Efficient robust output feedback MPC

Mark Cannon2 Shuang Li1 Qifeng Cheng Basil Kouvaritakis

Department of Engineering Science, University of Oxford

Abstract: This paper extends an efficient robust Model Predictive Control (MPC) methodology based on offline optimization of prediction dynamics in two respects. Firstly the case of output feedback via an observer is considered. Secondly, a method is proposed for imposing a bound on worst case performance using dynamic feedback which allows a nominal optimal feedback law to be implemented when this is feasible. The method is applicable to the case of uncertain systems which are not robustly stabilized by nominal LQ-optimal control. An efficient online optimization with feasibility and stability guarantees is proposed.

Keywords: robust predictive control, linear systems, constraints, output feedback.

1. INTRODUCTION

Model predictive control (MPC) can handle input/state constraints while providing near-optimal performance and guaranteeing stability. However its heavy computational burden reduces its usefulness in practice, particularly in the presence of model uncertainty. Kouvaritakis et al. [2000] achieved a significant reduction in online computational load by using an autonomous augmented prediction system to generate predictions, the state of which consists of a vector of the degrees of freedom (dof) in predictions appended to the plant state. With this reformulation, an ellipsoidal inner approximation of the robust invariant set for the plant state and dof in predictions can be computed offline. The online minimization of a quadratic nominal predicted performance index can then be conveniently performed subject to robust constraint satisfaction using a univariate Newton-Raphson iteration.

Using an approximate ellipsoidal constraint set can introduce suboptimality, and to minimize this the methodology was extended [Kouvaritakis et al., 2002; Li et al., 2010] by allowing the optimization to search outside the ellipsoidal constraint on dof subject to the satisfaction of system constraints at current time. In addition, Cannon and Kouvaritakis [2005] showed that the maximal ellipsoidal region of attraction for the plant state can be achieved without compromising performance by optimizing offline a dynamic feedback law incorporating the unconstrained nominal LQ-optimal feedback law. However these algorithms rely on the availability of plant state measurements. This paper extends the convex formulation to incorporate a linear state estimator, and uses robust constraint satisfaction given a polyhedral description of model uncertainty and knowledge of an ellipsoidal bound on the estimation error.

Furthermore, the approach of [Kouvaritakis et al., 2000, 2002; Cannon and Kouvaritakis, 2005] was only applicable to systems that are robustly stabilized by nominal LQ-optimal control. Moreover it could provide arbitrarily poor worst case performance. A number of methods [Kouvaritakis et al., 1992; Yoon and Clarke, 1995; Stoica et al., 2008] have been proposed for robustifying nominal predictive control strategies through the offline design of pre-stabilizing feedback laws which preserve nominal optimality. However all of these considered only the unconstrained case. Although Cheng et al. [2009] treated constraints, the approach was based on a heuristic which minimized the norm of a sensitivity function rather than a performance bound. The current paper addresses this by imposing a bound on worst case performance directly and incorporating this into the efficient MPC framework based on ellipsoidal sets. The online optimization is formulated as a convex problem, closed loop feasibility and stability properties are discussed, and the paper concludes with an illustrative numerical example.

2. PROBLEM STATEMENT

Consider the system with linear model and constraints,

\[ x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C x_k \tag{1} \]

\[ F x_k + G u_k \leq g \tag{2} \]

where \( x, u \) and \( y \) are the plant state, control input and measured output variables respectively, with \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^n \) and \( g \in \mathbb{R}^n \). The parameters \( A_k, B_k \) are unknown, but belong to a known polyhedral set:

\[ (A_k, B_k) \in \Omega = \text{co}\{ (A^j, B^j), j = 1, 2, \ldots, m \}. \]

The state \( x_k \) is not directly measured at time \( k \), but must be estimated from measurements \( \{y_0, y_1, \ldots, y_k\} \).

