Design of discrete-time adaptive PID controllers for a class of linear integrator systems

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Abstract: Proportional, Integral and Derivative (PID) controllers are widely employed in control applications. The Ziegler-Nichols method is one of the most widely used PID parameter tuning methods, even though it is known to have some drawbacks. Automatic or self-tuning of PID controllers has been extensively studied. However, the stability of the resulting system with the PID controller is not considered in most PID auto-tuning methods and the tuned PID parameters may not guarantee the stability of the control system after changes in the system. This paper presents a new method to design discrete-time stabilizing PID controllers with time varying (or adaptive) gains for a special class of MIMO linear systems. The technique, based on appropriate choices of control Lyapunov functions (CLF), determines the optimal values of the parameters, with respect to the CLF choice, which guarantee the asymptotic stability of the equilibrium of the considered system.

Keywords: PID Control; Lyapunov Functions; Linear Systems; MIMO Systems; Asymptotic Stability

1. INTRODUCTION

Proportional, Integral and Derivative (PID) controllers are widely employed in control applications. The Ziegler-Nichols method is one of the most widely used PID parameter tuning methods (Ziegler and Nichols [1942]). However it is known that the Ziegler-Nichols method has some drawbacks: it uses insufficient process information and the design criterion gives closed loop systems with poor robustness Aström and Hågglund [2001, 2004]. On the other hand, most PID controllers use fixed parameters which are tuned off line. With pre-determined parameters, stability and other desired control performance become difficult to maintain if there are some changes of system properties during operation. These problems resulted in an extensive study of automatic or self-tuning of PID controllers Aström and Hågglund [1988], Aström et al. [1993], Tan et al. [2001], Ho et al. [2003], Chang et al. [2003], Gyöngy and Clarke [2006], Ren et al. [2008]. However, the stability of the resulting system with the PID controller is not considered in most PID auto-tuning methods and the tuned PID parameters may not guarantee the stability of the control system after changes in the system. For linear continuous-time systems, Iwai et al. [2006], Tamura and Ohmori [2007] propose, respectively, auto-tuning and adaptive PID control strategies, based on the almost strictly positive real property (ASPR). A modified version of the auto-tuning and adaptive PID control based on the ASPR property was proposed in Mizumoto et al. [2010] for discrete-time SISO systems. The adaptive PID schemes based on the ASPR property of the system can guarantee the asymptotic stability of the resulting PID control system Mizumoto et al. [2010]. However, the ASPR conditions are not satisfied by most practical systems and the ASPR discrete-time system needs a direct input feedthrough, leading to difficulties such as causality problems which need to be compensated in the controller design Mizumoto et al. [2010].

This paper presents a new design method for discrete-time stabilizing PID controllers with time varying (or adaptive) gains for a special class of MIMO linear systems. The technique, based on adequate choices of control Lyapunov functions (CLF), determines the optimal values, by Lyapunov optimizing control (LOC), of the parameters which guarantee the asymptotic stability of the equilibrium of the considered system. This technique, proposed in Bhaya and Kaszkurewicz [2006] for the design of numerical algorithms, was subsequently applied in Diene and Bhaya [2006a,b, 2009, 2010] to the design of signal processing algorithms. In the present context, this technique is applied to the state-space representation of linear systems with PID controllers in unitary feedback configuration. The state-space representation permits design of PID controllers with time-varying (or adaptive) gains by choosing Lyapunov functions. The parameters of the controllers are determined such that the decrement of the Lyapunov function is as negative as possible and it is thus shown that the equilibrium of the resulting system is asymptotically stable. The same approach is applied for a class of SISO or MIMO linear systems which can be represented as (I, B, C, 0). This class of linear systems is found in some image and signal processing applications, and is a generalization of the linear system (I, I, C, 0) which represents the numerical algorithms considered in Bhaya and Kaszkurewicz [2006]. In order to verify the performance of the proposed controllers, numerical simulations are performed for some simple examples of SISO/MIMO systems.

