An approach to systematization of types of formal cognitive maps

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Abstract: Existing diversity of types of formal cognitive maps with corresponding theoretical models makes actual the problem of their systematization for further comparative analysis of their capabilities in solving applied problems and for development of the general theory of formal cognitive maps, when and if it is possible. The lack of the general theory of formal cognitive maps, orientation of many known theoretical models exclusively on simulation, semantic vagueness of some models have already led to incorrect formal statements and solutions of applied problems on the basis of cognitive maps. The general parametrical model of semantics of functional cognitive maps is developed to uniformly describe semantics of formal cognitive maps. The model has covered the major part of known types of formal cognitive maps and has enabled systemizing non-functional types of maps. Efficiency of the proposed approach to systematization of formal cognitive maps is shown on the representative set of types of such maps.

Keywords: formal cognitive map, cognitive map semantics, type of cognitive maps, functional cognitive map, systematization

INTRODUCTION

Up to date there are a lot of studies devoted to applications of cognitive maps to practical problems concerned with the analysis and control of ill-structured situations (see, for example, recent reviews in Peña et al. (2008), Avdeeva and Kovriga (2008)). In the variety of approaches in cognitive mapping the special place belongs to cognitive maps (CMs), that are aimed to represent the structure of causal (or, that is the same, cause-effect) influences in a mapped situation and are characterized by more or less formal semantics. It is natural to name such maps as formal ones. A formal CM is a model of a situation representing expert knowledge about the situation as a structure of cause-and-effect influences between factors (or, in other terms, concepts). Since a theoretical model defines a type of a formal CM in what follows the term “type of maps” is used meaning that there is a theoretical model behind a type of maps.

The diversity of types of formal CMs (Stylios and Groumpos (1999)) and the lack of general theory of such maps make actual the problem of their systematization for further comparative analysis of their capabilities in solving applied problems and for development of the general theory of formal CMs, when and if it is possible.

The analysis shows that the lack of general theory of CMs, orientation of many known theoretical models exclusively to simulation, semantic vagueness of some models have already led to incorrect formal statements and solutions of applied problems on the basis of CMs (Abramova and Kovriga (2008), Carvalho (2010)).

Preliminary analysis made by authors shows that both problems, (i) the problem of systematization of diversity of CMs and (ii) the revelation of practically significant incorrectness in known CM models, can be solved by means of uniform representation of various theoretical models of CMs in terms of a general model. The general model proposed here differs from conventional ways of describing theoretical models of formal CMs since it possessed more formality and rigidity.

In order to uniformly describe various theoretical models of formal CMs the model of functional scheme types and their external behavior (metamodel) (Abramova (1993)) was accepted. This model was adapted to systemize the diversity of formal CM types. It is worth underlining that earlier the model of functional schemes was successfully applied to uniform formalized description of families of such different types of objects as functional schemes of devices and algorithms, sequential programs, data flow programs and a number of others with subsequent development of algebraic methods of external behavior analysis.

The article presents an approach to the systematization of formal CM types based on the general parametrical model of semantics of functional CMs which is itself the adaptation of the model of functional schemes to formal CMs. (section 2). Efficiency of the proposed approach to systematization is shown on classification of the representative set of types of formal CMs (section 3). Section 3 is dedicated to the classification of the representative sets of formal CMs.
2. APPROACH TO SYSTEMATIZATION OF FORMAL COGNITIVE MAPS

The proposed unified approach to describe various theoretical models of formal CMs is based on the model of functional scheme types and their external behavior (Abramova (1993)).

2.1 Main features of the model of functional scheme types

The model (further denoted as $M$) serves as the general scheme for description of theoretical models of single types of functional schemes, schemes, nets and other structured objects consisting of elements of a small number of types, i.e. the model itself is the metamodel. The main features of $M$ are as follows.

Unlike conventional ways to describe theoretical models of formal CMs, $M$ follows the compositional approach: it represents a structured object of a certain type $K$ as a composition of elements of one or several types. Herewith (i) the structure $T_K$ of an arbitrary object $K$ of type $K$ is defined in terms of the structure of single elements, and (ii) the behavior of $K$ as a whole is defined via the behavior of its elements.