Nominal predicted performance is evaluated by the cost:

\[ J^0_k = \sum_{i=0}^{\infty} (\|x_{k+i|k}^0\|^2_Q + \|u_{k+i|k}^0\|^2_R). \tag{3} \]

Here \( \|x\|^2_Q = x^T Q x \), \( Q = C^T C \), \( R > 0 \), and \( x_{k+i|k}^0, u_{k+i|k}^0 \) denote nominal predicted trajectories of (1) at time \( k \), with \( x_{k|k}^0 = x_k \), \( x_{k+i|k}^0 = A^0 x_{k+i|k}^0 + B^0 u_{k+i|k}^0 \), \( i = 0, 1, \ldots \) for given nominal plant parameters \((A^0, B^0)\), assumed to satisfy \((A^0, B^0) \in \Omega\). The control problem is to optimally regulate the nominal cost (3) while satisfying constraints (2) and ensuring a specified bound on the worst case cost, defined for any given predicted control strategy by

\[ \bar{J}_k = \max_{(A_{k+i}, B_{k+i}) \in \Omega, i \geq 0} \sum_{i=0}^{\infty} (\|x_{k+i|k}\|^2_Q + \|u_{k+i|k}\|^2_R). \tag{4} \]
We assume that \((A_0,C)\) is observable and \(\{A_j,B_j\}\) is quadratically stabilizable. Hence there exist \(P,H\) such that
\[
P = (A_j + B_j H)^T \Sigma_j, H > 0,\]
for any \(\gamma > 0\) ([indicates a block of a symmetric matrix]).

**Definition 1.** Let \(\Sigma_j = (A_j, B_j, \gamma, Q, R, F, G, g)\) denote the set of all feasible \((P, H)\) for (5)-(6).

**Remark 2.** Conditions (5)-(6) imply that the worst cost for (1)-(2) under linear feedback \(u_{k+i|k} = H x_{k+i|k}\) satisfies \(J_\gamma < \gamma\) whenever \(x_k^T P x_k \leq 1\).

### 3. PREVIOUS WORK: FULL STATE FEEDBACK

Consider the case that \(x_k\) is known at each sample \(k\). To ensure future feasibility, constraints (2) must be robustly satisfied along predicted trajectories, i.e. for all \(i \geq 0\),
\[
F x_{k+i|k} + g u_{k+i|k} \leq g,
\]
(7)

Applying these constraints to closed loop predicted trajectories imposes a heavy computational load, but this can be reduced (at the expense of some suboptimality) through the use of a semi-open loop strategy based on a fixed feedback law. Thus Kouvaritakis et al. [2000] applied a perturbation sequence \(\{(c_i, i, 0, \ldots, N-1)\}\) containing the dof in predictions to a fixed linear feedback law with gain \(K\), defined as the LQ-optimal for the nominal cost (3):
\[
u_{k+i|k} = \begin{cases}
K x_{k+i|k} + c_k, & i = 0, 1, \ldots, N-1 \\
K x_{k+i|k} & i = N, N+1, \ldots
\end{cases}
\]

The corresponding predicted trajectories are generated by an autonomous system, the state of which is the predicted plant state augmented by the perturbation sequence:
\[
x_{k+i|k} = \begin{bmatrix}
A_{k+i} + B_{k+i} K & B_{k+i} C_c & c_{k+i} \\
A_{k+i} & x_{k+i|k}
\end{bmatrix}
\]

\[
c_k = \begin{bmatrix}
c_k, 0 \\
c_k, N-1
\end{bmatrix},
\]

Cannon and Kouvaritakis [2005] generalized this formulation by: (i) considering \(C_c\) and \(A_c\) as parameters to be optimized offline; (ii) replacing \(A_c\) in (9) with time-varying \(A_{c,k+i}\) \(\in \{A_{j,i}, j = 1, \ldots, m\}\), thus allowing the predicted perturbation sequence to depend on future model uncertainty while retaining the possibility of implementing LQ-optimal feedback. With these modifications the predicted inputs are generated by the dynamic state feedback law
\[
u_{k+i|k} = K x_{k+i|k} + C_c c_{k+i|k}, \quad i = 0, 1, \ldots
\]
with \(c_{k+i+1|k} \in \{A_{j,i} c_{k+i|k}, j = 1, \ldots, m\}\), for all \(i \geq 0\).