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The main design technique is denominated CLF/LOC, where LOC stands for Lyapunov optimizing control (Bhaya and Kaszkurewicz [2006]) and is summarized as follows:

1. Choose a candidate CLF: \( V(x) \), where \( x \) is the state variable.
2. Determine the discrete derivative along trajectories: \( \Delta V(x) = V(x_{k+1}) - V(x_k) \). The candidate CLF is valid if there exists a choice of \( u(x) \) that makes \( \Delta V(x) \) negative definite for the chosen candidate CLF.
3. The Lyapunov optimizing control (LOC) step is to choose the particular (optimal) \( u(x) \) that makes \( \Delta V(x) \) as negative as possible. This choice of \( u(x) \) is called optimal.

In the sequel, the word “optimal” will always be used in this specific CLF/LOC sense.

2. DESIGN OF DISCRETE-TIME ADAPTIVE PID CONTROLLERS FOR LINEAR INTEGRATOR SYSTEMS (I, B, C, 0)

The state-space representation of a linear discrete system without feedforward is given by

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k \\
e_k &= r - y_k \\
u_k &= f(e_k)
\end{align*}
\]

(1)

where \( x_k \in \mathbb{R}^n \) is the state variable, \( r \in \mathbb{R}^m \) the reference signal, \( y_k \in \mathbb{R}^m \) the output signal, \( u_k = f(e_k) \in \mathbb{R}^m \) the control law chosen in order to zero the control error \( e_k \in \mathbb{R}^m \), \( A \in \mathbb{R}^{n \times n} \) the state matrix, \( B \in \mathbb{R}^{n \times m} \) the input matrix and \( C \in \mathbb{R}^{m \times n} \) the output matrix. The evolution of the control error is given by the following equation

\[
e_{k+1} = r - y_{k+1} = r - Cx_{k+1} = r - CAx_k - CBu_k,
\]

which can be rewritten as

\[
e_{k+1} = r - Cx_k + C(I - A)x_k - CBu_k = e_k - CBu_k + C(I - A)x_k.
\]

where \( I \) is the identity matrix. For the class of the linear integrator systems the state matrix is \( A = I \), thus

\[
e_{k+1} = e_k - CBu_k.
\]

2.1 Discrete-time adaptive Proportional (P) Controller

The proportional controller is such that

\[
u_k = f(e_k) = g_p(k)e_k = \alpha_k e_k,
\]

(3)

where \( g_p(k) = \alpha_k \) is the time varying proportional gain. With the proportional controller, system (1) becomes

\[
\begin{align*}
x_{k+1} &= x_k + \alpha_k Bu_k \\
e_{k+1} &= (I - \alpha_k CB)e_k
\end{align*}
\]

(4)

Figure 1 shows the block diagram of the linear integrator system with the proportional controller in unitary feedback configuration.

For the dynamical system (4), we have:

**Theorem 1.** The choice

\[
\alpha_k = \frac{\langle e_k, CB e_k \rangle}{\langle CB e_k, CB e_k \rangle}
\]

(5)

is optimal in the sense of minimizing \( \Delta V \) for \( V \) chosen in (6) and ensures that the equilibrium of (4) is asymptotically stable (i.e. \( e_k \to 0 \)), if the product matrix \( CB \) is full rank.

**Proof of Theorem 1.** Choose the CLF

\[
V_e(k) = \langle e_k, e_k \rangle.
\]

(6)

Then

\[
\Delta V_e := V_e(k + 1) - V_e(k) = \langle e_{k+1}, e_{k+1} \rangle - \langle e_k, e_k \rangle
\]

(7)

From (4), substituting the value of \( e_{k+1} \) into (7) results in

\[
\Delta V_e = -2\alpha_k \langle e_k, CB e_k \rangle + \alpha_k^2 \langle CB e_k, CB e_k \rangle.
\]

(8)

The optimal value of \( \alpha_k \) (which makes \( \Delta V_e \) as negative as possible), given by (5), is determined calculating \( \partial \Delta V_e / \partial \alpha_k \) and setting it to zero. Substituting (5) into (8) yields

\[
\Delta V_e = - \frac{\langle e_k, CB e_k \rangle^2}{\langle CB e_k, CB e_k \rangle}.
\]

(9)

If \( CB \) is of full rank then \( \Delta V_e < 0 \), ensuring that the equilibrium of (4) is asymptotically stable.

