Adaptive Gains Super-Twisting Algorithm for Systems with Growing Perturbations

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Abstract: In this paper we propose a novel Lyapunov based approach for the gain adaptation of super-twisting algorithm ensuring global finite time convergence to the desired sliding surface for linear time invariant systems with absolutely continuous matched uncertainties/disturbances bounded together with their gradients by known functions which could be not globally Lipshitz. The proposed methodology makes the control strategy global but also provides chattering phenomenon alleviation.

Keywords: Sliding Mode Control, Lyapunov based design, Adaptive Control.

1. INTRODUCTION

Motivation. Sliding mode control (SMC) is a well-known tool for rejecting matched oscillations (see Utkin et al. (2009), for example). Classical (first order) sliding motion could be enforced by relay or unit control. The main disadvantage of those control strategy is the so called "chattering effect" (see Utkin et al. (2009), for example). The chattering magnitude is proportional to the relay or unit controller gains. For the systems with the matched uncertainties/disturbance bounded by known functions the relay or adaptive gains are proposed to deal with them (Hsu et al. (1997), Plestan et al. (2010)).

The radical solution allowing to adjust the chattering problem was suggested in (Levant (1993)) as a super-twisting algorithm (STA). This algorithm allows to substitute the relay/unit controllers via absolutely continuous one compensating exactly the Lipshitz perturbations/uncertainties.

Unfortunately, the first convergence proofs for the STA (Levant (1993)) are based on the idea geometrical design of majorant curves and homogeneity. Moreover, the homogeneous nature of the standard STA does not allow to compensate uncertainties/disturbances which may grow together with the state variables. On the other hand, in Boiko (2008) is shown that the chattering in STA still exists and it will be reasonable to decrease STA gains to adjust the chattering.

For that reason, it is very important to design a non-homogeneous extension of the standard STA guaranteeing exact compensation of smooth uncertainties/disturbances bounded together with their derivatives by known functions, which could grow and decrease together with the system states, i.e. design an adaptive gains STA (AGSTA).

Methodology. To prove the convergence of the second-order sliding mode twisting algorithm, the non smooth but Lipshitz non strict Lyapunov functions was used in (Orlov (2009)). A geometric approach to the Lyapunov design for STA was used in Utkin et al. (2009). In (Polyakov and Poznyak (2009)) the Zubov (Zubov (1964)) method was applied to prove STA convergence and convergence time estimation. Those approaches do not allow to prove finite time convergence for the case when the disturbances/uncertainties growing together with the system states and also the STA gains adaptation.

Contribution. In present paper we introduce the AGSTA with variable gains, using a Lyapunov based approach (Moreno (2009)) allowing

- to ensure exact compensation of the smooth uncertainties/disturbances bounded by known functions together with their derivatives;
- STA gains adaptation adjusting the chattering;
- convergence time estimation.

The rest of the paper is organized as follows. In section II the problem statement is given. Section III introduce the AGSTA and the main result of the paper are presented. In Section IV is proved the main result and in section V simulation results show the performance of the new algorithm. Finally, some conclusions are drawn.

2. PROBLEM STATEMENT

The analyze only consider a single input uncertain linear time invariant system (LTI)

$$\dot{x} = Ax + B(u + w(x, t))$$

(1)
where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input, \( A, B \) are constant matrices of appropriate dimensions, and \( w \) is a absolutely continuous uncertainties/disturbances in system (1). Without loss of generality, we can restrict our analyze to the single input case \((m = 1)\) (Utkin (1992)). The results can be easily extended to the multi-input case. The following assumptions are taken into account for the system (1):

(A1) rank \( B = m \),

(A2) the pair \((A, B)\) is controllable,

(A3) The function \( w \) together with its gradient is bounded by known continuous functions.

The supposition (A1) and (A2) allows to under a linear state transformation

\[
\begin{align*}
\dot{\eta} &= A_1\eta + A_2s + \mathcal{W}(\eta, \xi, t) \\
\dot{s} &= v + \tilde{w}(\eta, s + K\eta, t) \\
\end{align*}
\]

where \( \eta \in \mathbb{R}^{n-m} \) and \( \xi \in \mathbb{R}^m \). The main idea of the sliding mode control concerns to the design of a properly sliding surface, that is,

\[
s = \xi - K\eta = 0
\]

such that, when the motion is restricted to this manifold, the reduced-order model

\[
\dot{\eta} = (A_{11} + A_{12}K)\eta
\]

has the required performance. Since the pair \((A_{11}, A_{12})\) is controllable, matrix \( K \) can be designed using any linear control design method for system (4), for instance: (i) Eigenvalue assignment, or (ii) Optimal control (LQR), an strategy known as the Optimal sliding manifold design.