A significant reduction in online computation can be achieved by approximating the feasible set for the state \(\zeta_k = [\Sigma_{ijk}^T T_{ijk}]^T\) of (9) offline. Thus for example Cannon and Kouvaritakis [2005] approximate the feasible set using an ellipsoid, \(E = \{\zeta : \Sigma_{ijk}^T P \zeta \leq 1\}\), which is determined by maximizing \(E\) offline subject to the constraints:
\[
E = \{\zeta : \Sigma_{ijk}^T P \zeta \leq 1\}
\]

\[
P = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\Sigma_{ijk} = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\Sigma_{ijk} = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
geq \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\Sigma_{ijk} = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\Sigma_{ijk} = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\Sigma_{ijk} = \begin{bmatrix}
A_i^T + B_i K & B_i C_c & \Sigma_{ijk}^T \\
A_i & 0 & 0 \\
0 & A_c & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
with $Q = \text{diag} \{ Q, 0 \}$ and $F = [ F \ 0 ]$.

The constraints of (16) are not convex in variables $P$, $\{ A_j \}$, $C_c$, but methods such as sequential semidefinite programming [Insland et al., 2005], and convex LMI reformulation [Cannon and Kouvaritakis, 2005] have been proposed for handling similar constraints. Here we extend the latter to the prediction system with state estimator in (14). Defining a transformation of variables similar to that proposed in Scherer et al. [1997] for output feedback design problems, let $U, V \in \mathbb{R}^{2n_x \times n_{dx}}$, $M_1 \in \mathbb{R}^{2n_x \times 2n_x}$, $M_2 \in \mathbb{R}^{n_x \times 2n_x}$ and symmetric $X, Y \in \mathbb{R}^{2n_x \times 2n_x}$ be defined by

$$
P = \begin{bmatrix} X^{-1} & X^{-1} U \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & V \end{bmatrix} \tag{17}
$$

and only if: (i) $(Y^{-1} - H) \in F_1(\{ A \bar{V}, B \bar{V} \}, \gamma, Q, R, F, G, g)$ for some $H$; (ii) $\gamma \{ A \bar{V} + B \bar{V} \}$ is quadratically stable.

Proof: A congruence transformation applied to (19) yields

$$
\gamma I \begin{bmatrix} 0 & D & \bar{K}Y + M_2 \ Y & \Phi Y + B' M_2 \ & * & \Phi Y + B' M_2 \ \Phi Y + B' M_2 & \Phi Y + B' M_2 & \Phi Y + M_2' \ \Phi Y + M_2' & \Phi Y + M_2' & \Phi Y + M_2' \ \Phi Y + M_2' & \Phi Y + M_2' & \Phi Y + M_2' \ \end{bmatrix} > 0
$$

and (21) hence choosing $M_1 = \Phi (X - Y)$ for $j = 1, \ldots, m$ gives

$$
\gamma I \begin{bmatrix} 0 & D & \bar{K}Y + M_2 \ Y & \Phi Y + B' M_2 \ & * & \Phi Y + B' M_2 \ \Phi Y + B' M_2 & \Phi Y + B' M_2 & \Phi Y + (\Phi Y)' \Phi Y \ \Phi Y + \Phi Y & \Phi Y + \Phi Y & \Phi Y + \Phi Y \ \Phi Y + \Phi Y & \Phi Y + \Phi Y & \Phi Y + \Phi Y \ \end{bmatrix} > 0. \tag{21}
$$

By Schur complements, (21) and (20) are feasible iff

$$
\gamma I \begin{bmatrix} 0 & D & \bar{K}Y + M_2 \ Y & \Phi Y + B' M_2 \ & * & \Phi Y + B' M_2 \ \Phi Y + B' M_2 & \Phi Y + B' M_2 & \Phi Y + (\Phi Y)' \Phi Y \ \Phi Y + \Phi Y & \Phi Y + \Phi Y & \Phi Y + \Phi Y \ \Phi Y + \Phi Y & \Phi Y + \Phi Y & \Phi Y + \Phi Y \ \end{bmatrix} > 0. \tag{22}
$$

provided $X - \Phi^T \Phi > 0$ and $0 < \kappa < \epsilon I$ for sufficiently small $\epsilon > 0$, whereas feasibility of (22)-(23) is equivalent to $(Y^{-1} - K + M_2 Y^{-1}) \in F_1(\{ (A, B) \})$, $\gamma = Q, R, F, G, g$.

