Quickest Detection of State-Transition in Point Processes:
Application to Neuronal Activity

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Abstract: Quickest detection is the problem of detecting a change in the probability distribution of a sequence of random observations with as little delay as possible and with low probability of false alarm. To date, algorithms for quickest detection exist mainly for cases where the random observations are independent, and linear or exponential cost functions of the delay are used. We propose a dynamic programming-based algorithm to solve the quickest detection problem when dependencies exist among the observations, and for any nondecreasing cost function of the detection delay. We implement the algorithm for a Bayesian formulation (i.e., the change time \( \hat{T} \) in the probability distribution of the observations is a random variable with a priori fixed geometric distribution) when the observations distribute according to two distinct point processes. We apply the algorithm to spiking activity observations from neurons recorded in the subthalamic nucleus of Parkinson’s disease patients during the execution of a motor task. The algorithm exploits the point-process characterization of the spike trains before and during the movement (two states), and optimally detects the state transition at movement onset. Performances significantly (i.e., with a p-value \( p<0.05 \)) improve over a chance level predictor.

1. INTRODUCTION

Quickest detection is the problem of detecting a change in the probability distribution of a sequence of random observations with as little delay as possible and with low probability of false alarm. The general problem considers a sequence \( z_0, z_1, ..., z_M \) of random observations and a change time \( \hat{T} \) such that, given \( \tilde{T} \), \( z_0, z_1, ..., z_{\tilde{T}-1} \) are drawn from one probability distribution and \( z_{\tilde{T}}, z_{\til{T+1}}, ..., z_M \) are drawn from another distribution (Poor, 1998; Poor and Hadjiliadis, 2008). The goal is to construct an optimal feedback policy that detects the change as quickly as possibly avoiding a false alarm. This problem arises in a variety of applications, e.g., image processing (Trivedi and Chandramouli, 2005), epileptic seizure prediction (Mormann et al., 2007), neuronal coding (Yu, 2007), manufactory process monitoring (Wetherhill and Brown, 1991), and finance (Andreou and Ghysels, 2004).

Several techniques and strategies have been proposed for quickest detection (e.g., Basseville and Nikiforov, 1993; Brodsky and Darkhovsky, 1992; Crow and Schwartz, 1996; Lorden, 1971; Pelkowitz, 1987; Poor, 1998; Raghavan and Veeravalli, 2010; Teneketzis and Varaiya, 1984). Each of them involves the optimization of a trade-off between two performance measures, i.e., the delay \( (T_e - \hat{T}) \) between the time \( \hat{T} \) when a change occurs and the time \( T_e \) when the change is detected, and the frequency of false alarms (i.e., events of the type \( \{T_e < \hat{T}\} \)). In several studies (e.g., Lorden, 1971; Raghavan and Veeravalli, 2010; Teneketzis and Varaiya, 1984), the detection delay is penalized via a linear function of delay. In a few other studies (e.g., Crow and Schwartz, 1996; Pelkowitz, 1987; Poor, 1998), instead, nonlinear delay penalties have been considered. In (Poor, 1998), two formulations of the quickest detection problem with an exponential delay cost were considered. The first is a minimax formulation, wherein the delay penalty is a worst-case measure (Lorden, 1971), and a bound on the allowable mean time between consecutive false alarms is introduced. The worst-case delay is taken over all the possible realizations of the observations and over all the possible values of the change time \( \hat{T} \). The second formulation is Bayesian (Shiryayev, 1963), i.e., the change time \( \hat{T} \) has an a priori known distribution, and the opposing performance indices included in the cost function are the expected detection delay and the false-alarm probability. For both formulations, an optimal detection procedure is developed in (Poor, 1998), but independent observations are assumed.

In this paper, we solve the Bayesian quickest detection problem (Poor, 1998) with dependent observations and with an arbitrary nondecreasing cost function of the delay. To the purpose, we set up the problem as an optimal stopping problem and solve it using dynamic programming (DP) (Bertsekas, 2005). We then compute the solution in the case that the random observations distribute according two different point processes (Daley and Vere-Jones, 2003; Snyder and Miller, 1991), and the transition time \( \hat{T} \) from one distribution to one another is a random variable with a priori known geometric distribution.