Various statements of behavior analysis problems relating to an arbitrary object $K$ of type $K$ are defined in terms of external behavior model, $M_{ex}$. $M_{ex}$ restricts knowledge of behavior of object being studied to applied goals and to availability of appropriate input data. It is also required to prove the correctness of a chosen problem statement with respect to $M$ (i.e. in terms of (Abramova (1993))) to prove the theorem about formal applicability of $M_{ex}$.

External behavior models of formal CMs are not considered in the article due to the size limits. However, its worth to note that accounting for typical statements of behavior analysis problems for formal CMs has played a significant part in more adequate adaptation of $M$ to the diversity of formal CMs and in revelation of some incorrect problem statements.

2.2 Initial adaptation of model $M$ to the structure of formal CMs

The main structural element in any known theoretical model of CMs is an element, which in different traditions is called either the factor (as in this work) or the concept; in most models it is mathematically represented as a variable of some type. It is also supposed that there is cause-and-effect relation between this factor and other factors of the map, which is represented in many models as a function of a certain type. According to the compositional approach, it is relevant to represent such a factor as a bundle, i.e. this effect factor along with cause factors directly influencing it.

However in view of problems being practically solved with formal CMs one should account for external influences on a situation modeled with a CM, including the control and environment. In general, this results in three structural types of factors in CMs. A dependent factor whose value can change only due to cause-and-effect influences of its cause factors is named a strictly dependent factor, or a passive factor. A dependent factor whose value can change due to external influences on it is named a mixed factor. A factor not influenced by others is named a conditionally independent. In practice most of CMs contain such factors.

2.3 Main ideas of the proposed approach to systematization

The main idea of the proposed approach to systematization is in using analogies. When the base type of maps to carry out analogies is chosen it is then represented in terms of parametrical metamodel $M$ adapted to formal CMs. Every new type of formal CMs is considered as a parametrical modification of the base type.

2.4 Description of the base type of formal CMs ($K_0$)

As the base type of maps to carry out analogies a relatively simple type of CMs, $K_0$ is considered in which (i) a direct dependence of a dependent factor $y$ on other factors of a map (excluding itself) is represented by a linear function of type:

$$y(t+1) = \sum_{i=1}^{n} w_i \cdot x_i(t),$$

where $n$ is the number of factors directly influencing $y$, $y(t+1)$ is the value of factor $y$ at time $t+1$, $x_i(t)$ is the value of factor $x_i$ at time $t$, $w_i$ are the coefficients of a linear function, that are interpreted as the weighs of influences of $x_i$ on $y$, and (ii) the structure of cause-and-effect influences of factors can be represented as a directed tree, in which all arcs are directed from the leaves towards the root, leaves correspond to CI-factors.

2.5 Model of semantics of the base type ($K_0$)

The type $K_0$ in terms of the metamodel $M$ can be represented as the pair ($M_0^e$, $M_0^s$), where $M_0^e$ is basis, i.e. the set of models of types of factors (in $K_0$). $M_0^s$ is the general model of a map as a whole.

Basis of type $K_0$

The basis $M_0^e$ of maps of type $K_0$ consists of models of two types of factors: passive factors (type $D_0$) and CI-factors (type $I_0$). Thus, $M_0^e = \{M_{D_0}, M_{I_0}\}$, where $M_{D_0}$ is the general model of a passive factor, defining every possible factor of this type, $M_{I_0}$ is the general model of a CI-factor.

General model of a passive factor

The general model of a passive factor $M_{D_0}$ is represented as the pair

$$M_{D_0} = \{(\text{St}_D, B_D)\}.$$
where \( \text{St}_{D_0} \) is the general structural model of a passive factor, \( B_{D_0} = \left( I^M_{D_0}, P^I_{D_0} \right) \) is the general model of behavior of a passive factor, \( L^w_{D_0} \) is the type of influences aggregation function of a passive factor, \( P^l_{D_0} \) is the conception of dynamics of a passive factor.