From (17), the projection of $E = \{ \xi : \xi^T P \xi \leq 1 \}$ onto the $(x, e)$-subspace is given by $\{ (x, e) : [x^T \epsilon]^T [x^T \epsilon]^T ] \}$, and Corollary 7 therefore implies that this projection can be made as large as the maximal ellipsoidal set for the state of (14) under linear feedback $u = H(x^T \epsilon)^T$, such that constraints (2) and $J < \gamma$ hold. Furthermore, there are no restrictions on the structure of the feedback gain $H$ (i.e. $H$ is not required to provide feedback from the state estimate, unlike $\bar{K}$). It is therefore straightforward to show that $\hat{H}$ exists such that $Y^{-1}$ is feasible for $F_1(\{ (A, B) \}, \gamma, Q, R, F, G, g)$ and only if: (i) $H_u$ exists such that $Y_u^{-1}$ is feasible for $F_1(\{ (A', B') \}), \gamma, Q, R, F, G, g$, where $Y_u$ is the upper-left $n_x \times n_x$ block of $Y$; (ii) $A^0 - LC$ is strictly stable. Hence the projection of $E$ onto the $x$-subspace can be made as large as the maximal ellipsoidal feasible set for $x$ under full state feedback $u = H_x x$.

To maximize the projection of $E = \{ \xi : \xi^T P \xi \leq 1 \}$ onto the $x$-subspace, the offline computation of $P$, $\{ A_j \}$, $C_c$ can be performed by solving the SDP

$$
\text{maximize } \log \det(Y_x) \tag{24}
$$

subject to (19), (20), where $Y = \begin{bmatrix} Y_x & Y_ex \ Y_x & Y_{ex} \end{bmatrix}$ for $Y_x \in \mathbb{R}^{nx \times nx}$.

Remark 8. The proof of Corollary 7 implies that some or all of the eigenvalues of $X$ may be forced to be small in order to achieve the maximal volume ellipsoidal projection via (24). This would imply that the intersection of $E$ with the $(x, e)$-subspace is small, in which case, as discussed in Section 6, the estimate error that can be tolerated by the online optimization would also be small. This difficulty can be avoided if the objective of maximizing the projection
is balanced with the objective of maximizing the \((x,e)\)-subspace intersection, e.g. by adding \(\alpha X\) to the objective in (24), where \(\alpha > 0\) is a fixed weight and \(X \succeq \Delta X I\).

5. WORST CASE PERFORMANCE BOUND

Both the state feedback approach of Section 3 and the observer-based output feedback approach of Section 4 require the nominal LQ-optimal control law \(u = Kx\) to quadratically stabilize the uncertain model (1). This section extends the approach to problems in which the uncertain plant under nominal LQ-optimal control is either not quadratically stable, or has unacceptably poor worst-case performance. We propose a dynamic feedback controller to pre-stabilize the prediction system, and show that its dynamics can be optimized online using semidefinite programming so as to impose a bound on worst-case performance in the absence of constraints (2). The resulting pre-stabilized prediction system can be incorporated into the approach of Sections 3 or 4 in order to optimize the handling of robust constraints. As in Sections 3 and 4, the dynamic controller retains the capacity to generate the predictions of the nominal LQ-optimal control law.

