Balanced model reduction of gradient systems

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Abstract: Gradient systems are an important class of systems, e.g., a large class of linear and nonlinear electrical circuits are in fact gradient systems. The structure of the gradient system is important, and if model reduction is applied, it is thus of interest to preserve the gradient structure of the system, i.e., the reduced order model should (at least) be a gradient system again. In this paper we study balanced realizations for gradient systems. For linear gradient systems a balanced realization has a specific structure for the metric, implying that balancing based model order reduction preserves the gradient system structure of the system. For nonlinear gradient systems, such structure for the metric is not obtained, resulting in additional conditions to preserve the gradient structure of the system. Both balanced truncation and singular perturbation order reduction are studied. Finally, a relation with cross Gramians and balancing of gradient systems is studied.

Keywords: model order reduction, balancing, gradient systems

1. INTRODUCTION

In many areas of engineering and science one is increasingly led to high-order mathematical models. For many purposes, including simulation, control and design validation, it is of clear importance to have methodologies to derive lower-order approximations of the originally obtained high-dimensional model. An important theme in this model reduction endeavor is the search for methods which retain as much as possible the "structure" of the high-dimensional model. Of course, this is yet a rather vague statement, and, in fact, the question what kind of structure is desirably preserved in the process of model reduction is itself a research question. Nevertheless, in many cases, especially for physically well-defined models, it is clear what are desirable properties of the system to be retained under model reduction. For example we mention the properties of stability, passivity, presence of conservation laws, and symmetry. Note that in this context there is a clear link with system analysis and design; e.g. in the case of a large scale electrical circuit one would like to obtain an approximation which still can be realized as an electrical circuit.

In this paper we focus on the structure of a system as a gradient system. In the first part of the paper we concentrate on linear systems. In this case the property of being a gradient system is easily stated: it means that the transfer matrix of the system (or, equivalently, its impulse response matrix) is symmetric. In particular, any single-input single-output linear system is a gradient system. It is well-known, see especially the work of Willems (Willems [1972b]), that the symmetry property of the transfer matrix translates into the existence of a unique symmetric (possibly indefinite) matrix $G$ of full rank on the minimal state space of the system, satisfying certain properties with regard to the system realization $(A, B, C)$. In this paper we will emphasize the resulting system representation where the internal dynamics $\dot{x} = Ax$ is written as a (linear) gradient vector field $\dot{G}x = Px$, where $\frac{1}{2}x^TPx$ for some matrix $P = P^T$ defines the potential function and $G$ a (possibly indefinite) inner product. Our main result will be that balancing (based on the computation of the observability and controllability Gramian) is compatible with the gradient structure, in the strong sense that there exist balanced coordinates in which the inner product $G$ takes the form of a signature matrix. This has the immediate important consequence that the balanced truncation or singular perturbation truncation of a gradient system is again a gradient system with an inner product and potential function which is directly obtained from the inner product and potential function of the original (full-order) model. We also relate these balancing coordinates with the diagonalizing transformation of the cross-Gramian. The results obtained in our paper for balancing of linear gradient systems parallel to a certain extent the results for LQG-balancing in Opdenacker, Jonckheere [1985], and are related to some results in Liu et. al. [1998] for $G = I$.

The standard state space form of a linear gradient system defined with respect to an (possibly indefinite inner product and a potential function immediately extends to nonlinear state space systems, where the (indefinite) inner product structure extends to the notion of a (pseudo-)Riemannian metric on the nonlinear state space mani-
fold. An appealing example of such a gradient system is the Brayton-Moser-Smale description of (nonlinear) RLC-circuits (Brayton, Moser [1964a], Brayton, Moser [1964b], Smale [1972], Jeltsema et al. [2003]). In this section we report about ongoing research on the relation between nonlinear balancing and the gradient system structure.