The general structural model of a passive factor \( \text{St}_{D_0} \) reflects the links between this passive factor and other factors of map, and can be represented as a star graph with vertices marked with factor names (fig.1):

![Star Graph](image)

**Fig. 1 General structural model a passive factor**

The behavior of any passive factor is represented by some function of type \( L^w_{D_0} \), representing the dependence of this factor value on the values of cause factors and by conception of factor dynamics, \( P^l_{D_0} \).

Any passive factor is corresponded by a variable representing its dynamics. Influences aggregation function of any passive factor is a linear function of type:

\[
L^w_{D_0} : V_1 \times \ldots \times V_n \rightarrow V_y,
\]

where \( V_n \) is the set of every possible values of a factor \( x_i \), \( n \geq 1 \) is the number of factors directly influencing \( y \), \( V_y \) is the set of every possible values of factor \( y \), \( w_i \in V_w \) are the weights of influences, for maps of type \( K_0 \): \( V_u = R \).

The conception of dynamics of a passive factor, \( P^l_{D_0} \), specifies behavioral semantics of a passive factor in a form of logical statement and represents this semantics in a form of formalized expression which later can be used in a structure of theoretical proofs relating to dynamics of CMs and CM based models.

With the same influences aggregation function can be associated different conceptions of dynamics. For example one conception can be formulated in continuous time, another one – in discrete time.

For any passive factor \( y \) denote the set of map factors directly influencing \( y \) as \( X_y \), the set of values of these factors at time \( t \) denote as \( X_y(t) \). In discrete time the conception of dynamics of a passive factor, \( P^l_{D_0} \), can be formulated as follows:

\[
P^l_{D_0} : \text{For any time } t \text{ the value of a passive factor } y \text{ at time } t+1, \ y(t+1), \text{ is uniquely determined depending on } X_y(t)
\]

by influences aggregation function of the factor \( y \) of type \( L^w_{D_0} \).

Such maps for which the conception of dynamics of factors directly takes into account time are named dynamic maps.

**General model of a CI-factor**

Similarly to the general model of a passive factor, \( M_{D_0} \), the general model of a CI-factor, \( M_{I_0} \), is represented as the pair

\[
M_{I_0} = \left( \text{St}_{I_0}, B_{I_0} \right), \quad l_{B_{I_0}} = \left( C_{I_0}, P^l_{I_0} \right).
\]

Where the general structural model of a CI-factor \( y \), \( \text{St}_{I_0} \), is a named one-vertex graph.

If one tried to describe the dynamics of a CI-factor as a function representing the dependence of the value of this factor on the values of other factors of a map this function would have no arguments, i.e. the function would be a constant. Actually, the dynamics of a CI-factor is determined by external influences on this factor. So it is naturally to represent its dynamics as a time function \( f_y(t) \) of type \( C_{I_0} \). The conception of dynamics of a CI-factor is not added due to its triviality.

**General model of a map of type \( K_0 \) as a whole**

The general model of a map of type \( K_0 \) as a whole, \( M^0_S \), is represented as the pair

\[
M^0_S = \left( T, P^l_{K_0} \right),
\]

where \( T \) is the general structural model of a map of type \( K_0 \), \( P^l_{K_0} \) is the general model of behavior of a map of type \( K_0 \) as a whole.

The general structural model of a map of type \( K_0 \), \( T \), which determines the type of a map structure is a directed tree with named nodes and arcs directed from the leaves toward the root, which is considered as a composition of structural models of factors. The composition results from identification of nodes with the same name.

The general model of behavior of a map of type \( K_0 \) as a whole, \( P^l_{K_0} \), in a form of a logical statement uniquely defines the dynamics of a map of type \( K_0 \) in a certain space of behavior variants.

Consider two conceptions of behavior of a map as a whole which can be applicable to maps in a basis consisting of CI-factors and passive factors with a structure of type \( T \).

The following conception, \( P^l_{K_0} \), naturally results from models of behavior of elements (factors). This conception is based on the separation of the set of all factors of a map,
$A_K$ into CI-factors ($X_k$) and passive factors ($Y_k$), $A_k = X_k \cup Y_k; X_k \cap Y_k = \emptyset$, where values of passive factors are interpreted as states.