**Observation 1.** The CLF choice is not unique and each choice results in a different formula for \( \alpha_k \). For example, if \( CB \) is positive definite, the choice

\[
V_e(k) = \langle e_k, (CB)^{-1} e_k \rangle
\]

(10)

leads to the following optimal value of \( \alpha_k \)

\[
\alpha_k = \frac{\langle e_k, e_k \rangle}{\langle e_k, CB e_k \rangle}.
\]

(11)

2.2 Discrete-time adaptive Proportional and Integral (PI) Controller

The discrete-time proportional and integral controller is such that

\[
u_k = g_p(k)e_k + g_i(k)\sum_{i=0}^k e_i,
\]

(12)

where \( g_p(k) \) and \( g_i(k) \) are respectively the time varying proportional and integral gains. The integral part of (12) can be rewritten in terms of a state variable \( q_k \) as

\[
q_{k+1} = e_k + \beta_k q_k,
\]

(13)

where \( \beta_k \) is a gain introduced in order to ensure that the integrator state is zero at steady-state. The control law (12) is thus rewritten as

\[
u_k = g_p(k)e_k + g_i(k)q_k = \alpha_k p_k,
\]

(14)

where

\[
p_k = e_k + \gamma_k q_k,
\]

(15)

\( \alpha_k = g_p(k) \) is the proportional gain and \( \alpha_k \gamma_k = g_i(k) \) is the integral gain. With the proportional and integral controller, system (1) becomes:

\[
\begin{align*}
x_{k+1} &= x_k + \alpha_k Bp_k \\
e_{k+1} &= e_k - \alpha_k CBp_k \\
q_{k+1} &= e_k + \beta_k q_k \\
p_{k+1} &= e_{k+1} + \gamma_k q_{k+1}
\end{align*}
\]

(16)

Figure 2 shows the block diagram of the linear integrator system with the proportional and integral controller in unitary feedback configuration.
For the dynamical system (16), we have:

**Theorem 2.** The choices

\[
\alpha_k = -\frac{\langle e_k, CBp_k \rangle}{\langle CBp_k, CBp_k \rangle} \quad (17)
\]

\[
\beta_k = -\frac{\langle e_k, q_k \rangle}{\langle q_k, q_k \rangle} \quad (18)
\]

\[
\gamma_{k+1} = -\frac{\langle e_{k+1}, q_{k+1} \rangle}{\langle q_{k+1}, q_{k+1} \rangle} \quad (19)
\]

are optimal with respect to the quadratic CLFs (6), (22) and (24) and ensure that the equilibrium of (16) is asymptotically stable if CB is full rank.

**Proof of Theorem 2.** Choosing the CLF (6), it can be shown that

\[
\Delta V_e = -2\alpha_k \langle e_k, CBp_k \rangle + \alpha_k^2 \langle CBp_k, CBp_k \rangle. \quad (20)
\]

The minimal value of \(\Delta V_e\) is determined by finding the roots of its gradient with respect to \(\alpha_k\)

\[
\frac{\partial \Delta V_e}{\partial \alpha_k} = -2\langle e_k, CBp_k \rangle + 2\alpha_k \langle CBp_k, CBp_k \rangle = 0.
\]

For CB full rank, substituting (17) in (20) gives:

\[
\Delta V_e = -\frac{\langle e_k, CBp_k \rangle^2}{\langle CBp_k, CBp_k \rangle} < 0, \quad (21)
\]

implying that \(\|e_k\|_2 \to 0\).