2. LINEAR GRADIENT SYSTEMS

Consider a linear system \((A, B, C)\) given as
\[
\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m
\]
y = Cx

It is called a gradient system\(^1\) whenever there exists an invertible symmetric matrix \(G\) satisfying
\[
A^TG = GA, \quad B^TG = C
\]
It is well-known that a controllable and observable system is a gradient system if and only if its transfer matrix
\[
H(s) := C(I(sI - A)^{-1}B \text{ satisfies the symmetry (or, reciprocity) condition}^2
\]
\[
H(s) = H^T(s)
\]
This also implies that the matrix \(G\) satisfying (2) is unique whenever the system is controllable and observable.

Denoting \(P := -GA\) it follows that \(P = P^T\), and the gradient system \((A, B, C)\) can be rewritten into the standard gradient form
\[
\dot{G}x = -Pz + C^Tu, \quad G = G^T, P = P^T
\]
y = Cx

with \(\frac{1}{2}x^TPx\) the potential function of the gradient system, and the symmetric matrix \(G\) defining a (possibly indefinite) inner product on the state space. Conversely, it is immediately checked that the transfer matrix \(H(s)\) of any system in standard gradient form (4) satisfies \(H(s) = H^T(s)\) and thus is a gradient system. Gradient system formulations naturally arise in a number of contexts. For example, RLC-circuits can be represented as gradient systems (see the work of Brayton, Moser [1964a], Brayton, Moser [1964b], Smale [1972], Jeltsema et al. [2003]). In fact it is known from network synthesis theory that any positive real transfer matrix \(H(s)\) satisfying the symmetry property \(H(s) = H^T(s)\) can be realized by an electrical network consisting of capacitors, inductors, resistors and transformers, but without gyrators. In this case the potential function \(\frac{1}{2}x^TPx\) has the dimension of power.

The controllability Gramian \(W\) and the observability Gramian \(M\) of the systems \((A, B, C)\) are defined as the unique solutions of the Lyapunov equations
\[
AW + WA^T = -BB^T, \quad A^TM + MA = -C^TC
\]
Pre- and postmultiplying the first equation by \(G\) yields
\[
GAW + GW^TA = -GBB^TG
\]
which by substitution of (2) yields
\[
A^TG(WG) + (WG)A = -C^TC
\]

Thus by unicity of the solution to the second Lyapunov equation we obtain the following relation between the controllability and observability Gramian (compare with similar results in van der Schaft, Oeloff [1990], Opdenacker, Jonckheere [1985])
\[
GWG = M
\]
Now consider the system in balanced coordinates: that is, coordinates in which \(M = W = \Sigma\) with \(\Sigma\) the diagonal matrix containing the Hankel singular values of the system. In such balanced coordinates we therefore have
\[
G\Sigma G = \Sigma
\]

**Proposition 2.1.** Let \(G = C^T\) satisfy (7), where \(\Sigma\) is the diagonal matrix of Hankel singular values \((\text{all } > 0)\). Then the eigenvalues of \(G\) are \(+1\) or \(-1\). Furthermore, if all Hankel singular values are \(\text{distinct}\) then \(G\) is a signature matrix. If the Hankel singular values are not all distinct then there exist balanced coordinates in which \(G\) is a signature matrix. Finally, if \(G > 0\) then \(G = I\).

**Proof.** Following an argument in van der Schaft, Oeloff [1990] we conclude from \(G\Sigma G = \Sigma\) that the columns of \(G\) are orthogonal and in fact \(G^2 = I\). Hence the eigenvalues of \(G\) are \(+1\) or \(-1\). Since \(G^2 = I\) the equality \(G\Sigma G = \Sigma\) is equivalent with \(G\Sigma G = \Sigma G\). If all elements of \(\Sigma\) are distinct this implies (following an argument in Opdenacker, Jonckheere [1985]) that \(G\) is diagonal, and hence is a signature matrix. Suppose that all elements of \(\Sigma\) are equal. Then there exists an orthonormal matrix \(U\) such that \(U^TGU = D\) for some signature matrix \(D\). Define new coordinates \(\tilde{z}\) by \(z = U\tilde{z}\). It is immediately checked (since \(U^TU = I\)) that these are still balanced coordinates\(^3\). Furthermore, in these new coordinates \(G\) equals the signature matrix \(D\). The argument is easily extended to the case that some of the Hankel singular values are equal to each other, by considering the sub-blocks of \(\Sigma\) and \(G\) corresponding to the subsets of identical singular values.

