On the global convergence of identification of output error models

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Abstract: The Output Error Method is related to an optimization problem based on a multi-modal criterion. Iterative algorithms like the steepest descent are usually used to look for the global minimum of the criterion. These algorithms can get stuck at a local minimum. This paper presents sufficient conditions about the convergence of the steepest descent algorithm to the global minimum of the cost function. Moreover, it presents constraints to the input spectrum which ensure that the convergence conditions are satisfied. These constraints are convex and can easily be included in an experiment design approach to ensure the convergence of the iterative algorithms to the global minimum of the criterion.

Keywords: System identification, output error identification, convergence, global optimization.

1. INTRODUCTION

The optimization problem of the Output Error Method is based on a multi-modal criterion. Iterative algorithms based on the gradient of the criterion are usually used to find a minimum of the criterion. Since the criterion has several local minima, the algorithm can get stuck at a local minimum [Ljung, 2010]. It is very important to understand this local convergence problem to propose ways to avoid the convergence to local minima. The local convergence problem can be viewed from two perspectives: from the algorithm and from the cost function.

The algorithm is able to find the global minimum of the criterion if the initial condition is close to the global minimum. Hence, an alternative to avoid the local minima is to look for initial conditions close to the global minimum solving secondary optimization problems. This is usually utilized, normally using techniques like instrumental variables. However, these techniques may fail even when the noisy level is low [Ljung, 1987].

From the point of view of the cost function, it is noted that the shape of the criterion depends on several parameters [Söderström and Stoica, 1982]. Some of them cannot be changed by the user like the noise and the real plant. On the other hand, the parameters which depend on the experiment can be chosen: the input spectrum, sampling time and size of the experiment. Since the shape of the cost function depends on parameters which the user can change, an alternative to avoid the local minima is to choose these parameters in order to make the cost function “well behaved”. These ideas have been used in the context of adaptive control and data-based control [Riedle et al., 1986, Lee et al., 1995, Bazanella et al., 2008].

The choice of the input spectrum can be optimized utilizing an experiment design. The experiment design is utilized to minimize some objective related to the input spectrum while some restrictions are respected [Gevers and Bombois, 2006]. The objective can be to minimize the energy of the input spectrum or to minimize the variance of the estimates obtained with the identification method. Several quality restrictions can be included in the experiment design. The most utilized are energy constraints, frequency-by-frequency constraints and variance constraints. In this paper, we will include convergence constraints to the experiment design.

This work has two main objectives. The first is to present sufficient conditions about the convergence of the steepest descent algorithm to the global minimum of the criterion. These conditions should express how the parameters that the user can change are related to the convergence. Similar results have been obtained in Goodwin et al. [2003] for ARMAX structures and in Zou and Heath [2009] for OE, ARMAX and BJ structures. The second and main objective is to transform these convergence conditions in a way that they can be utilized in the experiment design framework. To do that, the conditions must be convex to keep the problem convex aiming to solve it with Linear Matrix Inequality solvers.

The Section 2 presents some definitions and the problem formulation. In the Section 3 sufficient conditions about the convergence of the steepest descent algorithm are presented. The convex restriction to the experiment design is presented in the Section 4. The Section 5 shows a didactic example and Section 6 is a brief conclusion.

2. PRELIMINARIES

Consider the identification of a linear time-invariant discrete-time single-input single-output system

\[ y(t) = G_0(q)u(t) + e(t). \]  \hspace{1cm} (1)

In (1) \( q \) is the forward-shift operator, \( G_0(q) \) is the process transfer functions, \( u(t) \) is the control input, and \( e(t) \) is a zero mean white noise sequence with variance \( \sigma_e^2 \). The transfer function \( G_0(q) \) is rational, stable and proper.