Denote the set of values of CI-factors at time $t$ as $X_k(t)$, analogically define $Y_k(t)$ for passive factors.

Taking into account assumed sources of required data this conception can be formulated as follows.

$P^1_{K_0}$: For any (defined from the outside) set of values of CI-factors at time $t$, $X_k(t)$, for any (known) set of states, $Y_k(t)$, for any passive factor, $y$, its state at time $t+1$, $y(t+1)$, is uniquely determined depending on the set $X_y(t), X_y(t) \subseteq A_k(t)$, according to influences aggregation function of the factor $y$, $I_y$, of type $L^w_{D_0}$.

In fact according to this conception a behavior of a formal CM of type $K_0$ can be represented by a model of nonautonomous states transformer:

$$F: V_X \times V_Y \rightarrow V_Y,$$

where $V_x$ is the set of every possible sets of values of CI-factors, $V_y$ is the set of every possible sets of values of passive factors. According to this conception to determine the state of some passive factor $y(t+m)$, on the basis of the prehistory from time $t = 0$, the knowledge of the initial state, $Y(0)$, and the sequence of values of CI-factors, $(X_k(1), \ldots, X_k(m-1))$, are required.

However there are no reasons to suppose that this sufficient structure and volume of data are always necessary. In particular, following the conception $P^1_{K_0}$, it is not difficult to prove that given sufficiently large $m$ (for maps of type $K_0$) the knowledge of (intermediate) states is not required.

Another common conception, $P^0_{K_0}$, interprets values of all factors as states ignoring the specificity of CI-factors.

This conception can be defined as follows.

$P^0_{K_0}$: For any factor of a map, $y$, its state at time $t+1$, $y(t+1)$, is uniquely determined depending on the set of values of factors $X_y(t), X_y(t) \subseteq A_k(t)$, according to influences aggregation function of the factor $y$, $I_y$, of type $L^w_{D_0}$.

This conception interprets a CM as an autonomous states transformer of type:

$$F^*: V_A \rightarrow V_A,$$

where $V_A$ is the set of every possible sets of values of all factors of a map.

For maps of type $K_0$ as for other types of maps having CI-factor this conception is wrong if one consider CI-factors as independent variables. Really if the influences aggregation function of some CI-factor $x, l_x$, is considered as a function of type $L^w_0$, with an empty set of arguments, the value of factor $x$ can be only independent of time constant (of type R). It is not difficult to prove that as a consequence there should not be any the dynamics of maps of type $K_0$.

In a typical case when a CM of type $K_0$ is represented as a matrix in which each raw represents the vector of coefficients of a corresponding linear influences aggregation function it is easy to see that for any CI-factor: $l_x = 0$.

The revealed failure of conception of CM behavior $L^0_{K_0}$ in a number of theoretical models is "corrected" in corresponding computational models. At the same time the dynamics of CI-factors is modeled as "impulse", i.e. the value of each CI-factor differs from zero only at the initial moment of modeling. However this correction is achieved thanks to the contradiction to the model of external behavior when solving problems of an analysis of maps dynamics.

Thus the accepted model of semantics of dynamic CMs of type $K_0$ can be represented as:

$$M_{K_0} = \{M^0, M^w\} = \{M_{t0}, M_{t1} = \{S_{t10}, B_{t1} = (L^w_{t0}, P^1_{t0})\}, \}$$

$$M_{t0} = \{S_{t10}, B_{t1} = (C_{t0}, P_{t0})\}, M^0 = \{T, P^0_{t0}\}$$

### 2.6 Family of functional cognitive maps

Systemizing the diversity of formal CMs in terms of metamodel $M$ adapted to the system of concepts and ideas of formal CMs enabled to single out open family $F$ of formal CM types which are named as functional maps.

Different types of maps in $F$ can be considered as parametrical modifications of base type of maps of this family, $K_0$: the deviation from $K_0$ is expressed by change of one or several parameters of the base model (1). Modifications can be of the form of generalizations of parameters or their alternatives (if generalization is impossible or unreasonable without loss of significant content). Thus one can speak about general parametrical model of semantics of functional CMs.