Finally, if \(G > 0\) then the eigenvalues of \(G\) are all \(+1\), and since \(G\) is symmetric it has a basis of eigenvectors, implying that any vector can be expressed as a linear combination of eigenvectors with eigenvalue \(+1\), and thus \(G = I\).

**Remark 2.2.** If the Hankel singular values are not all distinct then one cannot conclude that \(G = G^T\) satisfying (7) is necessarily a signature matrix. A counterexample is \(\Sigma = I_2\) and
\[
G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

**Remark 2.3.** Order the balanced coordinates in such a way that \(G = \text{diag}(I_k, I_{n-k})\), and partition the matrices \(P\) and \(C\) correspondingly as
\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}
\]
Then the system is additionally passive if \(P_{11} \geq 0, P_{22} \leq 0\) and \(C_2 = 0\). This follows by considering the candidate storage function \(\frac{1}{2}^T|x|^2\).

\(^1\) In some references the terminology ‘gradient system’ is restricted to systems with \(G > 0\), whereas for arbitrary indefinite \(G\) the system is called a pseudo-gradient system.

\(^2\) This can be directly extended to the parity reciprocity condition \(\Sigma_s H(s) = H^T(s)\Sigma_s\), where \(\Sigma_s\) is a signature matrix on the space of inputs and outputs. For details we refer to Willems [1972b].
Proposition 2.1 has the following direct implication with regard to structure-preserving model reduction of gradient systems.

**Proposition 2.4.** Consider a gradient system (4), with singular values \( \sigma_1 \geq \cdots \geq \sigma_k \gg \sigma_{k+1} \geq \cdots \geq \sigma_n \). Choose (based on Proposition 2.1) balanced coordinates such that \( G \) is a signature matrix. Apply balanced truncation by leaving out the state components \( x_{k+1}, \ldots, x_n \). Then the reduced order model is still a gradient system

\[
\begin{align*}
\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{P}} \hat{\mathbf{x}} + \hat{\mathbf{C}}^T \hat{\mathbf{u}}, \quad \hat{\mathbf{x}} \in \mathbb{R}^k \\
y &= \hat{\mathbf{C}} \hat{\mathbf{x}}
\end{align*}
\]  

(8)

Here the reduced signature matrix \( \hat{G} \) is defined as the diagonal matrix consisting of the first \( k \) diagonal elements of \( G \). Furthermore, partitioning \( P \) correspondingly as

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix}
\]

we have \( \hat{P} = P_{11} \), while partitioning \( C = [C_1 \ C_2] \) we have \( \hat{C} = C_1 \).

In the case of singular perturbation balanced truncation we also obtain a reduced order gradient system (8), with the same \( \hat{G} \) but with \( \hat{P} \) the symmetric matrix given as the Schur complement

\[
\hat{P} = P_{11} - P_{12} P_{22}^{-1} P_{21}
\]

Furthermore, in this case the output equation of the reduced-order model becomes

\[
y = \hat{C} \hat{\mathbf{x}} + \hat{D} \hat{\mathbf{u}}
\]

where \( \hat{D} := C_1 - C_2 P_{21}^{-1} P_{22} \).

**Proof.** Since \( P = P^T \) it follows that also \( P_{11} \) is symmetric. The same holds for the Schur complement of \( P \).

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**Example 2.5.** A linear RLC-circuit with external sources can be formulated as a gradient system, where the state vector consists of the currents \( I \) through the inductors and the voltages \( V \) over the capacitors (all assumed to be independent). Then \( G \) is given as the diagonal matrix with elements being the inductances of the inductors and minus the capacitances of the capacitors. Furthermore, the potential function is of the form

\[
\frac{1}{2} \mathbf{I}^T \mathbf{R} \mathbf{I} - \frac{1}{2} \mathbf{V}^T \mathbf{G} \mathbf{V} + \mathbf{I}^T \mathbf{A} \mathbf{V}
\]

(9)

for some positive-definite matrices \( \mathbf{R} \) (resistances corresponding to the current-controlled resistors) and \( \mathbf{G} \) (conductances corresponding to the voltage-controlled resistors). Finally, the matrix \( \mathbf{A} \) (consisting of \( 0, 1, -1 \) elements) is determined by the topology of the circuit (Kirchhoff's laws). It follows that in balanced coordinates the matrix \( G \) transforms into a signature matrix (of course having the same signature as the original \( G \) matrix; thus corresponding to the same number of inductors and capacitors). In general the potential function will transform under balancing into a more general form than (9), that is, the realization of the truncated model may also require additional transformers.