We apply the optimal policy to the spiking activity of 27 neurons recorded in the subthalamic nucleus (STN) from seven Parkinson’s disease (PD) patients before and during the execution of a movement. Several studies (e.g., Bergman et
al., 1994; Gale et al., 2009; Williams et al., 2005) have shown that STN neurons in PD conditions modulate their activity during the execution of voluntary motor tasks. Further, (Sarma et al., 2010) showed that, for each neuron in this data set, a point process model can be estimated from the spike trains and that the parameters of this model significantly change before vs. during the execution of a movement. For each patient and recorded neuron, the occurrence of a movement is random. Our detection algorithm runs on the acquired spike trains and optimally detects the changes in spike distribution from no movement to movement (state transition).

2. METHODS

2.1 HMM and Information State Variable

We formulate the quickest detection problem as a $M$-stage optimal stopping problem (i.e., stopping is mandatory at or before a given stage $M$) with imperfect state information and solve it via the DP algorithm (Bertsekas, 2005).

We consider a generic bi-state system $\mathcal{S}$ and model its evolution as a Hidden Markov Model (HMM) (Elliott et al., 1995), with the state being $x_k \in \{0,1\}$ at each stage $0 \leq k \leq M$ (Fig. 1). We assume that $x_0 = 0$, and that $\mathcal{S}$ switches from state 0 to state 1 at some stage $0 < T < M$, where $\bar{T}$ is a geometric random variable with distribution known a priori.

In a HMM the state is inaccessible or “hidden”. However, output observations $z_k, k = 0,1,2,...$ are available and depend probabilistically on the states. One can think of $z_k$ as a “noisy” observation of $x_k$. We assume that for any $k$, $z_k$ belongs to a countable set $\mathcal{Z}_k$, and that the probability mass function $q_x(z|x|H) \triangleq P(z_k = z|x_k = x, H_k = H)$ is known for $x \in \{0,1\}, z \in \mathcal{Z}_k$ and $H \in \mathcal{H} \times Z_0 \times Z_1 \times \ldots \times Z_{k-1}$, where $\mathcal{H}$ is the past history of observations at stage $k$, i.e., $H_k = \{z_0,...,z_{k-1}\}$. See Fig. 1. Note that, differently from (Basseville and Nikiforov, 1993) and (Poor, 1998), we allow observations $z_0,...,z_k$ to be dependent and we account for such dependency in the probability functions $q_x(z|x|H)$.

Because the state $x_k$ is inaccessible we introduce the new variable $\pi_k \triangleq P(x_k = 1|z_0,z_1,...,z_k) = P(\bar{T} \leq k|z_0,z_1,...,z_k)$, i.e., the probability of being in state 1 at stage $k$ conditioned to the observations up to and including stage $k$. $\pi_k$ is called the “information state variable” (Poor, 1998) and can be estimated at each stage $k$ given the observations $z_0,...,z_k$, the functions $q_x(z|x|H)$ and the parameter $\rho$. In particular, the evolution equation of $\pi_k$ can be obtained by induction for any $0 \leq k \leq M$ (Basseville and Nikiforov, 1993; Poor, 1998):

$$\pi_0 = P(x_0 = 1|z_0) = 0,$$

$$\pi_{k+1} = \frac{L(z_{k+1}|\pi_k + (1 - \pi_k)\rho)}{(1 - \pi_k)(1 - \rho) + L(z_{k+1}|\pi_k + (1 - \pi_k)\rho)}$$

where we have introduced the likelihood ratio $L(z_k) \triangleq q_1(z_k|H_k)/q_0(z_k|H_0)$, and we have applied the Bayes’ rule (Papoulis and Pillai, 2002).