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The system (1) is identified using a model structure parametrized by a vector \( \theta \in \mathbb{R}^p \):

\[
y(t) = G(q, \theta)u(t) + e(t)
\]

where

\[
G(q, \theta) = \frac{B^T(q)\theta}{1 + F^T(q)\theta}
\]

and \( B^T(q) = [q^{-d} q^{-1-d} q^{-2-d} \ldots q^{-m-d} 0_1 \times n] \), \( F^T(q) = [0_1 \times m+1 q^{-1} q^{-2} \ldots q^{-n}] \), \( \theta \) is the vector of unknown parameters which we want to estimate, \( m \) is the number of model zeros, \( n \) is the number of model poles and \( d \) is the delay. For a given \( \theta \in \mathbb{R}^p \), \( G(q, \theta) \) is called a model, while the model structure is defined as a differentiable mapping from a connected open subset \( \mathcal{D} \in \mathbb{R}^p \) to a model set \( \mathcal{M}^* \):

\[
\mathcal{M} : \theta \in \mathbb{R}^p \to G(q, \theta) \in \mathcal{M}^*.
\]

The true system is said to belong to this model set, \( \mathcal{S} \in \mathcal{M}^* \), if there is a \( \theta_i \in \mathcal{D} \) such that \( G(q, \theta_i) = G_0(q) \). This assumption will be utilized in all results of this work. Let us define \( \Gamma \) the set of all \( \theta \) for which \( G(q, \theta) \) is stable. In a prediction error identification framework, a model uniquely defines the one-step-ahead predictor of given all input/output data up to time. With the proposed model structure we have the following one-step-ahead predictor [Ljung, 1987]

\[
\hat{y}(t|t-1, \theta) = G(q, \theta)u(t).
\]

The one-step-ahead prediction error is defined as

\[
\varepsilon(t, \theta) = y(t) - \hat{y}(t|t-1, \theta) = (G_0(q) - G(q, \theta))u(t) + e(t).
\]

We are interested in looking for the vector parameter that minimizes the following cost function

\[
J(\theta) = E[\varepsilon(t, \theta)^2]
\]

which can be rewritten using the Parseval’s Theorem for all \( \theta \in \Gamma \) as

\[
J(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(G_0(e^{j\omega}) - G(e^{j\omega}, \theta_i))|^2 \Phi_\omega(\omega) d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_\omega(\omega) d\omega
\]

where \( \Phi_\omega(\omega) \) and \( \Phi_\omega(\omega) \) are the power spectrum of the input and of the noise. It should be noted that in this formulation it is used asymptotic results. In practice, they are just valid if the number of collected samples is large.

In this paper we are interested in finding the global minimum of the cost function \( J(\theta) \). This cost function is not convex and a possible way to solve the optimization problem is the use of gradient descent algorithms. In this paper we focus in the steepest descent algorithm defined by

\[
\theta_{i+1} = \theta_i - \gamma_i \nabla J(\theta_i)
\]

where \( \gamma_i > 0 \) is the step size of the algorithm and \( \nabla J(\theta_i) \) is the gradient of the cost function with respect to \( \theta_i \):

\[
\nabla J(\theta_i) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Re \left\{ (G_0(e^{j\omega}) - G(e^{j\omega}, \theta_i))^* \nabla (G_0(e^{j\omega}) - G(e^{j\omega}, \theta_i)) \right\} \Phi_\omega(\omega) d\omega.
\]

We would like such an optimization algorithm to converge to the global optimum for the largest possible set of initial conditions. A set of initial conditions for which the algorithm converges is called a domain of attraction (DOA) of the algorithm.

**Definition 1.** Let \( \theta_* \) be the global minimum of the function \( J(\theta) : \mathbb{R}^p \to \mathbb{R}^+ \). A set \( \Omega \subset \mathbb{R}^p \) is a domain of attraction of the algorithm (4) for the function \( J(\theta) \) if \( \lim_{i \to \infty} \theta_i = \theta_* \) \( \forall \theta_0 \in \Omega \).

In the next section the convergence of the algorithm (4) to the global minimum \( \theta_* \) will be analyzed.

## 3. CONVERGENCE OF THE ALGORITHM

This section presents some results about the global convergence of the steepest descent algorithm.

**Lemma 1.** [Bazanella et al., 2008]. Let \( \theta_* \) be the global minimum of \( J(\theta) \) and define a set

\[
\mathcal{B}(\theta_*) = \left\{ \theta : (\theta - \theta_*)^T(\theta - \theta_*) < \alpha \right\}
\]

If

\[
(\theta_i - \theta_*)^T \nabla J(\theta_i) > 0 \forall \theta_i \in \mathcal{B}(\theta_*), \theta \neq \theta_* \quad (5)
\]

then there exists a sequence \( \gamma_i, i = 1, \ldots, \infty \) such that \( \mathcal{B}(\theta_* \gamma_i) \) is a DOA of the algorithm (4) for \( J(\theta) \).