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\[\text{Footnote 4: The definition of a gradient system is easily extended to systems with a feedthrough term } \hat{D} \hat{\mathbf{u}} \text{ by additionally requiring that } \hat{D} \text{ is symmetric. This is obviously equivalent to requiring the non-strict proper transfer matrix } H(s) \text{ to satisfy } H(s) = H^T(s).\]

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In case of a ladder RL-network (for three inductors \( L_1, L_2, L_3 \) and three resistors \( R_1, R_2, R_3 \)) the gradient system equations are given by

\[
\begin{bmatrix}
L_1 & 0 & 0 \\
0 & L_2 & 0 \\
0 & 0 & L_3
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
= \begin{bmatrix}
-R_1 & -R_1 & 0 \\
-R_1 + R_2 & -R_2 & 0 \\
0 & -R_2 & R_2 + R_3
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
+ \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\hat{\mathbf{u}}
\]

(10)

with input \( \hat{\mathbf{u}} \) being the voltage over the voltage source, and \( y \) the current through the branch of the voltage source.

Balancing leads to \( G = I \). Reduction to a 1-dimensional truncated model corresponds to another RL-network with one inductor and one resistor.

Reduction to a 2-dimensional truncated model will be again a gradient system. However, it is not immediately clear under which conditions this is realizable as an RL-network.

**Remark 2.6.** Since we apply the usual balanced truncation procedure, the error bounds are given by the standard balanced truncation error bounds, see e.g., Antoulas [2005].

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### 2.1 The cross-Gramian

It has been shown in Laub et al. [1983] that for a gradient system \((A, B, C)\) with observability Gramian \( M \) and controllability Gramian \( W \)

\[
WM = X^2,
\]

(11)

where the square matrix \( X \) is the so-called cross-Gramian, defined as the unique solution of the Sylvester equation

\[
AX + XA = BC
\]

(12)

In fact, see (Ionescu [2009], Ionescu et. al. [2010]), it can be shown that the cross-Gramian \( X \) satisfies

\[
X = WG = G^{-1} M
\]

(13)

(which is in obvious accordance with the earlier obtained equality (6)). It follows that in balanced coordinates the cross-Gramian is given

\[
X = \Sigma G,
\]

(14)

implying that \( X \) is the diagonal matrix with \( i \)-th diagonal element being \( \pm \sigma_i \), depending on the sign of the \( i \)-th diagonal element of \( G \).

As a corollary we obtain the following converse result showing that diagonalizing coordinates for \( W \) are, up to scaling, necessarily balanced coordinates, at least when the singular values are all distinct.

**Corollary 2.7.** Assume the Hankel singular values of the system \((A, B, C)\) are all different. Let \( \hat{x} = Sx \) be a diagonalizing transformation for the cross-Gramian \( X \), i.e., \( SXS^{-1} = \text{diag}(\pm \sigma_1, \pm \sigma_2, \ldots, \pm \sigma_n) \). Then there exists a diagonal matrix \( D \) such that \( DS \) is a balancing transformation.

**Proof** We know that in balanced coordinates \( X = \text{diag}(\pm \sigma_1, \pm \sigma_2, \cdots, \pm \sigma_n) \). Consider any transformation \( R \) such that \( RXR^{-1} \) is still equal to the diagonal matrix
$X = \text{diag}(\pm \sigma_1, \pm \sigma_2, \ldots, \pm \sigma_n)$. Then $RX = XR$, which implies, since the diagonal elements of $X$ are all assumed to be different, that $R$ is a diagonal matrix.