Consider first the case that \(x_k\) is measured directly, and define predicted inputs
\[
u_{k+i|k} = Kx_{k+i|k} + f_{k+i|k} \quad i = 0, 1, \ldots \tag{25}
\]
where \(f_{k+i|k} \in \text{co}\{M^i x_{k+i|k} + N^i, f_{k+i|k}, j = 1, \ldots, m\}\) for all \(i \geq 0\), and \(f_{k+i|k} \in \mathbb{R}^{n_u}\) is a variable in the online optimization at time \(k\). The set of quadratically stabilizing \((M^i, N^i)\) can be obtained from the set \((P, \{M^i, N^i\})\) satisfying \(P > 0\) and
\[
P - \left[\begin{array}{cc} A^T + B^T K & B^T \\ M^j & N^j \end{array}\right] P \left[\begin{array}{cc} A^T + B^T K & B^T \\ M^j & N^j \end{array}\right]^T > \left[\begin{array}{cc} 0 & 0 \\ \frac{K^T}{I_{n_u}} & \frac{K^T}{I_{n_u}} \end{array}\right], \quad j = 1, \ldots, m
\]
\[
\text{Definition 9. Let } F_j(\{A^T, B^T\}, K, Q, R) \text{ denote the set of all feasible } (P, \{M^i, N^i\}) \text{ for (26).}
\]

The following theorem shows that \((M^i, N^i)\) can be chosen so that the minimum over \(f_{k+i|k}\) of the worst case cost \(J_k\) along predicted trajectories of (1), (25) for given \(x_k\) is equal to the minimum \(J_k\) achievable using linear state feedback.

\[
\text{Theorem 10. There exist } \{M^i, N^i\}, S_{xf} \text{ and } S_f \text{ such that}\]
\[
\left(\begin{array}{cc} S_{xf} & S_f \\ S_f & N_f \end{array}\right) \in F_j(\{A^T, B^T\}, K, Q, R) \text{ if and only if there exists a feedback gain that satisfies:}
\]
\[
S_{xf}^{-1} - (A^T + B^T H)^T S_{xf}^{-1} (A^T + B^T H) > Q + H^T R H \quad (27)
\]

\[
\text{Proof: Let } P^{-1} = \left[\begin{array}{cc} S_{xf} & S_f \\ S_f & N_f \end{array}\right] \text{ and define } \left[\begin{array}{cc} \Gamma_x & \Gamma_f \\ Y_x & Y_f \end{array}\right] \text{ by}
\]
\[
\left[\begin{array}{cc} \Gamma_x & \Gamma_f \\ Y_x & Y_f \end{array}\right] = \left[\begin{array}{cc} (A^T B^T K) S_{xf} + B^T S_{xf}^T (A^T B^T K) S_{xf} + B^T S_f \\ M^i S_{xf} + N^i S_{xf}^T & M^i S_{xf} + N^i S_f \end{array}\right].
\]

Also let \([\Sigma_x \Sigma_f] = [D_x S_x + D_f S_f^T \quad D_x S_f + D_f S_f] \text{ where}
\]
\[
[D_x D_f] = \left[\begin{array}{cc} 0 & 0 \\ I_{n_u} & R \end{array}\right], \quad [\Gamma_x \Gamma_f] = [K^T \quad I_{n_u}].
\]

The conditions of (26) can be written for \(j = 1, \ldots, m\) as
\[
\left[\begin{array}{cc} I & \Sigma_x \\ \Sigma_x & \Gamma_f \\ \Gamma_x & \Gamma_f \end{array}\right] \succeq 0.
\]

By Schur complements, (28) is equivalent to \(S_{xf} > 0\) and
\[
\left[\begin{array}{cc} I - \Sigma_x S_{xf}^{-1} & -\Sigma_x S_{xf}^{-1} \Gamma_x^T \\ -\Sigma_x S_{xf}^{-1} \Gamma_f & \Gamma_f S_{xf}^{-1} \Gamma_f + B^T P_f \end{array}\right] > 0
\]
where \(P_f^{-1} = S_f - S_{xf} S_{xf}^T \). Choosing \((M^j, N^j)\) so that \(Y_j = S_{xf}^{-1} \Gamma_j, Y_f = S_{xf}^{-1} \Gamma_f\), and considering Schur complements again, conditions are given that
\[
\left[\begin{array}{cc} I & \Sigma_x (\Xi^T)^{-1} \Sigma_x^{-1} \Gamma_x^T \\ \Sigma_x (\Xi^T)^{-1} \Gamma_f & \Gamma_f S_{xf}^{-1} \Gamma_f + B^T P_f \end{array}\right] > 0
\]
This condition is satisfied if \(P_f > \epsilon I\) for sufficiently small \(\epsilon\) and \(I - \Sigma_x (S_{xf} - \Gamma_j S_{xf}^T) - \Sigma_x \Gamma_j^T > 0\), which in turn is equivalent to (27) if \(H = K + S_{xf}^{-1}\).