**Proof:** Let \( V(\theta) = (\theta - \theta_*)^T(\theta - \theta_*) \) be a Lyapunov function for the discrete-time system (4). Then

\[
V(\theta_{i+1}) - V(\theta_i) = (\theta_i - \gamma_i \nabla J(\theta_i) - \theta_*)^T \times
\]

\[
(\theta_i - \gamma_i \nabla J(\theta_i) - \theta_*) - (\theta_i - \theta_*)^T(\theta_i - \theta_*)
\]

\[
= -2\gamma_i (\theta_i - \theta_*)^T \nabla J(\theta_i) + \gamma_i^2 \nabla J^T(\theta_*) \nabla J(\theta_i)
\]

which is negative provided that

\[
0 < \gamma_i < 2(\theta_i - \theta_*)^T \nabla J(\theta_i) \frac{1}{\nabla J^T(\theta_*) \nabla J(\theta_i) - \gamma_i^2}.
\]

For \( \theta_i \in \mathcal{B}(\theta_*) \) the existence of such \( \gamma_i \) is guaranteed by condition (5), which also implies that \( \nabla J(\theta_i) \) is nonzero for all \( \theta_i \neq \theta_* \). Recall that any connected and bounded level set of a Lyapunov function is a DOA if the Lyapunov difference is strictly negative in its interior. Then the proof is completed by noting that \( \mathcal{B}(\theta_*) \) is a connected and bounded level set of the Lyapunov function \( V(\theta) \).

The above lemma shows that if condition (5) is satisfied then for all initial conditions inside the set \( \mathcal{B}(\theta_*) \) there is a step size sequence \( \gamma_i \) which ensures the convergence of the algorithm to the global minimum \( \theta_* \). Actual convergence also involves the proper choice of the sequence \( \gamma_i \), an issue which we do not address in this paper. This means that the cost function is “well-behaved” inside the set \( \mathcal{B}(\theta_*) \) and then we say that this set is a candidate DOA. This concept is formalized below.

**Definition 2.** Let \( \theta_* \) be the global minimum of the function \( J(\theta) \). A ball \( \mathcal{B}(\theta_*) \) is a candidate DOA for the function \( J(\theta) \) if (5) is satisfied.

In the next theorem some properties of the cost function are explored to obtain sufficient conditions which ensure a set is a candidate DOA.

**Theorem 2.** Let \( \mathcal{B}(\theta_*) \) be a connected set such that:

\[
\Re \left\{ 1 + F^T(e^{j\omega}) \theta_* \right\} \Phi_\omega(\omega) d\omega > 0 \forall \omega, \forall \theta_i \in \mathcal{B}(\theta_*), \theta \neq \theta_*.
\]

Then \( \mathcal{B}(\theta_*) \) is a candidate DOA.

**Proof:** The gradient of the cost function can be rewritten to

\[
\nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re \left\{ (G_0(e^{j\omega}) - G(e^{j\omega}, \theta_i))^* \nabla G(e^{j\omega}, \theta_i) \right\} \Phi_\omega(\omega) d\omega
\]
and the gradient of the model is
\[ \nabla G(e^{j\omega}, \theta) = \frac{B(e^{j\omega})}{1 + F^T(e^{j\omega})\theta} - \frac{BT(e^{j\omega})\theta F(e^{j\omega})}{(1 + F^T(e^{j\omega})\theta)^2}. \]

So, we can write
\[ \nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re\left\{ \left( G_0(e^{j\omega}) - \frac{BT(e^{j\omega})\theta}{1 + F^T(e^{j\omega})\theta} \right)^* \left( \frac{B(e^{j\omega})}{1 + F^T(e^{j\omega})\theta} - \frac{BT(e^{j\omega})\theta F(e^{j\omega})}{(1 + F^T(e^{j\omega})\theta)^2} \right) \Phi_u(\omega) d\omega. \]