3. NONLINEAR GRADIENT SYSTEMS

Nonlinear gradient systems are defined in local coordinates $x$ for some state space manifold $\mathcal{X}$ as systems of the form\footnote{\( \frac{\partial P}{\partial x} \) denotes the matrix with $j$-th column being the column vector of partial derivatives of $h_j$. For a linear system with $y = h(x) = Cx$, we have $\frac{\partial P}{\partial x} = C^T$.}

\[
G(x)\dot{x} = -\frac{\partial P}{\partial x}(x) + \frac{\partial h}{\partial x}(x)u
\]
\[y = h(x)\]
where the invertible matrix
\[
G(x) = G^T(x)
\]
defines a (pseudo-) Riemannian metric on $\mathcal{X}$, cf. Crouch [1981], van der Schaft [1984], Cortes et al. [2005]. The function $P : \mathcal{X} \to \mathbb{R}$ is called the potential function. This constitutes an obvious extension to the linear case considered before, where $G$ is a constant matrix, the potential function $P$ is the quadratic function $\frac{1}{2}x^TPx$, and $h(x)$ is the linear function $Cx$. An external characterization of nonlinear gradient systems has been given in Cortes et al. [2005]. This characterization entails two different prolongations of the system, namely the standard prolongation to the tangent space (the variational system) and the gradient prolongation to the cotangent space endowed with an indefinite Riemannian metric (defined with respect to an a priori assumed connection on the state space). The property of being a gradient system then translates into the equality of the input-output maps of these two prolongations.

Motivating examples of nonlinear gradient systems are so-called topologically complete nonlinear RL-C circuits, also called Brayton-Moser-Smale systems (Brayton, Moser [1964a], Brayton, Moser [1964b], Smale [1972], Jeltsena et al. [2003]).

3.1 Balanced truncation for gradient systems

Model reduction based on balanced realizations for nonlinear systems, while preserving the balanced structure is presented in Fujimoto, Scherpen [2010]. The extension of the linear controllability and observability Gramians are given by the following definition:

**Definition 3.1.** The controllability and observability function of a nonlinear gradient system (15) are

\[
L_c(x_0) = \min_{u \in L_2(-\infty, 0)} \frac{1}{2} \int_{-\infty}^0 \| u(t) \|^2 \, dt, \tag{17}
\]

and

\[
L_o(x_0) = \frac{1}{2} \int_0^{\infty} \| y(t) \|^2 \, dt, \tag{18}
\]

for $x(0) = x_0, \ u(t) \equiv 0, \ 0 \leq t < \infty$ respectively.

We assume that

- the gradient system (15) is zero-state observable on some neighborhood of 0,
- $L_c$ and $L_o$ exist and are smooth on that neighborhood,
- the Hessians of $L_c$ and $L_o$ are positive definite,
- $P(x)$ is positive definite, and $\frac{\partial P}{\partial x}(x) \neq 0 \ \text{for} \ x \neq 0$.

For a linear system we have that $L_o(x) = \frac{1}{2}x^TW^{-1}x$ and $L_c(x) = \frac{1}{2}x^TW^{-1}x$. There exists characterizations that can be seen as nonlinear generalizations of the Lyapunov equations (5), in terms of a Lyapunov and Hamilton-Jacobi equation, see e.g., Fujimoto, Scherpen [2010]. With $T(x) := G(x)^{-1}$ it follows that $L_o(x)$ is the solution of

\[
-\frac{\partial L_o}{\partial x}(x)T(x)\frac{\partial P}{\partial x}(x) + \frac{1}{2}x^THx = 0, \ L_o(0) = 0, \tag{19}
\]

and $L_c(x)$ is the solution of

\[
\frac{\partial T}{\partial x}(x)\frac{\partial L_c}{\partial x}(x) + \frac{1}{2}x^THxT(x)\frac{\partial L_c}{\partial x}(x) = 0, \tag{20}
\]

with $L_c(0) = 0$, such that

\[
T(x)\frac{\partial P}{\partial x}(x) + \frac{\partial h}{\partial x}(x)\frac{\partial T}{\partial x}(x)\frac{\partial h}{\partial x}(x)T(x)\frac{\partial L_c}{\partial x}(x) = 0,
\]

is asymptotically stable.