2.2 Problem Formulation

We aim at detecting the actual change time $\bar{T}$ with the lowest possible delay and probability of false alarms based on the sequential observations $z_k, k = 0,1,2,...$. We formulate this problem as a stopping problem, i.e., at each stage $k < M$ we have to decide whether to stop our searching and say that the change point $\bar{T}$ occurred ($u_k = \bar{T}$) or keep going ($u_k = \hat{S}$). To this purpose, we introduce a cost per stage that penalizes depending on the decision we take:

$$g_k(x_k, u_k) = \left\{ \begin{array}{cl} E_f[\xi(k - \bar{T})] & x_k = 1 \land u_k = \bar{T} \\ 1 & x_k = 0 \land u_k = \hat{S} \\ 0 & \text{otherwise} \end{array} \right.$$  (2)

with $0 \leq k < M$, and the terminal cost:

$$g_M(x_M) = \left\{ \begin{array}{cl} 1 & x_M = 0 \land u_{M-1} = \hat{S} \\ 0 & \text{otherwise} \end{array} \right.$$  (2)

The cost $g_k(x_k)$, with $0 \leq k < M$, accounts for false alarms (i.e., $x_k = 0 \land u_k = \hat{S}$) and delays in transition detection (i.e., $x_k = 1 \land u_k = \bar{T}$), and it is 0 if the decision of stopping is taken before stage $k$. The terminal cost $g_M(x_M)$, instead, is chosen equal to the stopping cost. $\phi(\epsilon)$ is any function of the input $\epsilon$, and the expectation $E_f[\xi(k - \bar{T})]$ is taken over $\bar{T}$.

Because the state $x_k$ is inaccessible, we average the cost per stage over all the possible values of $x_k$, i.e., we introduce the new function $G_k(\pi_k, u_k) \triangleq E_f[z_k|u_k]g_k(x_k, u_k)$. It can be shown that:

$$G_k(\pi_k, u_k) = \left\{ \begin{array}{cl} E_f[\xi(k - \bar{T})]\pi_k & u_k = \hat{S} \\ 1 - \pi_k & u_k = \bar{T} \\ 0 & \text{otherwise} \end{array} \right.$$  (3)

with $0 \leq k < M$, and

$$G_M(\pi_M) = \left\{ \begin{array}{cl} 1 - \pi_M & u_{M-1} = \hat{S} \\ 0 & \text{otherwise} \end{array} \right.$$  (3)

Now, the stopping problem consists in choosing the stage $T_k = k'$, with $0 < k' < M$, such that the decision policy ($u_0 = \hat{S},...,u_{k'-1} = \hat{S}, u_{k'} = \bar{T}$) minimizes the overall cost

$$J_0 = E_f[G_M(\pi_M) + \sum_{k=0}^{M-1} G_k(\pi_k, u_k)]$$

(4)

where the expectation $E_f$ is taken over all the observations $Z = \{z_0,...,z_M\}$ (Bertsekas, 2005). Note that (4) is the expected value of the sum of all the stage costs, and can be shown to be equivalent to

$$J_0 = E_f[\bar{T} - \bar{T}^*] + P(T_s < \bar{T})$$

(4)
2.3 Optimal Solution

Minimization of the cost (4) is achieved by using the DP algorithm (Bertsekas, 2005):

\[
J_M(\pi_M) = G_M(\pi_M),
\]

\[
J_k(\pi_k) = \min \left \{ G_k(\pi_k, S), G_k(\pi_k, S) + E_{x_{k+1}} \left \{ J_{k+1}(\pi_{k+1}) | \pi_k \right \} \right \}. \tag{6}
\]

It can be proved that, by using (3), the optimal solution satisfies the threshold policy:

\[
k^* = \min \left \{ 0 < k < M \left | \pi_k > \frac{1 - A_k(\pi_k)}{1 + E_{x}(\varphi(k - \bar{T}))} \right \} \tag{7}
\]

where \( \pi_k \) is given by (1) and \( A_k(\pi_k) \) is computed by using the backward DP recursion (Bertsekas, 2005).

In order to guarantee that a finite-time solution exists (i.e., the optimal stage \( k^* \) exists) we require that the cost \( J_k \) is a monotonic function of \( k \), i.e., \( J_k(\cdot) \leq J_{k+1}(\cdot) \), for all \( k \) (Bertsekas, 2005). Given the evolution (1) and assuming that the change time has geometric distribution, this condition is satisfied by any nondecreasing function \( \varphi \) of the delay. It can also be shown that, under the aforementioned assumptions, the function \( A_k \) is monotonically increasing with \( k \), i.e., \( A_k(\epsilon) \leq A_{k+1}(\epsilon) \) for any \( 0 \leq \epsilon \leq 1 \), and from (1), (3), (6) it follows that \( A_k(\epsilon) \) is convex for every \( k \) with \( A_k(1) = 0 \).