Using \((\eta^T, s)\) as state variables, and applying the controller

\[
u = -(A_2(I + K) + K(A_{11} + A_{12}))\eta - A_{22} - KA_{12}s + v^T
\]

system (2) takes the form

\[
\begin{align*}
\dot{\eta} &= (A_{11} + A_{12}K)\eta + A_{12}s \\
\dot{s} &= v + \tilde{w}(\eta, s + K\eta, t) \\
\end{align*}
\]

For the case, when the uncertainties/disturbances is bounded by a known function \( \varrho(x) \), that is, when \( u(x, t) \) satisfies the restriction \( |u(x, t)| \leq \varrho(x) \), the classical (first order) sliding mode can be enforced by a variable gain controller

\[
v = -\varrho'(x)(s)\varrho(s)
\]

with positive scalar \( \varrho > 0 \). Alternatively, unit controllers can also be used for this purpose (see, for example Utkin (1992)). The main disadvantage of these controllers is that they produce the chattering effect, that is as large as the large of the uncertainty bound \( \varrho(x) \).

The homogeneous nature of the STA does not allow to compensate the classes of uncertainties/disturbances (7). In this paper the Lyapunov based design of the STA (Moreno (2009)) is resumed. The reason is including (i) high-degree (non homogeneous) terms (terms with degree bigger than one), and (ii) variable gains in order to alleviate the semi-global nature of the standard STA. Since the geometric or homogeneity proofs of the standard STA does not allow to deal with these extensions, the use of the Lyapunov method becomes a good tool to design the AGSTA.

3. AN ADAPTIVE GAINS SUPER-TWISTING ALGORITHM

The Adaptive Gains Super-Twisting Algorithm (AGSTA) proposed has the following form

\[
v = -k_1(t, x)\phi_1(s) - \int_0^t k_2(t, x)\phi_2(s) \, dt
\]

where

\[
\phi_1(s) = |s|^\frac{2}{3}\text{sign}(s) + k_3|s|^\frac{2}{3}\text{sign}(s) , \quad k_3 > 0
\]

\[
\phi_2(s) = \frac{1}{2}|s|^2 + 2k_3s + \frac{3}{2}k_3^2|s|^2\text{sign}(s)
\]

It is clear to see that the standard STA can be recovered, when \( k_3 = 0 \) and the gains \( k_1 \) and \( k_2 \) are constant. \( k_3 > 0 \) allows to faster convergence and deal with uncertainties/disturbances growing with degree bigger than a linear term in \( s \), i.e. outside of the sliding surface. The variable gains \( k_1 \) and \( k_2 \) make possible to render the sliding surface insensitive to perturbations growing with bounds given by known functions. Note that the perturbations can always be written as

\[
\tilde{w}(\eta, s + K\eta, t) = |\tilde{w}(\eta, s + K\eta, t)| = \varrho(\eta, s + K\eta, t)
\]

where \( \varrho(\eta, s, t) = 0 \) when \( s = 0 \). We assume that the perturbation \( \tilde{w}(\eta, s + K\eta, t) \) is bounded by

\[
|\varrho_1(\eta, s, t)| \leq \varrho_1(\eta, s + K\eta, t)\varrho_1(\eta) \]

\[
\frac{d}{dt}\varrho_2(\eta, s + K\eta, t) \leq \varrho_2(\eta, s + K\eta, t)\varrho_2(\eta)
\]

where \( \varrho_1(\eta, s + K\eta, t) = \varrho_1(t, x) \geq 0 \), \( \varrho_2(\eta, s + K\eta, t) = \varrho_2(t, x) \geq 0 \) are known continuous functions. The system (6) with the AGSTA (8) can be rewritten as

\[
\begin{align*}
\dot{\eta} &= (A_1 + A_2K)\eta + A_2s \\
\dot{s} &= -k_1(t, x)\phi_1(s) + x + g_1(\eta, s, t) \\
\dot{z} &= -k_2(t, x)\phi_2(s) + \frac{d}{dt}g_2(\eta, t)
\end{align*}
\]

The solutions of (10) are all trajectories in the sense of (Filippov (1988)).