Definition 10. Let \(F_j(\{A^T, B^T\}, K, Q, R) \text{ denote the set of all feasible } (P, \{M^i, N^i\}) \text{ for (26).}

The following theorem shows that \((M^i, N^i)\) can be chosen so that the minimum over \(f_{k+i|k}\) of the worst case cost \(J_k\) along predicted trajectories of (1), (25) for given \(x_k\) is equal to the minimum \(J_k\) achievable using linear state feedback.

\[
\text{Theorem 11. There exist } \{M^i, N^i\}, S_{xf} \text{ and } S_f \text{ such that}\]
\[
\left(\begin{array}{cc} S_{xf} & S_f \\ S_f & N_f \end{array}\right) \in F_j(\{A^T, B^T\}, K, Q, R) \text{ if and only if } S_{xf}
\]
\[
S_{xf}^{-1} - (A^T + B^T H)^T S_{xf}^{-1} (A^T + B^T H) > Q + H^T R H \quad (27)
\]

\[
\text{subject to (28) for } j = 1, \ldots, m \text{ and } S_{xf} \geq \lambda_x I. \text{ Likewise, replacing } S_x, S_{xf} \text{ with } S_x, S_{xf} \text{ in (28) and (32) allows}
\]
\[
\{M^i, N^i\} \text{ in (29)-(30) to be determined so as to minimize the maximum worst case cost over } \{x^T T, Y^T T\} \in B_{1,2}^m.
\]

Remark 12. The preceding formulation does not explicitly ensure that \(f_{k+i|k} = 0\) can be implemented in (25) for all \(k\)
\[ x_{k+1}^e \leq x_k^e + v_{k+1}^e - \beta_k \quad \forall e \in E_{c,k} \] (35)

subject to
\[
\begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} P_x & P_{xe} & P_{xv} \\ * & P_e & P_{ev} \\ * & * & P_v \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \leq \beta_k \quad \forall e \in E_{c,k}
\]

and \( x = \hat{x}_k \), where \( P = \begin{bmatrix} P_x & P_{xe} & P_{xv} \\ * & P_e & P_{ev} \\ * & * & P_v \end{bmatrix} \). The quadratic constraint on \((x, e, v)\) in (35) implies robust feasibility of predictions, while the RHS of this inequality is chosen so as to ensure convergence of the closed loop system. Thus \( \beta_k \) evolves according to

\[ \beta_{k+1} = \beta_k - \frac{1}{\gamma} \left( \|y_k\|^2 + \|u_k\|^2 \right), \quad \beta_0 = 1. \] (36)

The following lemma gives a convenient form for the MPC optimization (34)-(35) and demonstrates that it is convex.

**Lemma 14.** The constraints of (35) are equivalent to

\[ \begin{bmatrix} \hat{x}_k^T \\ v \end{bmatrix} \begin{bmatrix} \hat{P}_x & \hat{P}_e \\ * & \hat{P}_v \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ v \end{bmatrix} \leq \beta_k - \lambda, \] (37)

\[ \begin{bmatrix} \hat{x}_k^T \\ v \end{bmatrix} \begin{bmatrix} \hat{P}_x & \hat{P}_e \\ * & \hat{P}_v \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ v \end{bmatrix} \leq \beta_k - \lambda, \] (38)