2.4 Point Processes

We apply the general framework of Section 2.1-2.3 to the case where \( S \) is a spiking neuron and the observations \( z_k \) represent the number of spikes generated in consecutive short time windows. We use windows small enough such that \( z_k \) can be either 0 or 1 for every \( k \), and we assume that the sequence \( z_0, z_1, z_2, \ldots \) follows a point process model in each hidden state.

A point process is a 0-1 stochastic process defined in continuous time and characterized entirely by the conditional intensity function (CIF) (Daley and Vere-Jones, 2003; Snyder and Miller, 1991). Consider the time interval \((0, T]\) as the continuum, let \( t_1, t_2, \ldots, t_n \) denote the times that 1 occurs, with \( t_1 < t_2 < \cdots < t_n \leq T \), and let \( N(t) \) be the number of 1s occurring in \((0, t]\). Then, the CIF is defined as

\[
\lambda(t | H_t) = \lim_{\Delta t \to 0} \frac{P(N(t + \Delta t) - N(t) = 1 | H_t)}{\Delta t},
\]

where \( H_t \) is the history of the sample path up to \( t \in (0, T] \). \( \lambda(t | H_t) \) is a generalized history-dependent rate function and \( \lambda(t | H_t) \Delta t \) approximately gives the probability that a 1 occurs at time \( t \), provided that \( \Delta t \) is small (Snyder and Miller, 1991).

As in (Brown et al., 2003; Snyder and Miller, 1991), we bin the time interval \((0, T]\) with a fixed \( \Delta t \) and denote with \( z_k \) the number of 1s observed in the interval \((k\Delta t, (k + 1)\Delta t]\) (stage \( k \)), for \( k \geq 0 \). \( \Delta t \) is chosen such that \( z_k \in [0,1] \) for every \( k \geq 0 \). Then, we approximate the history of the path up to time \( k\Delta t \) with \( H_k = \{x_0, \ldots, x_{k-1}\} \) and set \( P(x_k = 1 | H_k) \equiv \lambda(k\Delta t | H_k) \Delta t \) (Brown et al., 2003).

<table>
<thead>
<tr>
<th>Subject</th>
<th># of Trials</th>
<th># of Neurons</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD-1</td>
<td>152</td>
<td>4</td>
</tr>
<tr>
<td>PD-2</td>
<td>52</td>
<td>2</td>
</tr>
<tr>
<td>PD-3</td>
<td>164</td>
<td>5</td>
</tr>
<tr>
<td>PD-4</td>
<td>45</td>
<td>2</td>
</tr>
<tr>
<td>PD-5</td>
<td>108</td>
<td>4</td>
</tr>
<tr>
<td>PD-6</td>
<td>220</td>
<td>7</td>
</tr>
<tr>
<td>PD-7</td>
<td>40</td>
<td>3</td>
</tr>
</tbody>
</table>

For the quickest detection problem formulated in Section 2.1-2.3, we assume that the observations distribute according to different point processes in different states, and denote with \( \lambda_x(\cdot) \) the CIF when the system is in state \( x \), with \( x \in \{0, 1\} \). Then, for any \( 0 \leq k < M \), we approximate

\[
q_x(z | H_k) = \begin{cases} 
\lambda_x(k\Delta t | H_k) \Delta t & z = 1 \\
1 - \lambda_x(k\Delta t | H_k) \Delta t & z = 0 
\end{cases}
\]

2.5 Simulation Example

We preliminarily tested the approach developed in Section 2.1-2.4 to simulated data. 5000 spike trains (i.e., trains of 0s and 1s), each spanning \( M = 1000 \) bins, were generated. For each train, a bin \( 0 < \bar{T} < M \) was randomly picked and the spikes occurring before \( \bar{T} \) (state 0) distribute differently from the spikes occurring after \( \bar{T} \) (state 1). In particular, spikes in state 0 and 1 follow a Poisson process (which is a special point process) with constant CIF function \( \lambda_0 = 0.1 \) and \( \lambda_1 = 0.02 \), respectively. The change time \( \bar{T} \), instead, follows a geometric distribution with \( \rho = 0.001 \). We implemented the optimal policy (7) with

\[
\varphi(k - \bar{T}) = \begin{cases} 
\frac{c \cdot [2(k - \bar{T}) - 1]}{ \bar{T} \leq k} & 0 \\
0 & \bar{T} > k 
\end{cases}
\]

where \( c > 0 \) is a constant to be chosen and \( \varphi(k - \bar{T}) \) is such that the overall cost \( J_0 \) in (4) weights both the squared detection delay and the occurrence of false alarms (Poor, 1998).