For example, in addition to dynamic CMs, which constitute the overwhelming majority in $F$, there are static maps in $F$. These maps constitute an alternative type, $K_S$, with respect to $K_0$. With respect to this type of CMs (which is applied to various quality models) there is no sense to speak about delay in influence between factors. This leads to changes in almost all parameters of the model (1), except the structural model $T$. On the contrary, generalizing transition from model $T$ to the model of any loop-free graph, $G$, in (1), without changing of other parameters, generates a new type of CMs.
To date the overwhelming majority of formal CM types belong to family F. The basic property of functional maps is that factors are associated with variables, and dependent factors are associated with the functions representing their dependence on cause factors.

It is worth noting that conceptions of behavior of factors and CM as a whole are rarely strictly formulated in modern traditions of the description of theoretical models of CMs. However the revelation and analysis of typical conceptions in the diversity of modern models of functional CMs (implicit and/or indistinct) have shown that typical concepts of behavior of a CM as a whole (the conception of «wave distribution» and the conception of states transformer), as well as an ignoring of models of behavior of CI-factors, serve as risk factors for validity of results of practical application of theoretical models of CMs. However this analysis is beyond the scope of this article.

3. CLASSIFICATION OF REPRESENTATIVE SET OF TYPES OF FORMAL COGNITIVE MAPS

To systematize the diversity of formal CM types the representative set \( R \) consisting of 21 types of maps from 35 types of formal CMs known to authors (excepting very similar types) is segregated.

As a result of the analysis of set \( R \) it is established that the majority of types of maps from \( R \) (19 of 21) belong to the open family \( F \) described earlier, i.e. to functional maps. It means that they can be presented as parametrical modifications of the base model (1) of \( K_0 \).

There are two types of formal CMs besides the functional ones that have been segregated in \( R \) (see Fig. 2): maps of analysis of influences of one factor on another (type \( I \)) (Kosko (1986)) and logical maps (type \( L \)) (Axelrod (1976)). In these types of maps there are no variables associated with factors, and it seems that the dependence of the effect on cause factors in a bundle can not be presented by a function.

The majority of functional CM types from \( R \) have two “degrees of freedom” in expressing the strength of influence of factors: via the values of cause factors and via the weight of their influences. As opposed to them, in the maps «in the spirit of B. Kosko» (Kosko (1986)) of analysis of influences the strength of influence of one factor on another can be expressed only via the weight of influence of a cause factor. In logical CMs there are no any specified “degrees of freedom” for expression of the strength of influence of factors; the influence of one factor on another is characterized only by a sign of the influence.

For preliminary classification of the types of functional CMs from \( R \) it has appeared sufficient to consider only some parameters of the model (1), namely, the basis consisting of a set of models of factor types.

In the majority of types from \( R \) the scales of factor values (and scales of influences) are double consisting of the internal scale, \( V^{\text{in}}_x \), for factor values, the external scale, \( V^{\text{out}}_x \), for their interpretation and mappings in both directions between the scales. At the same time, as a rule, the external scale is a finite ordinal verbal (qualitative) one, and the internal scale is numerical. Double linguistic scales in which every verbal value from the external scale is associated with a fuzzy set from the internal scale are also applied. As a rule, for the values of all factors of a map the same scale (symmetric with respect to zero or non-negative) is used. Practically useful extension is proposed in (Kulinich (2010)) where there is a possibility of different scales for different factors.

With respect to the influences aggregation function of the passive factor it is possible to segregate the following classes of functional CMs from \( R \): (1) linear (Roberts (1976), Kulba et al. (2004)); (2) pseudo-linear (Kornoushenko and Maximov (2001)); (3) pseudo-fuzzy (Stylios et al. (2008), Taber (1994), Zhang (2003), Tsadiras (2003), Kosko and Dickerson (1994), Papakostas et al. (2008), Aguilar (2004), Stylios and Groumpos (1999)); (4) truly fuzzy (Carvalho and Tom (2000), Fedulov (2005), Markovskii (2008))

Influences aggregation function in pseudo-linear maps is a modification of a linear function which is defined on finite numerical scales of factor values. Despite the fact that pseudo-fuzzy maps are called by authors as "fuzzy" influences aggregation function in these maps is also defined on finite numerical scales. Truly fuzzy CMs in contrast with pseudo-fuzzy CMs really use fuzzy mathematics.