**Theorem 3.2.** (Fujimoto, Scherpen [2010]), Consider the gradient system (15). Suppose that the Jacobian linearization of the system has $n$ distinct Hankel singular values. Then there exists coordinates in a neighborhood of the origin such that the system is in balanced form satisfying

\[
L_c(z) = \frac{1}{2} \sum_{i=1}^n z_i^2 \sigma_i(z) \tag{17}
\]

and

\[
L_o(z) = \frac{1}{2} \sum_{i=1}^n z_i^2 \sigma_i(z) \tag{18}
\]

Suppose that the gradient system is in balanced form. Suppose moreover that

\[
\min \{\sigma_k(s), \sigma_k(-s)\} \gg \max \{\sigma_{k+1}(s), \sigma_k(-s)\}
\]

holds with a certain $k$ ($1 \leq k < n$). Divide the state space accordingly, i.e., $x = (x^a, x^b), \ x^a := (x_1, \ldots, x_k) \in \mathbb{R}^k, \ x^b := (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$, and the inverse metric

\[
T(x) = \begin{pmatrix}
T^{aa}(x^a, x^b) & T^{ab}(x^a, x^b) \\
T^{ba}(x^a, x^b) & T^{bb}(x^a, x^b)
\end{pmatrix}
\]

and truncate the state by substituting $x^b = 0$. Then we obtain a $k$-dimensional state-space model $\Sigma^a$ with state $(x^a, 0)$ (with a $(n-k)$-dimensional residual model $\Sigma^b$ with the state $(0, x^b)$). This procedure is called balanced truncation. The obtained reduced order models have preserved the following properties.

**Theorem 3.3.** (Fujimoto, Scherpen [2010]), The controllability and observability functions of the reduced order models $\Sigma^a$ and $\Sigma^b$ denoted by $L^a_c, L^a_o$ and $L^b_c, L^b_o$ respectively, satisfy the following equations

\[
L^a_c(x^a) = L_c(x^a, 0), \quad L^a_o(x^a) = L_o(x^a, 0)
\]

and

\[
L^b_c(x^b) = L_c(0, x^b), \quad L^b_o(x^b) = L_o(0, x^b)
\]

with $\sigma_i$'s the singular value functions of the system $\Sigma^a$ and $\sigma_i$'s the singular value functions of the system $\Sigma^b$.\]
Hence, balanced truncation preserves the balanced form, or in other words, the reduced order system is in balanced coordinates, and the singular value functions are preserved as well. In order to preserve the gradient system structure, we need an additional condition, i.e.,

$$T_{ab}(x^a, 0) \frac{\partial P}{\partial x^a}(x^a, 0) = 0,$$

and

$$T_{ab}(x^a, 0) \frac{\partial h}{\partial x^a}(x^a, 0) = 0,$$

then the reduced order system is a gradient system with pseudo-Riemannian metric $G^a(x^a) = T_{ab}(x^a, 0)^{-1}$

**Proof.** The reduced order system can be written as follows

$$\Sigma^a:
\begin{cases}
\dot{x}^a = T_{ao}(x^a, 0) \frac{\partial P}{\partial x^a}(x^a, 0) + T_{bo}(x^a, 0) \frac{\partial h}{\partial x^a}(x^a, 0) u^a \\
y^a = h(x^a, 0)
\end{cases}
$$

Under the above conditions, the system reduces to

$$\Sigma^a:
\begin{cases}
\dot{x}^a = T_{ao}(x^a, 0) \frac{\partial P}{\partial x^a}(x^a, 0) + T_{bo}(x^a, 0) \frac{\partial h}{\partial x^a}(x^a, 0) u^a \\
y^a = h(x^a, 0)
\end{cases}
$$

and the result follows immediately.