Now it is possible to compute
\[ (\theta - \theta_s)^T \nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re\left\{ \left( G_0(e^{j\omega}) - \frac{BT(e^{j\omega})\theta}{1 + F^T(e^{j\omega})\theta} \right)^* \left( \frac{B(e^{j\omega})}{1 + F^T(e^{j\omega})\theta} - \frac{BT(e^{j\omega})\theta F(e^{j\omega})}{(1 + F^T(e^{j\omega})\theta)^2} \right) \Phi_u(\omega) d\omega. \]

which we want to make positive definite. Expanding some terms of the above condition we obtain
\[ (\theta - \theta_s)^T \nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re\left\{ \left( G_0(e^{j\omega}) - \frac{BT(e^{j\omega})\theta}{1 + F^T(e^{j\omega})\theta} \right)^* \left( \frac{B(e^{j\omega})}{1 + F^T(e^{j\omega})\theta} - \frac{BT(e^{j\omega})\theta F(e^{j\omega})}{(1 + F^T(e^{j\omega})\theta)^2} \right) \Phi_u(\omega) d\omega. \]

If we use the relation \( S \in M \) we can write
\[ (\theta - \theta_s)^T \nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re\left\{ \left( G_0(e^{j\omega}) - \frac{BT(e^{j\omega})\theta}{1 + F^T(e^{j\omega})\theta} \right)^* \left( \frac{B(e^{j\omega})}{1 + F^T(e^{j\omega})\theta} - \frac{BT(e^{j\omega})\theta F(e^{j\omega})}{(1 + F^T(e^{j\omega})\theta)^2} \right) \Phi_u(\omega) d\omega. \]

which can be rearranged to
\[ (\theta - \theta_s)^T \nabla J = -\frac{1}{\pi} \int_{-\pi}^{\pi} \Re\left\{ \left( G_0(e^{j\omega}) - G(e^{j\omega}) \right)^2 \right\} d\omega. \]

Using the condition of the theorem we ensure that \((\theta - \theta_s)^T \nabla J > 0, \forall \theta \in B(\theta_s), \theta \neq \theta_s.\]

The above theorem presents a sufficient condition to ensure that the set \( B(\theta_s) \) is a candidate domain of attraction. As usual, the convergence depends on a strictly positive real (SPR) condition \((7)\). This condition is related to the real process and to the parameters of the model. In the sequence, this result will be used to obtain several conclusions about the convergence of the Output Error Method.

Corollary 4. Consider that the order of the process is one \((p = 1)\). For all \( \theta \in \Gamma \) the condition \((7)\) is ensured.

Proof: Let us define the transfer function
\[ T(e^{j\omega}, \theta) = \frac{1 + F^T(e^{j\omega})\theta}{1 + F^T(e^{j\omega})\theta}. \]

Using the conditions of the theorem \( T(e^{j\omega}, \theta) \) has one pole, has one zero, is stable and minimum phase. Then it is easy to see that \(-\frac{\pi}{2}rad < \angle T(e^{j\omega}, \theta) < \frac{\pi}{2}rad\) which means that \( \Re \{ T(e^{j\omega}, \theta) \} > 0 \forall \omega. \)

The result shows that for every first order process that is possible to compute \((\theta - \theta_s)^T \nabla J \rightarrow 0\).

This condition can only be satisfied if the order of the process is two or more. Moreover, the condition \((9)\) is not respected only if \( \theta \) is not close to \( \theta_s.\]

It has been proven that the SPR property \((7)\) of a particular transfer function is sufficient for ensuring the convergence of the steepest descent algorithm within a given set. However, the SPR condition is not a necessary condition; it can be circumvented by a proper manipulation of the input \( u(t).\) To realize how to do that, we first explore the properties of the function that enter the SPR condition \((7)\). The function \( T(e^{j\omega}, \theta) \) can be written as
\[ T(e^{j\omega}, \theta) = \frac{1 + p_1^* e^{-j\omega}}{(1 + p_1)(1 + p_2) \cdots (1 + p_n)} \]

where \( p_i^* \), \( i = 1, 2, n \) are the poles of the process and \( p_i \), \( i = 1, 2, n \) are the poles of the model. The response at frequency \( \omega \) is
\[ T(e^{j\omega}, \theta) \]

and the response at frequency \( \pi \) is
\[ T(e^{j\pi}, \theta) \]

Since all the poles of the process and all the poles of the model are inside the unitary circle, both \( T(e^{j\omega}, \theta) \) and \( T(e^{j\pi}, \theta) \) are real and positive. From the property above and the continuity of \( T(e^{j\omega}, \theta) \) we can also conclude that the phase of \( T(e^{j\omega}, \theta) \) is small for frequencies close to \( \omega = 0 \) and \( \omega = \pi.\)