2.6 Experimental Set Up

We applied the approach developed in Section 2.1-2.4 to spike trains acquired in vivo from PD patients. Seven patients undergoing deep brain stimulator placement for the treatment of PD were included in this study. Details about surgical procedure, data collection, motor task, and clinical evaluation of the subjects are in (Sarma et al., 2010).

Briefly, subjects were awake in the operative room with no anti-Parkinsonian medications or sedatives while single unit recordings were acquired from the STN. Subjects viewed a computer monitor and moved a joystick with the hand contralateral to the recorded site. The joystick controlled a cursor on the monitor and was mounted such that movements of the grasping hand were in a horizontal orientation with the elbow flexed at approximately 45°. The task consisted in fixing the monitor till a target point was displayed on it (target onset) and, then, move the joystick to reach the target with the cursor. After the target onset, the subject was
required to wait for a visual cue before he could initiate the movement. Each subject repeated the task several times while multiple STN neurons were simultaneously recorded. For each trial, the position of the target on the monitor and the delay (ranging from 500 to 1000 ms) between the task onset, the target onset, and the display of the visual cue, were randomly chosen. The onset of each movement was determined by examining the joystick voltages acquired during the task.

The STN neuronal activity was recorded through tungsten microelectrodes, band-pass filtered (0.3–6 kHz) and sampled at 20 kHz. Spikes were sorted offline using a standardized template-matching algorithm (Cambridge Electronics Design, Cambridge, England). Table 1 shows the number of neurons and trials per patient included in this study.

2.7 Parameter Estimation for STN Neurons

For each neuron in the experimental data set, both before and during movement, the CIF was defined to be a function of the neuron’s own spiking history in the previous 150 ms (Sarma et al., 2010). Denoted with $x = 0$ and $x = 1$ the state before and during movement respectively, we adopted the following multiplicative structure (Kass and Ventura, 2001; Sarma et al., 2010):

$$
\lambda_x(k \Delta t | H_k, \theta_x) = e^{\alpha_x} \cdot \lambda_{H}^D(k \Delta t | H_k, \theta_x)
$$

where $x \in \{0, 1\}$, $e^{\alpha_x}$ (in spikes/s) accounts for the history-independent, direction-tuned activity, $\lambda_{H}^D(\cdot)$ describes the effect of the spiking history on the neural response, and $\theta_x$ is a vector of parameters to be fitted on the data. We set $\Delta t = 1$ ms and defined the generalized linear model (McCullagh and Nelder, 1990):

$$
\log \lambda_{H}^D(k \Delta t | H_k, \theta_x) = \sum_{i=1}^{10} \beta_{x,i} Y(k-i:k-i+1) + \sum_{j=1}^{14} y_{x,j} Y(k-10(j+1):k-10j)
$$

where $Y(a:b) \equiv N(b \Delta t) - N(a \Delta t)$ is the number of spikes in $(a \Delta t, b \Delta t]$ and $\Theta_x = [\alpha_x, \{\beta_{x,i}\}_{i=1}^{10}, \{y_{x,j}\}_{j=1}^{14}]^{T}$. Motivations for this model are in (Sarma et al., 2010).

For each neuron, a maximum-likelihood (ML) estimation of the parameter vectors $\theta_1$ and $\theta_0$ was computed by using 500 ms-long windows of recorded neuronal activity centered at the movement onset (250 ms before movement onset for $\theta_0$, the rest for $\theta_1$). For each neuron and state $x \in \{0, 1\}$, a different value of $\alpha_x$ was estimated for each available movement direction (4 options), while $[\beta_{x,i}]_{i=1}^{10}$ and $[y_{x,j}]_{j=1}^{14}$ were independent from the movement direction. The probability mass functions $q_x(z|H), x \in \{0, 1\}$, were given by (8) with $\lambda_x$ given by (9).