**Remark 3.5.** For a linear gradient system in balanced form the conditions of Theorem 3.4 are automatically fulfilled, since according to Proposition 2.1 $G = G^{-1}$ is a signature matrix, and thus automatically $T_{ab} = 0$. □

**Remark 3.6.** For nonlinear gradient systems we did not establish a similar result as in Proposition 2.1 for linear systems, since the linear result relies heavily on the symmetry characterization of the system. For nonlinear systems an extension of the result relying on the notion of symmetry is less clear. As mentioned above, an external symmetry characterization is given in Cortes et al. [2005], and the pseudo-metric $G(x)$ connects the variational extension and the prolongation, implying that the linear result is not easily extended. This is related to the discussions about cross Gramians given in Ionescu [2009], Ionescu et al. [2009], Ionescu et al. [2010]. In Ionescu [2009], Ionescu et al. [2010] another notion of symmetry for a nonlinear system is given. This notion is based on the observability and controllability operators connected to the observability and controllability functions defined in Definition 3.1. We come back to this in the section about the cross-Gramian. □

### 3.2 Balanced order reduction via singular perturbation

To find a nonlinear analogue of the second part of Theorem 2.4, we assume that the system is in balanced form, that similar as for the truncation

$$\min\{\sigma_k(s), \sigma_{k+1}(s)\} \gg \max\{\sigma_{k+1}(s), \sigma_{k+1}(-s)\}$$

holds with a certain $k$ $(1 \leq k < n)$, and also similar to the truncation, we divide the state space accordingly $x = (x^a, x^b)$. For singular perturbation, we put $\dot{x}^b = 0$, i.e.,

$$0 = T_{ba}(x) \frac{\partial P}{\partial x^a}(x) + T_{bb}(x) \frac{\partial P}{\partial x^b}(x) + T_{ba}(x) \frac{\partial h}{\partial x^a}(x) u + T_{bb}(x) \frac{\partial h}{\partial x^b}(x) u$$

and with $T_{bb}(x)$ invertible, we have

$$\frac{\partial P}{\partial x^b}(x) = -T_{ba}^{-1}(x) T_{ba}(x) \frac{\partial P}{\partial x^a}(x) - \left( T_{ba}^{-1}(x) T_{ba}(x) \frac{\partial h}{\partial x^a}(x) - \frac{\partial h}{\partial x^b}(x) \right) u$$

resulting in

$$\dot{x}^a = \ddot{T}(x) \frac{\partial P}{\partial x^a}(x) + \ddot{\dot{T}}(x) \frac{\partial h}{\partial x^a}(x) u + \dot{\dot{h}}(x^a) + Du$$

with $\dot{T}(x) = T_{aa}(x) - T_{ab}(x) T_{bb}^{-1}(x) T_{ba}(x)$ being symmetric since $T(x)$ is symmetric. Still, system (22) is not necessarily a gradient system. The following proposition now follows straightforwardly:

**Proposition 3.7.** Suppose that $T_{ab}(x) = (T_{bb}(x))^T = 0$, i.e., $T(x)$ is block diagonal, and $\ddot{T}$ does not depend on $x^b$. Suppose that $P(x)$ quadratic in $x^b$ and that $h(x)$ linear in $x^b$, then the reduced order system

$$\dot{x}^a = \ddot{T}(x) \frac{\partial P}{\partial x^a}(x^a) + \ddot{\dot{T}}(x^a) \frac{\partial h}{\partial x^a}(x^a) u,$$

and

$$\dot{\dot{h}}(x^a) + Du$$

with $\dot{P}(x^a)$, $\dot{h}(x^a)$ following from solving $x^b$ from

$$\frac{\partial P}{\partial x^b}(x) + \frac{\partial h}{\partial x^b}(x) u = 0,$$

and $\ddot{D}$ as in Proposition 2.4 is a gradient system.

**Proof.** Since we have assumed linearity in $x^b$, the result follows immediately.

**Remark 3.8.** The above result is rather restrictive in the sense that full linearity in $x^b$ is assumed. We conjecture that following nonlinear "Schur complement" type arguments we can remove the restrictive assumptions. This is a topic of ongoing research. □

### 3.3 The cross-Gramian

As mentioned in Remark 3.6, there currently exist two definitions of cross Gramians for nonlinear systems. One is based on the gradient system and two extensions, and the other definition is based upon direct symmetry between zero-state observability and asymptotic reachability (as generalizations observability and controllability for linear systems). These concepts are treated in Ionescu, Scherpen [2009], Ionescu et al. [2010], Ionescu [2009]. Here, we focus on the definition for gradient systems, as given in Ionescu, Scherpen [2009].