for some \( \lambda \in [0, \beta_k] \). Applying the S-procedure, this is equivalent to the condition that the matrix

\[ \begin{bmatrix} \lambda E_x & - (P_x + P_{xe} + P_{xv}^T + P_{xe}^T + P_{xe}^T + P_{xe}^T + P_{xe}^T + P_{xe}^T) \hat{x}_k + (P_x + P_{xe} + P_{xe}^T + P_{xe}^T + P_{xe}^T) v \end{bmatrix} \]

is positive definite for some \( \lambda \geq 0 \). The Schur complement of the upper left block of this matrix gives (37), whereas expanding the lower right block using Schur complements yields a LMI in \( \hat{x}_k, v, \lambda \), implying that (37) is convex.

---

6. MODEL PREDICTIVE CONTROL

The MPC law described in this section is based on the receding horizon minimization of a nominal predicted cost. We describe the robust handling of constraints based on the offline computation of Algorithm 1, propose an efficient optimization procedure, and discuss closed loop stability.

To extend the definition of nominal cost to the case of feedback based on state estimates, we set the estimation error \( \{e_k, k \in \mathbb{Z} \} \) to zero along nominal predicted trajectories.

\[ J^0_{k} = \frac{1}{\gamma} \int_{t_k}^{T_k} W_x r_x + v_{k+1}^e (R \hat{R} C_e) v_{k+1} \quad (33) \]

where \( \hat{R} = B^T W_x B + R \) and \( W_x, W_e \) are solutions of

\[ W_x - (A^T + B^T K) W_x (A^T + B^T K) = Q + K^T R K \]

in order to ensure robust satisfaction of constraints, we assume that a quadratic bound on estimation error

\[ e_k \in E_{c,k} = \{ e \in \mathbb{R}^n \mid e \leq 1 \}, \]

is known for each \( k \). Given the state estimate \( \hat{x}_k \), and \( P \) computed offline in Step 2 of Algorithm 1, the online optimization of dof in predictions at time \( k \) is specified by

\[ v^*_{k|k} = \text{arg min}_{v} v^T (R \hat{R} C_e) v \quad (34) \]

subject to
The bound (38) on closed loop performance is due to (36), which implies \( \sum_{k=0}^{N} (\|x_k\|^2_{Q} + \|u_k\|^2_{R}) = \beta_0 - \beta_N \leq 1 \) for any \( N > 0 \). Since \( Q = C^T C \), (38) implies \( (y_k, u_k) \rightarrow (0, 0) \) as \( k \rightarrow \infty \).

**Remark 16.** From the proof of Lemma 14, a necessary condition for feasibility of (37) is \( \mathcal{A} \mathcal{E}_k \geq P_s + P_{xe} + P_{xe}^* + P_e \). But (37) also requires that \( \lambda \leq \beta_k \), so this condition limits the size of the estimation error bound \( \mathcal{E}_{e,k} \) that can be tolerated in the online MPC optimization. Remark 8 discusses how to make this condition less stringent via the offline optimization of \( P \). The decreasing sequence of values for \( \beta_k \) necessitates that \( \mathcal{E}_{e,k} \) also decreases with \( k \). Note however that \( \mathcal{E}_{e,k} \) is a subset of \( \{ e : \| (x, e, v) \|_F^2 \leq \beta_k \} \), which clearly decreases at the same rate as \( \beta_k \).

**Remark 17.** For fixed \( \lambda \), the minimization of (34) subject to (37) can be performed efficiently using a Newton-Raphson iteration [see e.g. Kouvaritakis et al., 2002]. This suggests an efficient online optimization procedure which searches over \( \lambda \in [0, \beta_k] \) using Newton-Raphson iteration to optimize sequentially over \( v(k) \). Convergence to the optimum solution is ensured by convexity of (34) and (37).