In most types of functional maps from \( R \) there is only one degree of freedom in the situation control: through external influences on factors. The exception is the so-called «advanced cognitive maps» (Vasiliev and Ilyasov (2009)), in which there is a possibility of control to weights of influences between factors.

Rather unexpected was the adjacency of the pseudo-linear CMs, which are modifications of the maps proposed by Roberts (Roberts (1976)), and pseudo-fuzzy CMs, which are modifications of the maps of Kosko (Kosko (1988)).

Influences aggregation function of a passive factor in pseudo-linear and pseudo-fuzzy, as well as most types of functional CMs from \( R \) (12 of 19) can be represented as a superposition of linear and threshold function, \( f \), which is a special case of the function of type:

\[
V^{t}{y}_{t/\Delta t} = f \left( k_1 \sum_{x \in X_y} w_{x \rightarrow y} V^{t-1}_{x/\Delta t} + k_2 V^{t-1}_{y/\Delta t} \right),
\]

where \( V^{t}{y}_{t/\Delta t} \) is the value (or increment) of the factor \( y \) at time \( t \); \( V^{t-1}_{x/\Delta t} \) is the value (or increment) of the factor \( x \) at time \( t \); \( X_y \) is the set of cause factor in a bundle, \( k_2 \geq 0 \) defines the presence and the weight of influence of the effect factor in a bundle on itself, \( w_{x \rightarrow y} \) is the weight of influence of the factor \( x \) on the factor \( y \).
Fig. 2. Preliminary classification of formal CMs

The analysis shows that the type of influences aggregation function is less significant with respect to the behavior of the dependent factor and the dynamics of the map as a whole than whether or not there is a loop on the dependent factor and the way of representing the cause-and-effect dependency in the bundle: the cause factors values influence the effect factor value, or the cause factors increments influence the effect factor increment.

4. CONCLUSIONS

In the article the unified approach to the formal description and systematization of the diversity of formal CM types in terms of the general parametrical model of functional CM semantics (the metamodel) is proposed. The approach has enabled to consider the overwhelming majority of known types of CMs as the kind of functional scheme known in the control theory.

Formalization of known theoretical models in terms of the metamodel enabled a systematic way to identify a number of typical incorrectnesses in these models. First of all, the problem is in the incorrectness of the general model of behavior of a map which considers the map as a state transformer (autonomous on default) in the case of maps interacting with external environment and subjected to the control. Also, among incorrectnesses revealed in this way there are those found by Carvalho (Carvalho, 2010) and some others remaining beyond the scope of the article. Such incorrectnesses serve as risk factors for validity of results of practical application of theoretical models of CMs.

Thus it is possible to speak about verification of theoretical models of formal CMs by embedding an initial model description into the proposed general model of semantics in order to reveal incompleteness, ambiguity, contradictions.

The following problems are being solved by authors and their colleagues on the basis of the proposed representation of the different types of CMs in the form of unified family of functional CMs:

- comparative analysis of the capabilities of different types of CMs in solving practical problems;
- development of the general theory of formal CMs, in particular, functional-algebraic approach to the analysis of situation under research;
- development of formally correct typical statements of CM external behavior analysis problems for various types of CMs;
- analysis and verification of representative structures of cause-and-effect influences in typical socio-economic situations.

It is important to emphasize that formalization of mathematical semantics of functional CMs in terms of the proposed metamodel does not exhaust their semantics: conceptual semantics of CMs is not taken into account. Without it the problem of validity of application of formal CMs to modeling of ill-structured situations is assumed not solvable. Researches on the verification of applied CMs give the evidence of this assumption. (Abramova and Kovriga (2009), Abramova and Kovriga (2011)). Conceptual semantics of CMs is not studied yet and is a subject of further research.

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