In order to determine \(\beta_k\), choose the CLF

\[
V_q(k) = \langle q_k, q_k \rangle, \quad (22)
\]

for which

\[
V_q(k+1) = \langle e_k + \beta_k q_k, e_k + \beta_k q_k \rangle = \langle e_k, e_k \rangle + 2\beta_k \langle e_k, q_k \rangle + \beta_k^2 \langle q_k, q_k \rangle.
\]

Determining the roots of the gradient of \(V_q(k+1)\) with respect to \(\beta_k\)

\[
\frac{\partial V_q(k+1)}{\partial \beta_k} = 2\langle e_k, q_k \rangle + 2\beta_k \langle q_k, q_k \rangle = 0,
\]

it can be seen that \(\beta_k\) given by (18) is optimal and results in

\[
\|q_{k+1}\|^2 = \|e_k\|^2 - \frac{(\langle e_k, q_k \rangle)^2}{\langle q_k, q_k \rangle} < \|e_k\|^2 \quad (23)
\]

From (21) and (23), it can be concluded that \(q_k\) decreases in 2-norm.

In order to determine \(\gamma_{k+1}\), choose the CLF

\[
V_p(k) = \langle p_k, p_k \rangle. \quad (24)
\]

As for \(\beta_k\), determining the roots of the gradient of \(V_p(k+1)\) in relation with \(\gamma_{k+1}\), it can be seen that \(\gamma_{k+1}\) given by (19) is optimal and results in

\[
\|p_{k+1}\|^2 = \|e_{k+1}\|^2 - \frac{(\langle e_{k+1}, q_{k+1} \rangle)^2}{\langle q_{k+1}, q_{k+1} \rangle} < \|e_{k+1}\|^2 \quad (25)
\]

As \(e_{k+1}\) decreases in 2-norm, (25) ensures that \(p_{k+1}\) decreases.

**Observation 2.** As the CLF choices are not unique, it can be shown that if CB is positive definite the choices

\[
\alpha_k = \frac{\langle e_k, p_k \rangle}{\langle p_k, CBp_k \rangle}, \quad (29)
\]

\[
\beta_k = -\frac{\langle e_k, CBq_k \rangle}{\langle q_k, CBq_k \rangle}, \quad (30)
\]

\[
\gamma_{k+1} = -\frac{\langle e_{k+1}, CBq_{k+1} \rangle}{\langle q_{k+1}, CBq_{k+1} \rangle}. \quad (31)
\]

### 2.3 Discrete-time adaptive Proportional and Derivative (PD) Controller

In order to reduce the effects of perturbations and/or abrupt changes on the reference signal, it is a common practice to use the derivative of the state variable or the output variable instead of the derivative of the error Åström and Hägglund [1988]. A discrete-time proportional and derivative controller can be written in terms of the derivative of the state variable such as

\[
u_k = g_p(k)e_k + g_d(k)(x_k - x_{k-1}), \quad (32)
\]

where \(g_p(k)\) and \(g_d(k)\) are respectively the time-varying proportional and derivative gains. Defining

\[
p_k = e_k + \gamma_k(x_k - x_{k-1}), \quad (33)
\]

the state equation can be written as

\[
x_k = x_{k-1} + \alpha_k - 1 B p_{k-1}, \quad (34)
\]

From (33) and (34) it can be seen that

\[
p_k = e_k + \gamma_k \alpha_{k-1} B p_{k-1} = e_k + \beta_{k-1} B p_{k-1}. \quad (35)
\]

With the proportional and derivative controller, system (1) becomes:

\[
\begin{cases}
x_{k+1} = x_k + \alpha_k B p_k \cr e_{k+1} = e_k - \alpha_k CB p_k \cr p_{k+1} = e_k + \beta_k B p_k,
\end{cases} \quad (36)
\]

where \(\alpha_k\) and \(\beta_k\) are, respectively, the proportional and the derivative gains. Figure 3 shows the block diagram of the linear integrator system with the proportional and derivative controller in unitary feedback configuration.