Lemma 5. For all \( \theta \in \Gamma, \exists \omega_l, \omega_h \) such that:
\[ |\angle T(e^{j\omega}, \theta)| < \frac{\pi}{2}rad \forall \omega \leq \omega_l, \]
\[ |\angle T(e^{j\omega}, \theta)| < \frac{\pi}{2}rad \forall \omega \geq \omega_h. \]

According to the previous results, if the input spectrum contains only those frequencies for which \( |\angle T(z, \theta)| \) is small, then the integral in \((8)\) will be bounded away from zero, even if the phase difference exceeds \( \frac{\pi}{2}rad \) in some frequency ranges. For instance, if the input spectrum is concentrated at the borders of the frequency spectrum - where the phase is always small - then the integral \((8)\) will be positive definite. This means that there isn’t any local minimum inside the stable set \( \Gamma.\) This also means that the
set $B(\theta^*) \subset \Gamma$ is a candidate domain of attraction. These facts are formalized in the following theorem.

Theorem 6. If $\Phi_{\alpha}(\omega) = 0 \forall \omega \in (\omega_l, \omega_h)$, where $\omega_l, \omega_h$ respect (10) then the any ball $B(\theta^*) \subset \Gamma$ is a candidate DOA.

Proof: Using the conditions of the theorem the integral (8) is positive definite $\forall \theta \in \Gamma$. Using the Lemma 1 we conclude that $B(\theta^*) \subset \Gamma$ is a candidate DOA.

This result is very powerful because one can always choose the spectrum of the input signal. However, the frequencies $\omega_l$ and $\omega_h$ are not known a priori. The next section will show a method to choose the frequencies of the input spectrum in order to respect the restrictions of the above theorem.

4. EXPERIMENT DESIGN

In the last section, it was shown that the user always can choose an input spectrum such that the global convergence is achieved. In the literature, the choice of the input spectrum is normally called Experiment Design. The input spectrum is chosen with the experiment design by solving an optimization problem. The optimization problem can have several restrictions, which must be met. The restrictions can impose spectrum constraints, quality constraints and even convergence constraints. Generally, the optimization problem is described by [Lindqvist and Hjalmarsson, 2001]

$$\min \Phi_{\alpha}(\omega) \quad \text{s.t.} \quad \text{spectra, quality and convergence constraints}.$$  

Objectives commonly used are the energy of the signal or the variance of the estimates. The spectrum constraints can be an upper-bound to the signal energy or frequency-by-frequency constraints ($\alpha(\omega) < \Phi(\omega) < \beta(\omega)$). The convergence constraints will be introduced in this article.

The optimization problem as posed is not convex and is infinite dimensional [Jansson and Hjalmarsson, 2005]. However, it is possible to parametrize the input spectrum and to describe the constraints in a way that the problem be convex and finite dimensional. Each time we want to add a new constraint to the problem, it is important to keep the optimization problem convex. In the sequel, it will be presented convergence constraints to the experiment design which keep the optimization problem convex.

The input spectrum is parametrized by

$$\Phi_{\alpha}(\omega) = \sum_{i=1}^{N} a_i^2 \delta(\omega_i)$$

where the frequencies $\omega_i$, $i = 1, \ldots, N$ are chosen by the user. A realization to this spectrum is $u(t) = \sum_{i=1}^{N} a_i \sin(\omega_i t)$.

Let us define the set

$$\mathcal{E} = \{\theta| (\theta - \theta_*)^T P (\theta - \theta_*) \leq 1\}.$$  

We would like that $\forall \theta \in \mathcal{E}$ the integral (8) be positive definite. Doing that, any ball $B(\theta^*) \subset \mathcal{E}$ is a candidate DOA. It is possible to ensure that the integral (8) be positive inside the set $\mathcal{E}$, making $a_i = 0$, $\forall \omega_i, \forall \theta \in \mathcal{E}$ such that $\Re \{1 + \frac{\alpha}{e^{\omega_i}}\} < 0$.