The prolongation of the gradient system (15) is defined by

$$(y, y_p) = \Sigma_p(x, v, u, u_p) :$$

$$\begin{cases}
\dot{x} = f(x) + g(x) u \\
\dot{v} = \frac{\partial f(x)}{\partial x} v + \sum_{j=1}^m u_j \frac{\partial g_j(x)}{\partial x} v + g(x) u_p \\
y = h(x), \ y_p = \frac{\partial T h(x)}{\partial x} v
\end{cases}$$

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with \( f(x) = T(x)\partial P/\partial x(x) \), \( g(x) = \partial h/\partial x(x) \), and where \( u_p, y_p \) are small variations of the input and the output of the system, respectively. \( v \in TM \) are small variations of the state, where \( TM \) is the tangent bundle of the state space manifold \( M \). Next, we consider the gradient extension. The gradient extension is tedious to construct, since the definition of a metric on the dual space (cotangent bundle) \( T^*M \) is non-trivial and not straightforward. For further details we refer the reader to Cortes et al. [2005]. Let all the geometric terms obtained from writing the gradient with respect to the \( T^*M \) metric be described by the function \( F(\Gamma_{jk}, X_k, p_i) = \sum_{i,j,k} (2p_i \Gamma_{jk} X_k) \), with \( X_k \in \{ f, g, \ldots, g_n \} \). The \( \Gamma_{jk} \) are defined by relation [Cortes et al., 2005, (2.8)]. The gradient extension system associated to (15) with \( f(x) = T(x)\partial P/\partial x(x) \), \( g(x) = \partial h/\partial x(x) \), is defined by \( (y, y_p) = \Sigma_q(x,v, u_p, y_p) \):

\[
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
\dot{p} &= \partial^T f(x) + g(x)u) \partial_x p + F(\Gamma_{ij}, f_k(x), g(x), p) \\
\dot{y} &= h(x), \quad y_p = g^T(x)p, \text{ } i, j, k = 1\ldots n.
\end{aligned}
\] (25)

It is known (Cortes et. al. [2005]) that for a gradient control system (15) the relation between \( v \) and \( p \) is given by \( p = G(v)\dot{v} \), or \( v = T(x)p \). From there, it follows that

\[
G(x)g(x) = \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial^T h}{\partial x} T(x)p = g^T(x)p.
\]

which indeed corresponds to the gradient system proper ties. Now consider the observability function \( \odot (x) \) and its observability function \( G(x) \). The cross Gramian of the nonlinear gradient system (15) is now defined by

\[
X(x) = T(x)L(x)
\]

Corollary 3.9. Consider the gradient system (15). The nonlinear cross Gramian fulfills the following Sylvester like equation.

\[
p^T T(x)L(x) \frac{\partial f}{\partial x}(x) + \frac{1}{2} p^T \frac{\partial f}{\partial x}(x) \frac{\partial^T h}{\partial x}(x)v = -v^T \frac{\partial^T L(x)}{\partial x}(x,v)f(x) + \frac{\partial^T L(x)}{\partial x}(x,v)f(x) - F^T L(x)v
\]

with \( F = F - T(x) \hat{G}(x)v \).

Proof. The proof follows by substituting the gradient system expressions in the cross Gramian expression from Ionescu, Scherpen [2009].

Remark 3.10. For a linear system, Corollary 3.9 results in the standard Sylvester equation (12). Linearization also immediately yields the results for linear gradient systems, see e.g., Ionescu, Scherpen [2009].

4. CONCLUDING REMARKS

In this paper, we study balancing based structure preserving model order reduction of linear and nonlinear gradient systems. For nonlinear gradient systems, the structure preservation is less straightforward, and further research on the singular perturbation method and on the cross Gramian and its usefulness for balancing is ongoing.

REFERENCES

A.C. Antoulas, Approximation of large-scale dynamical systems, SIAM, Philadelphia (2005).