4. A LYAPUNOV BASED APPROACH

4.1 Finite Time Convergence to the Sliding Surface

Extending the ideas of (Moreno (2009)) we consider as candidate Lyapunov function the quadratic form

\[
V(s, z) = \zeta^TP\zeta
\]

where

\[
\zeta = [\phi_1(s), z] \quad \text{and} \quad P = \begin{bmatrix} p_1 & p_3 \\ p_2 & p_2 \end{bmatrix} = \begin{bmatrix} \beta_0 + 4\epsilon^2 & -2\epsilon \\ -2\epsilon & \epsilon \end{bmatrix}
\]

with arbitrary positive constants \( \beta_0 > 0, \epsilon > 0 \).

**Proposition 1.** Consider the system (10). Suppose that the perturbations satisfies Assumption (A3), for some known continuous functions \( \varrho_1(t, x) \geq 0 \), \( \varrho_2(t, x) \geq 0 \). Then for any initial condition \((\eta_0, s_0, z_0)\) the sliding surface \( s = 0 \)
will be reached in finite time if the variable gains are selected as
\[ k_1(t, x) = \delta_0 + \frac{1}{\beta_0} \left\{ 1 + 2(2 + \zeta_1 + \zeta_2) \right\} \]
\[ + \frac{2 + 2(2 + \zeta_1 + \zeta_2) (\beta_0 + \lambda \zeta_2^2 + \zeta_2^2) \right\} \]
where \( \beta_0 > 0, \epsilon > 0, \delta_0 > 0 \) are arbitrary positive constants. The reaching time to the sliding surface can be estimated by
\[ T = \frac{2}{\gamma_1} V^{\frac{1}{2}} (s_0, z_0) \]
where \( V(s, z) \) is given by (11), and \( \gamma_1 \) is as in (14).

