a feasible control measure \( \mu \) may have unbounded total variation but its total variation on
any bounded set is finite, i.e., \( \bar{\mu}(A) < \infty \) for any finite \( A \subset [0, \infty) \) (here and in what follows \( \bar{\mu} \) denotes the total variation measure associated with \( \mu \)). Obviously, intrinsic to the well-posedness of this problem is the existence of equilibrium points as \( t \to \infty \). The point \( \xi \in \mathbb{R}^n \) is an equilibrium of the dynamic system (6) as \( t \to \infty \) if there exists a feasible control process \( (x(t), \mu(dt)) \) satisfying

\[
\lim_{t \to \infty} x(t) = \xi,
\]

and for which

\[
0 \in \lim_{t \to \infty} \left\{ F(t, x(t)) dt + G(t, x(t)) \mu(dt) \right\}.
\]

This limit is considered in the sense of Kuratowski. This optimal control problem is a natural impulsive extension of the optimal control problem addressed in Pereira and Silva [2006] for which a control formulation with only ordinary Borel measurable controls was considered, that is, one with the data as in \( (P) \) with \( K = \{0\} \).

Now we recall the adopted concept of solution to the measure driven differential inclusion that was defined in Pereira and Silva [2006] and which can be regarded as an infinite horizon extension of the one developed in Silva and Vinter [1996, 1997], Pereira and Silva [2000].

**Definition 1.** The trajectory \( x_t \), with \( x(0) = x_0 \), is admissible for (6) if \( x(t) = x_{ac}(t) + x_a(t) \quad \forall t \in [0, \infty) \), where

\[
\begin{align*}
\dot{x}_{ac}(t) &= f(t) + G_{ac}(t) \cdot w_{ac}(t) \quad \text{a.e.} \\
x_a(t) &= \int_{[0,t]} G_r(\tau) w_{sa}(\tau) \, d\bar{\mu}_{sa}(\tau) + \int_{[0,t]} g_{sa}(\tau) \, d\bar{\mu}(\tau).
\end{align*}
\]

(9)

Here, \( \bar{\mu} \) is the total variation measure associated with \( \mu_{ac} \), \( \mu_{sa} \) and \( \mu_{ac} \) are, respectively, the singular continuous, the singular atomic, and the absolutely continuous components of \( \mu_{ac} \), \( \mu_{sa} \) is the Radon-Nikodým derivative of \( \mu_{ac} \) with respect to its total variation, \( f(\cdot) \) and \( G_{ac}(\cdot) \) are Lebesgue measurable selections of, respectively, \( F(\cdot, x(\cdot)) \) and \( G(\cdot, x(\cdot)) \). \( G_0(\cdot) \) is a \( \mu_{ac} \) measurable selection of \( G(\cdot, x(\cdot)) \) and \( g(\cdot) \) is a \( \mu_{sa} \) measurable selection of the multifunction

\[
\bar{G}(t, x(t^-); \mu(\{t\})), \quad \forall t \in [0, \infty) \times \mathbb{R}^n \times K \to \mathcal{P}(\mathbb{R}^n),
\]

specifying the reachable set of the singular dynamics at \( (t, x(t^-)) \) when the control measure has an atom of “weight” \( \mu(\{t\}) \).

More specifically, for \( |\alpha| = 0 \), \( \bar{G}(t, z; \alpha) \) is given by \( \{G(t, z)w(t)\} \), and, for \( |\alpha| > 0 \), by the set of all vectors

\[
\{\eta(t) - \xi(\eta(t^-)) | \alpha \}
\]

where \( (\xi(\cdot), \gamma(\cdot)) \) satisfies \( (\xi(s), \gamma(s)) = (G(t, \xi(s))v(s), v(s)) \), \( \eta(t) \)-a.e., being \( G(t, \xi(s)) \in G(t, \xi(s)) \), \( v(s) \in V \) a.e. in \( \eta(t) \), with \( \xi(\eta(t^-)) = z \), and \( \gamma(\eta(t^-)) = \eta(\eta(t^-)) = \alpha \). Here, given \( a \), possibly unbounded, time \( T \) and a measure \( \mu \) supported on \( T \), a new time parameterization is defined by associating with \( t \) the range of \( \eta(t) \) given by \( \eta(t^-), \eta(t) \) if \( \mu(t^-) > 0 \) and by \( \eta(t) \) otherwise, being

- \( \eta(t) := t + \sum_{i=1}^q M_i(t) \), and
- \( M(\cdot) = \text{coll}(M_1(\cdot), \ldots, M_q(\cdot)) \), with \( M_i(0) = 0 \), and
- \( M_i(t) = \int_{[0,t]} \mu_i(ds) \), \( \forall t > 0 \),

In Silva and Vinter [1996], with extensions in Pereira and Silva [2000] and Pereira and Silva [2006], important properties of this solution concept are shown, notably, robustness and the equivalence relationship between the impulsive control problem and the associated conventional control problem obtained by re-parametrization.

5. **THE NECESSARY CONDITIONS OF OPTIMALITY**

In this section, we state necessary conditions of optimality that are proved under the following set of assumptions.

**SI1** \( h \) is continuously differentiable.

**SI2** \( F \) is measurable Haudorff Lipschitz with constant \( K_F(\cdot) \in L^1 \).

**SI3** \( G \) is continuous in \( t \) and Haudorff Lipschitz with constant \( K_G \).

**SI4** \( F \) and \( G \) are convex and compact valued set-valued maps with closed graphs.

**SI5** \( C \) and \( S \) are compact sets in \( \mathbb{R}^n \) and \( K \subset \mathbb{R}^n \) is a positive, pointed, convex cone.