The movement onset times were forced to follow a geometric distribution. In particular, since each spike train includes 250 ms before and 250 ms after the movement onset, we run a geometric process with $\rho = 0.0125$ for each train, picked a number $0 < T < 250$ from this process, and set the first observation $z_0$ of the train such that the movement onset occurs at $T$ (Fig. 2). All the computations were performed in MATLAB® (Mathworks, Natick, MA).

3. RESULTS

3.1 Simulated Spike Trains

Fig. 3 summarizes results achieved by applying the DP-based QD policy (7) (DP-QD) to the simulated spike trains. Results were compared with the chance level (CL) monitor $T_M^D = 1/\rho$ with $\rho = 0.001$ as reported in Section 2.5.

DP-QD significantly (t-test, $p<0.05$) reduced the distance $|T_M - T|$ between the estimated change time $T_M$ and the actual change time $T$ (Fig. 3a). This was consistently achieved both in case of delay (i.e., events of type $T_M > T$), Fig. 3b) and anticipation (i.e., events of type $T_M < T$), Fig. 3c). We note that, even though the anticipation was smaller with the DP-QD than with the CL predictor, the t-test was not passed. This lack of significance, however, can be explained by noting that the DP-QD significantly reduced the incidence of the anticipation events (777 out of 5000 spike trains for DP-QD vs. 1650 out of 5000 for the CL predictor), which results because the DP-QD explicitly minimizes the probability of false alarms.
3.2 Spike Trains from STN Neurons

The DP-QD policy (7) was applied on the spike trains from the STN neurons. The policy was implemented with a cost function \( \varphi \) as in Section 2.5. In addition, for computational efficiency, we computed offline the sequence \( A_k(\cdot) \) for each stage \( k < M \) (\( M \) varied from 277 to 500) by using the backward DP recursion (Bertsekas, 2005). For this computation, we grid the probability interval \([0,1]\) and approximated the CIF in (9) with \( \lambda_k(k\Delta\tau|H_k,\Theta_\tau) \equiv e^{\alpha_k}. \) Results are reported in Fig. 4.

With this approximation, the algorithm detected a change time in every spike train (i.e., \( k^* < M \) always) and guaranteed a low distance between the actual and estimated change time (Fig 4a). Anticipation events occurred in 4.6 out of 42 trials per neuron on average (min: 1 out of 76 trials; max: 7 out of 18 trials) and, in each of these events, the temporal distance between the estimated and actual change time ranged from 1 ms to 166 ms (18.3 ± 24.43 ms, mean ± standard deviation). In case of delayed detection (i.e., \( T_s > T \)), instead, the delay was 0.6 ± 0.3 ms (range: 0 ms to 25 ms). Average values across the neuronal population are reported in Fig. 4b,c (black bars).

Performances significantly improve over the CL predictor (t-test, \( p<0.05 \)): the average absolute distance (Fig. 4a), delay (Fig. 4b) and anticipation (Fig. 4c) decrease with the DP-QD policy and such a reduction is consistent across the whole neuronal population (average distance and delay decreases in 27 out of 27 neurons, anticipation decreases in 25 out of 27 neurons). DP-QD also significantly decreased the incidence of the anticipation events, which were 15.4 out of 42 trials per neuron on average for the CL predictor (min: 2 out of 12 trials; max: 12 out of 23 trials).

4. CONCLUSIONS

We proposed a DP-based algorithm to solve the Bayesian quickest detection problem when dependencies exist among the observations and any nondecreasing cost function of the detection delay is introduced. We implemented this algorithm when the observations in each state were fitted by a point process model, and we tested the algorithm on the spike trains of 27 STN neurons recorded in PD patients performing a motor task. Results were compared with a chance level predictor and showed significantly lower distance between actual and estimated change times.

We are currently working to further improve our detection algorithm by (i) analyzing the impact of the cost function on the detection performances, (ii) investigating different stochastic distributions of the change time, and (iii) studying the impact of the spiking history. We are also investigating more efficient and refined approaches for the computation offline of the sequence \( A_k(\cdot) \) for each stage \( k < M \).

REFERENCES