### 7. NUMERICAL EXAMPLE

First consider the offline computation of \( \{ M^j, N^j \} \) and the minimization of the performance bound in Section 5. The system model (from Cheng et al. [2009]) is given by

\[
A^0 = \begin{bmatrix} -0.8 & 1 \\ -0.3 & 0 \end{bmatrix}, \quad A^1 = A^2 = \begin{bmatrix} -0.96 & 1 \\ -0.36 & 0 \end{bmatrix}, \quad A^2 = A^4 = \begin{bmatrix} -0.96 & 1 \\ -0.36 & 0 \end{bmatrix}
\]

\[
B^0 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad B^1 = B^4 = \begin{bmatrix} 1.02 \\ 0.612 \end{bmatrix}, \quad B^2 = B^3 = \begin{bmatrix} 0.98 \\ 0.588 \end{bmatrix}
\]

with \( C = [1 \ 0] \) and cost weight \( R = 0.01 \). Table 1 compares worst case costs by giving the eigenvalues of \( P \), where \( J_k = \|x_k\|^2_F \), for four controllers: (i). Minimax linear feedback \( u = Hx \), where \( H \) minimizes \( \lambda_{\max}(S_x^{-1}) \) subject to (27) and \( P = S_x^{-1} \); (ii). Equation (25), with \( \{ M^j, N^j \} \) computed by minimizing (32) subject to (28) with the nominal LQ-optimal gain \( K = [0.8 \ -1] \); (iii). Equation (29) with \( K = [0.8 \ -1] \), observer gain \( L = [-2 \ 0.05] \) chosen to place eigenvalues of \( A^0 - LC \) at \([0.5, 0.7]\), and \( \{ M^j, N^j \} \) computed by Step 1 of Alg. 1; (iv). Nominal LQ-optimal feedback with a Youla parameter which is optimized so as to minimize the \( H_{\infty} \)-norm of the closed loop system sensitivity transfer function (see Cheng et al. [2009]).

<table>
<thead>
<tr>
<th>Optimization method</th>
<th>( \lambda(P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( H ) optimized subject to (27)</td>
<td>1.0393, 0.0224</td>
</tr>
<tr>
<td>(ii) ( { M^j, N^j } ) optimized via (32) s.t. (28)</td>
<td>1.0396, 0.0241</td>
</tr>
<tr>
<td>(iii) ( { M^j, N^j } ) from Step 1 of Alg. 1</td>
<td>1.0396, 0.0243</td>
</tr>
<tr>
<td>(iv) ( H_{\infty} )-optimal Youla parameter</td>
<td>1.0464, 0.0549</td>
</tr>
</tbody>
</table>

Table 1. Eigenvalues of \( P \), where \( J_k = \|x_k\|^2_F \).

Clearly the cost bound is almost identical for (i), (ii) and (iii). The slight increase in (ii) and (iii) arises because of the additional constraint discussed in Remark 12 which allows for \( \{ M^0, N^0 \} = (0, 0) \) and \( \{ M^0, N^0 \} = (0, 0) \). The bound for (iv) is larger because the \( H_{\infty} \) design methodology is intended to minimize sensitivity of output predictions to additive disturbances, so does not directly address the problem of minimizing \( J_k \).

To illustrate the optimization of prediction dynamics in Step 2 of Algorithm 1, we consider the same system with model input constraint \( |u_k| \leq 1 \) and cost bound \( \gamma = 10^3 \). Figure 1 shows: (a). The maximal ellipsoid \( \{ x : \|x\|^2_F \leq 1 \} \) for linear state feedback \( u = Hx \) satisfying (5)-(6); (b). The maximal x-subspace projection of the ellipsoidal set satisfying (11)-(12), computed via (24); (c). The x-subspace projection of the ellipsoidal set for the plant state, observer error and dof in predictions, computed in Step 2 of Alg. 1. The slight reduction in size of the feasible set in (c) is due to the trade-off discussed in Remark 8. Choosing a weight of \( \alpha = 10^8 \) results in \( P_e + P_{xe} + P_{xe}^* + P_e = [6.11 \ 65] \), thus allowing for initial estimation error bounds in the MPC optimization of Algorithm 2 of magnitude \( |e_0| \leq 0.22 \ 0.08 \times 10^2 \).

### REFERENCES