**Proof.** Function (11) is a constant, global, robust and strong (strict) Lyapunov function for the subsystem (10).
It shows finite time convergence and is positive definite, everywhere continuous and differentiable everywhere except on the set \( S = \{ (s, z) \in \mathbb{R}^2 | s = 0 \} \). Note that due to Assumption (A3) we can write \( g_1(\eta, s, t) = \alpha_1(t, x) \phi_1(s) \), and \( \frac{d}{dt} \phi_2(\eta, t) = \alpha_2(t, x) \phi_2(s) \) for some functions \( \alpha_1(t, x) \leq g_1(t, x) \) and \( \alpha_2(t, x) \leq g_2(t, x) \). Using these functions and noting that \( \phi_2(s) = \phi_1(s) \phi_1(s) \) one can show that
\[ \dot{\zeta} = \left[ \begin{array}{c} \phi_1'(s) (\beta_0 - \alpha_1(t, x) - 2(\beta_0 + \lambda \zeta_2^2 + \zeta_2^2) \right\} \]
\[ - \phi_2'(s) \phi_2(s) + \frac{d}{dt} \phi_2(s, x, t) \]
\[ = \phi_1'(s) \left[ \begin{array}{c} - (k_1(t, x) - \alpha_1(t, x)) + (k_2(t, x) - \alpha_2(t, x)) \right\} \]
\[ - k_2(t, x) \phi_2(s, x, t) + 1 \]
\[ = \phi_1'(s) A(t, x) \zeta \]
for every point in \( \mathbb{R}^2 \setminus S \), where this derivative exists. Similarly one can calculate the derivative of \( V(s, \zeta) \) on the same set as
\[ \dot{V}(s, \zeta) = \phi_1'(s) \zeta^T (A^T(t, x) P + PA(t, x)) \zeta \]
where
\[ Q(t, x) = \left[ \begin{array}{cc} 2(k_1(t, x) - \alpha_1(t, x)) & 2(k_2(t, x) - \alpha_2(t, x)) \\
-2(k_2(t, x) - \alpha_2(t, x)) & -2 \end{array} \right] \]
With the selection of \( P \) in (4.1) and the gains in (12) it follows that (the arguments of the functions were left out)
\[ Q - 2 \epsilon I = \left[ \begin{array}{cc} 2(2k_1(t, x) - \alpha_1(t, x)) & 2(k_2(t, x) - \alpha_2(t, x)) \\
-2(k_2(t, x) - \alpha_2(t, x)) & -2 \end{array} \right] \]
that is positive definite for every value of \( (t, x) \). This shows that
\[ \dot{V} = -\phi_1'(s) \zeta^T Q(t, x) \zeta \leq -2\epsilon \phi_1'(s) \zeta^T \zeta \]
\[ = -2\epsilon \left( \frac{1}{2|s|^2} + \frac{3}{2k_3 \zeta^2} \right) \zeta^T \zeta \]
Since
\[ \lambda_{\min}\{P\} \|\zeta\|_2 \leq \zeta^T P \zeta \leq \lambda_{\max}\{P\} \|\zeta\|_2 \]
where
\[ \|\zeta\|_2^2 = \zeta_1^2 + \zeta_2^2 = |s|^2 + 2k_3 |s|^2 + k_3^2 s^2 + z^2 \]
is the Euclidean norm of \( \zeta \), and
\[ \frac{1}{|s|^2} \leq 1 \]
\[ \frac{1}{|s|^2} \leq 1 \]
it follows that
\[ \dot{V} \leq -\epsilon \|\zeta\|_2 - 3k_3 \|\zeta\|_2^2 \leq -\gamma_1 V^{\frac{1}{2}} (s, z) \]
\[ \leq -\gamma_1 V^{\frac{1}{2}} (s, z) - 2 \gamma_2 s^2 V (s, z) \leq -\gamma_1 V^{\frac{1}{2}} (s, z) \]
where
\[ \gamma_1 = \frac{\epsilon}{\lambda_{\max}\{P\}} \text{ and } \gamma_2 = \frac{3k_3}{\lambda_{\max}\{P\}} . \]
Noting that the trajectories of the AGSTA cannot stay on the set \( S = \{ (s, z) \in \mathbb{R}^2 | s = 0 \} \), one concludes by means of Zubov’s stability theorem with the continuous Lyapunov function candidate that the equilibrium point \( (s, z) = 0 \) is reached in finite time from every initial condition. Since the solution of the differential equation
\[ \dot{v} = -\gamma_1 v^{\frac{1}{2}} , \quad v(0) = v_0 \geq 0 \]
exists and converges to zero in finite time and reaches that value before a time given by (13).

5. SIMULATIONS RESULTS

We are considering the linearized model of the Pendulum-Car motion (Utkin et al. (2009)), see Fig. 1. The motion equation considered here was linearized around \( x^* = (0, 0) \in \mathbb{R}^2 \) and can be written in the form of LTI system with matched perturbations
\[ \dot{x} = Ax + B(u + w(x, t)) \]
where
\[ A = \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
o_{32} & 0 & 0 & 0 \\
o_{42} & 0 & 0 & 0 \end{array} \right] , \quad B = \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & a_{32} & 0 & 0 \\
0 & a_{42} & 0 & 0 \end{array} \right] \]
and
\[ a_{32} = -3(C - mga)/(4Mm) \]
\[ a_{42} = -3(4M + m)(C - mga)/(4M + m) \]
\[ b_4 = 4/(4M + m) \]
\[ b_3 = 3/(4M + m) \]
\[ M \text{ and } m \text{ are the car mass and the pendulum mass, } a \text{ is the length of the pendulum center mass from the pivot}, \]
\[ C \text{ is the spring modulus, } g \text{ is the gravitational acceleration constant, } u \text{ and } w(t) \text{ are the control and the perturbations.} \]
The physical parameters of the Pendulum-car are defined in Table 1.
Table 1. Pendulum-car Parameters

<table>
<thead>
<tr>
<th>Notation</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>5</td>
<td>Kg</td>
</tr>
<tr>
<td>$m$</td>
<td>1</td>
<td>Kg</td>
</tr>
<tr>
<td>$C$</td>
<td>1</td>
<td>N/m</td>
</tr>
<tr>
<td>$a$</td>
<td>1</td>
<td>m</td>
</tr>
<tr>
<td>$g$</td>
<td>9.810</td>
<td>m/s²</td>
</tr>
</tbody>
</table>