Let us first analyze the set $\mathcal{E}$. The matrix $P \in \mathbb{R}^{p \times p}$ defines the shape and the size of the ellipsoid $\mathcal{E}$. This set can also be expressed in a matrix form by

$$\mathcal{E} = \{\theta| \begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} -P & \rho \theta_* \\ \rho^T \theta_* & P - \rho^T \rho \rho^T \theta_* \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} \geq 0 \}.$$  

Let us also analyze the expression

$$\Re \left\{1 + \frac{\alpha}{e^{\omega_i}}\right\} = \frac{1 + F^T(e^{\omega_i})\rho}{1 + F^T(e^{\omega_i})\rho} \geq 0$$

where $F^*(e^{\omega_i})$ is the transpose conjugate of $F(e^{\omega_i})$ and $F^T(e^{\omega_i})$ is the transpose of $F(e^{\omega_i})$. The denominator of this expression is positive definite by construction so,

$$\Re \left\{1 + \frac{\alpha}{e^{\omega_i}}\right\} \geq 0 \iff (1 + F^T(e^{\omega_i})\rho)(1 + F^*(e^{\omega_i})\rho) \geq 0.$$  

This condition can also be expressed in a matrix form by

$$\begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \Re\{F^T + F^T F^* \rho\} \\ \Re\{F^T + F^T F^* \rho\} & 2\Re\{1 + F^T \theta_*\} \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} \geq 0,$$

where the frequency dependence was omitted for reason of space.

We would like to verify the above condition for every frequency $\omega_i, i = 1, \ldots, N$ and for all $\theta \in \mathcal{E}$. For each frequency that the above condition is not satisfied, the $a_i$ must be zero to ensure that the integral (8) be positive definite.

It is possible to test if the above condition is satisfied for $\forall \theta \in \mathcal{E}$ using the S-procedure.

$$\begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & \Re\{F^T + F^T F^* \rho\} \\ \Re\{F^T + F^T F^* \rho\} & 2\Re\{1 + F^T \theta_*\} \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} \geq 0 - \sigma \begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} -P & \rho \theta_* \\ \rho^T \theta_* - P - \rho^T \rho \rho^T \theta_* \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} \geq 0,$$

where $\sigma > 0$ is an additional variable. This condition is quadratic and it is satisfied if

$$\begin{bmatrix} 0 & \Re\{F^T + F^T F^* \rho\} \\ \Re\{F^T + F^T F^* \rho\} & \Re\{F^T + F^T F^* \rho\} \end{bmatrix} \geq 0.$$  

If $\sigma > 0$ that satisfies the above condition then $\forall \theta \in \mathcal{E}$ the condition $\Re \left\{1 + \frac{\alpha}{e^{\omega_i}}\right\} \geq 0$ is satisfied for the frequency $\omega$. This condition can be tested for all frequencies $\omega_i$. For all frequencies that the above condition is not satisfied the parameter $a_i$ must be zero to ensure that the integral (8) be positive definite.

The condition (11) is a Linear Matrix Inequality (LMI) which has just one free variable $\sigma$. To check this condition is straightforward with any LMI software. It should be notice that it is necessary to know the parameter $\theta_*$ to solve this problem, which is unknown by assumption. In practice, it is necessary to obtain some approximation of this value. However, this is a usual drawback in all experiment design.
5. EXAMPLE

Consider that we have the following system
\[ y(t) = 0.0030q^{-3} - 1 - 2.5500q^{-1} + 2.1650q^{-2} - 0.6120q^{-3} + e(t) \]
where \( e(t) \) is white noise with variance \( \sigma_e^2 = 0.01 \). This system can also be represented by
\[ y(t) = \frac{0.0030q^{-3}}{(1 - 0.9q^{-1})(1 - 0.85q^{-1})(1 - 0.8q^{-1})}u(t) + e(t). \]
The step response of this system is shown in the Figure 1.

![Step Response](image)

Fig. 1. Step response of the system.

Let us consider that the system is unknown, and we want to identify the parameters of the following model with data collected from the system.

\[ G(q, \theta) = \frac{B^T(q)\theta}{1 + F^T(q)\theta}, \]

\[ B^T(q) = \begin{bmatrix} q^{-3} & 0 & 0 & 0 \end{bmatrix}, \quad F^T(q) = \begin{bmatrix} 0 & q^{-1} & q^{-2} & q^{-3} \end{bmatrix}, \]

Note that this structure can represent the real system when \( \theta = \theta_* = \begin{bmatrix} 0.0030 & -2.5500 & 2.1650 & -0.6120 \end{bmatrix}^T \).

Let us then design the input spectrum of the signal to identify this process. This will be done by

\[ \min_{\Phi_u(\omega)} \text{energy of the input signal} \]

s.t. quality and convergence constraints.