These are by no means the weakest assumptions under which necessary conditions of optimality for impulsive control problems can be proved. As in the previous section, we need to posit the equivalent to HI-H3, i.e.,

**H1** For all \( \mu(dt) \in K \), the \( \lim_{t \to \infty} F(t, x(t)) + G(t, x(t)) \mu(dt) \) exists in a sense of Haudorff and is denoted by \( F(\xi(t))dt + G(\xi(t)) \mu(t) \) where \( \xi(t) : \lim_{t \to \infty} x(t) \).

**H2** There exists \( \delta > 0 \) such that \( \forall x \in \xi + \delta B, \quad 0 \in \text{Int} \lim_{t \to \infty} F(t, x(t))dt + G(t, x(t))dK \).

**H3** \( \exists \eta(t) \in \mathbb{R}^n \) satisfying

\[
\eta(t) \in \text{Int}[G(0, x^*(0))dt + G(0, x^*(0))dK],
\]

and such that either \( x^*(0) \in \text{Int}C_0 \), or \( \eta(t, 0_0) < 0 \), \( \forall t_0 \in \mathbb{N} \cap (x^*(0)) \).

**Theorem 2.** Let \( (x^*, \mu^*) \) be a solution to problem \( (P) \) whose data satisfies the hypotheses stated above. Then, there exist a nonnegative number \( \lambda \) and a function of bounded variation \( p \) satisfying \( \lambda + |p| \leq 0 \) and

\[
\begin{align*}
-\langle dp, dx^* \rangle(t) &= \partial H_F(t) + H_G(t)w_{ac}(t) \rangle dt + \partial H_G(t)w_{sa}(t) \rangle dt \\
&+ \partial \partial H_G(t, x^*(t), \mu_{ac}(t)) \rangle dt \forall t \in [0, \infty),
\end{align*}
\]

Moreover, for \( t \in \text{Supp} (\mu_{sa}^*), \exists \xi(t), \zeta(t) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^n \times K_1, \quad K_1 := K \cap B_1(0) \) satisfying:

\[
\begin{align*}
&\langle \xi(t), \zeta(t) \rangle \in \partial H_G(t, x^*(t), \mu_{ac}(t)) v_1^*(t) \rangle \eta(t) \rangle - \eta(t) \rangle \rangle \in \partial H_G(t, x^*(t), \mu_{ac}(t)) \rangle \eta(t) \rangle \\
&= H_G(t, x^*(t), \mu_{ac}(t)) \rangle \eta(t) \rangle \\
&= (\xi(t), \zeta(t)) \langle \eta(t) \rangle - (x^*(t), p(t)) \rangle \\
&\in \text{Int} \lim_{t \to \infty} F(t, x(t))dt + G(t, x(t))dK.
\end{align*}
\]
Here,
• η and ¯η are as defined in the previous section;
• µ∗(dt) = w∗ ac(t)dt + µ∗ sc(dt) + µ∗ sa(dt) is the usual canonical decomposition of the measure µ∗, whose continuous part is denoted by dp∗c, and the Radon-Nykodim derivative of dp∗c w.r.t. its total variation measure, µ∗ c, by v∗ c;
• H∗F(t) and H∗G(t) are the Hamiltonian functions HF and HG defined along (t, x∗(t), p(t)), respectively, by:
  \[ H∗F(t) = \max \{ \langle p(t), v⟩ : v ∈ F(F t, x∗(t)) \} \]
  \[ H∗G(t) = \begin{cases} \langle hG(t), v⟩ & \text{if t ∈ Supp(µ∗ c)} \\
    \langle hG(s) : s ∈ η(t)⟩ & \text{if t ∈ Supp(µ∗ sa)} \end{cases} \]
where
  \[ \langle hG(t), v⟩(t) = \max \{ \langle p(t), G(t, x∗(t))w⟩ \} \]
and
  \[ \langle hG(s), v⟩(s) = \max \{ \langle G(t, ξ∗(s))w⟩ \text{w} \in K, G ∈ G \} \]
Here, v∗ c(t) denotes w∗ c(t) and v∗ sa(t) on the supports of the, respectively, absolutely continuous and singular continuous components of µ∗ c;
• The generalized gradients of the Hamiltonian functions are taken w.r.t. (x∗, p) or their graph completions at time t, (ξ∗, G);
• σK(·) is the usual support function to K;
• Supp(µ) is the support of µ.

The proof is based on approximating the impulsive optimal control problem by a sequence of conventional problems for which a perturbed sequence of problems has minimizers. We then apply necessary conditions of optimality that we just provided for conventional problems and take the limit to get the desired impulsive optimality conditions.

6. CONCLUSIONS

In this article, necessary conditions of optimality in the form of the Hamiltonian inclusions and featuring a novel transversality condition were given for an infinite horizon control problem whose state trajectories are required to be asymptotically stable and the equilibrium point is constrained to a given closed set. Various comments relating the obtained result are included.

In a second part, this result is extended for impulsive control problems whose dynamics are given by measure driven differential inclusions. Once again, a critical role is played by the solution concept which, together with the notion of equilibrium, is introduced for such a class of problems. A sketch of the proof is outlined.

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