For the simulations, we set initial conditions sufficiently close to the equilibrium point $x^*$. Sliding surface (3) now is a scalar and has a form $s = c^T \dot{x}$, $c^T$ could be found via the Ackermann-Utkin formula (Ackermann and Utkin (1998)) for the desired eigenvalues assignment as follows: Let $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$ be the desired eigenvalues of the sliding motion. For the given values of the parameters in matrices $A$ and $B$

$$c^T = [-4.77, 48.4, -8.75, 18.7].$$

The next part is dedicated to show the simulation results of the AGSTA based controller and puts in evidence the chattering alleviation and the high accuracy and the strong robustness of the control strategy in presence of very strong perturbations. Four controllers were tested with above described pendulum-car system to illustrate the performance of the proposed AGSTA based controller. The control law (5) is implemented with the:

1.- First-order sliding mode (FOSM) controller with $M_0 = \text{const}$,

$$v = -M_0 \text{sign}(s)$$

2.- STA with constant gains $k_1$ and $k_2$

$$v = -k_1 |s|^2 \text{sign}(s) - k_2 \int_0^t \text{sign}(s) dt$$

3.- First-order sliding mode controller with variable gain (VGSM)

$$v = -\beta_1 (|x| + |\dot{x}| + |\alpha| + |\dot{\alpha}| + \delta_1) \text{sign}(s),$$

where $\beta_1$ and $\delta_1$ are positive values which can be selected arbitrary.

4.- AGSTA based controller

$$v = -k_1 (t, x(t)) \phi_1(s) - \int_0^t k_2 (t, x(t)) \phi_2(s) dt,$$

where $k_1 (t, x)$ and $k_2 (t, x)$ were selected as in (12).

For the simulation, Eulers method ode solution was used with a fixed step size of 0.001 seconds and the main parameters was selected as follow: $M_0 = 70$, $k_2 = 40$, $k_1 = 20$, $\beta_0 = 4$, $\delta_0 = 0$, $\beta_1 = 60$, $\delta_1 = 0.5$ and finally, $p_1$ and $P_2$ can be selected in such a way that $P = P^T > 0$, for this case, choosing $\epsilon = 1$, the elements of the matrix $P$ are $p_1 = 8$, $p_2 = 4$, $p_3 = -2$ and $p_4 = 1$. Finally, $g_1 = k_1 \kappa$ and $g_2 = k_2 \kappa$, with $\kappa = |x| + 0.5 |\dot{x}| + 0.05 |\alpha| + 0.05 |\dot{\alpha}|$.

The simulations are divided in two parts. Firstly, we consider the following initial conditions $x(0) = [0.5 \ 0.2 \ 0 \ 0]$, and $w(x, t) = 0.5 \sin(3t)$. Simulations results of the pendulum angle and the car position controlled by the different control inputs $v$. In the figures 2 and 3 are depicted the simulation results. It easily to see from both figures that all controllers provide the exact convergence of the system states (2) and finite-time exact convergence to the sliding surface (3), left column. It can be easily seen from fig.(3) that VGSM reduces the level of chattering w.r.t. FOSM, STA w.r.t. VGSM, and AGSTA w.r.t. STA.

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Fig. 4. Adaptive Gains, $k_1(t, x)$ (lower line) and $k_2(t, x)$.

Fig. 5. Pendulum angle and position: a) FOSM, b) STA, c) VGSM, d) AGSTA.

Fig. 6. Control action and the chattering effect: a) FOSM, b) STA, c) VGSM, d) AGSTA.

Fig. 7. Adaptive Gains, $k_1(t, x)$ (lower line) and $k_2(t, x)$.

6. CONCLUSION

The AGSTA proposed in the paper ensures

- the exact compensation of absolutely continuous uncertainties/disturbances bounded together with their gradients by known functions which are not obligatory globally Lipschitz;
- global finite time convergence to the desired sliding surface;
- the chattering adjustment via STA gains adaptation.

The convergence time to the sliding surface can be estimated using a Lyapunov function. The proposed results was validated with simulations and compared with other known algorithms. Also, the proposed algorithm can be applied to a other class of system. For example, a class of nonlinear system which can be represented as a chain of integrators, for instance, second order systems, as the mechanical systems. These class of system can be rewritten...
in regular form in the state space. And the design is made in similar way to the one developed here.

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