It is considered that the input spectrum is

\[ \Phi_u(\omega) = \sum_{i=1}^{N} a_i^2 \delta(\omega_i) \]

where \( \omega = \pi 10^{3i(i-1)}/100 - 3 \), \( i = 1, \ldots, 100 \). This signal covers the spectrum from 0.001\( \pi \) rad/s until 0.93\( \pi \) rad/s. The quality constraint will be the variance of the estimates. We want that all parameters be estimated with precision better than 0.001 in a confidence interval of 95%. Two experiment design will be done. In the first the convergence constraints will not be applied. In the second we will utilize the convergence constraints.

5.1 Experiment Design without Convergence Constraints

We design the input spectrum to minimize the energy of the input signal and to ensure that the precision of the estimated parameters be better than 0.001 in a confidence interval of 95%. A realization of the optimum input spectrum is

\[ u_1(t) = 71.01\sin(0.0433t) + 137\sin(0.1982t). \]

This signal can be view in the Figure 2. The input signal \( u_1(t) \) was applied to the system and the output was collected. The input and output signals were utilized to estimate the parameters of the model. The initial condition \( \theta_1 = [0.0196 -2.5461 2.1701 -0.6044]^T \) was used in the steepest descent algorithm. The algorithm run 10000 iterations and it obtained the following estimate to the parameters of the model \( \theta = [-0.0004 -2.5447 2.1703 -0.6033]^T \). These parameters represent the model

\[ G(q, \theta) = \frac{-0.00042015q^{-3}}{(1 - 0.6063q^{-1})(1 - 1.939q^{-1} + 0.9952q^{-2})}. \]

This model is far away from the real system. The estimated cost was

\[ J = \sum_{i=1}^{1000} (e(t, i))^2 = 2289. \]

In fact, the algorithm converged to a local minimum. Observe the Bode Diagram of \( T(e^{j\omega}, \theta_1) \) in the Figure 3. At the frequency 0.1982 rad/s the function \( T(e^{j\omega}, \theta_1) \) has phase higher than \( \pi/2 \) rad. Most of the energy of the input signal is in this frequency. Hence, it is easy to see that the integral (8) could not be positive definite. This also indicates that the algorithm could converge to a local minimum.

![Bode Diagram](image)

Fig. 3. Bode Diagram of \( T(e^{j\omega}, \theta_1) \).

5.2 Experiment Design with Convergence Constraints

Let us define the set \( \mathcal{E} = \{ \theta \mid (\theta - \theta_*)^TP(\theta - \theta_*) \leq 1 \} \) where

\[ P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 847885 & -198919 & -296429 \\ 0 & -198919 & 739874 & -387637 \\ 0 & -296429 & -387637 & 422343 \end{bmatrix}. \]

This set represents the region for which we want to ensure that the integral (8) be definite positive. Note that just the parameters related to the poles of the systems are restricted in this set. It is done because only the poles affect the convergence of the algorithm. This set is also
represented in the Figure 4. Note also that the initial condition $\theta_i$ is inside this set.

We run the experiment design with the convergence constraints presented in the last section. A realization of the optimum input spectrum is

$$u_2(t) = 47.01 \sin(0.0703t) + 163 \sin(0.2275t).$$

This signal can be viewed in the Figure 5. The input signal $u_0(t)$ was applied to the system and the output was collected. The input and output signals were utilized to estimate the parameters of the model. The initial condition was the same of the previous example. The algorithm ran 10,000 iterations, and it obtained the following estimate to the parameters of the model

$$\theta = [0.0029 -2.5496 2.1644 -0.6116].$$

Now the algorithm converged to the global minimum, as can be seen with the estimate of the cost function

$$J = \sum_{i=1}^{1000} (\epsilon(t, \theta))^2 = 0.2188.$$

Note that the quality constraints were respected, all parameters were estimated with precision better than 0.001. To achieve a better precision, we would need an input signal with more energy. The step response of this model obtained with the experiment design with the convergence constraints avoids the frequencies which $\mathbb{R}\{T(e^{j\omega}, \theta)\} < 0$. Doing that the integral (8) is positive definite for all $\theta \in \mathcal{E}$.

6. CONCLUSION

This paper presented sufficient conditions about the global convergence of identification of output error models. This article also presented convex constraints to the experiment design which ensure the convergence to the global minimum. A didactic example showed how the proposed method can be utilized. In the sequence, we intend to extend these results to other model structures.

REFERENCES