The time-varying gains \(\alpha_k\) and \(\beta_k\) are given by the following corollary:

**Corollary 1.** The choices
In regard to other design methods, we note that the decoupled PID design method proposed in Aström et al. [2002] is not immediately applicable to determine the optimal gains as follows:

\[
\alpha_k = \frac{\langle e_k, CBp_k \rangle}{\langle CBp_k, CBp_k \rangle} \quad (46)
\]
\[
\beta_k = -\frac{\langle e_{k+1}, Bp_k \rangle}{\langle Bp_k, Bp_k \rangle} \quad (47)
\]
\[
\gamma_k = -\frac{\langle e_{k+1}, Bp_k + q_{k+1}, Bp_k \rangle}{\langle Bp_k, Bp_k \rangle} \quad (48)
\]

are optimal and ensure that the equilibrium of (45) is asymptotically stable.

**Proof of Corollary 2.** Choosing the CLF (6), (22) and (24) the proof follows as the proof of theorem 2. ■

For the sub-class of linear integrator systems such that CB and B are positive definite, the Lyapunov functions (26), (27) and (39) allow to determine the optimal gains as follows:

\[
\alpha_k = \frac{\langle e_k, p_k \rangle}{\langle p_k, CBp_k \rangle} \quad (49)
\]
\[
\beta_k = -\frac{\langle e_{k+1}, CBq_k \rangle}{\langle q_k, CBq_k \rangle} \quad (50)
\]
\[
\gamma_k = -\frac{\langle e_{k+1}, p_k \rangle + \langle q_{k+1}, p_k \rangle}{\langle p_k, Bp_k \rangle} \quad (51)
\]

### 2.4 Discrete-time adaptive Proportional, Integral and Derivative (PID) Controller

The proportional, integral and derivative controller is such that

\[
u_k = g_p(k)e_k + g_i(k)\sum_{i=0}^{k} e_i + g_d(k)(x_k - x_{k-1}),
\]

where \(g_p(k), g_i(k)\) and \(g_d(k)\) are respectively the proportional, integral and derivative gains. Defining

\[p_k = e_k + \delta_k(x_k - x_{k-1}) + q_k,
\]

where \(q_k\) is given by (13), and as for the PD controller it can be seen from (34) that

\[p_k = e_k + \delta_k \alpha_{k-1} Bp_{k-1} + q_k = e_k + \gamma_{k-1} Bp_{k-1} + q_k.
\]

Therefore, with the proportional, integral and derivative controller, system (1) results in

\[
\begin{align*}
x_{k+1} &= x_k + \alpha_k Bp_k \\
q_{k+1} &= e_k - \alpha_k CBp_k \\
\delta_{k+1} &= e_k + \beta_k q_k \\
p_{k+1} &= e_k + \delta_k (x_k - x_{k-1}) + q_k.
\end{align*}
\]

where \(\alpha_k = g_p(k), \beta_k = g_i(k)\) and \(\alpha_k \gamma_k = g_d(k)\). Figure 4 shows the block diagram of the linear integrator system with the proportional, integral and derivative controller in unitary feedback configuration. The time varying gains \(\alpha_k, \beta_k\) and \(\gamma_k\) are given by the following corollary.

**Corollary 2.** The choices

\[
\alpha_k = \frac{\langle e_k, CBp_k \rangle}{\langle CBp_k, CBp_k \rangle} \quad (46)
\]
\[
\beta_k = -\frac{\langle e_{k+1}, Bp_k \rangle}{\langle Bp_k, Bp_k \rangle} \quad (47)
\]
\[
\gamma_k = -\frac{\langle e_{k+1}, Bp_k + q_{k+1}, Bp_k \rangle}{\langle Bp_k, Bp_k \rangle} \quad (48)
\]

are optimal and ensure that the equilibrium of (45) is asymptotically stable.

In order to verify the performance of the designed controllers, numerical simulations are performed for SISO and MIMO systems.

#### 3.1 SISO Systems

The SISO system considered is given as follows

\[
\begin{align*}
x_{k+1} &= x_k + 3u_k \\
y_k &= 2x_k.
\end{align*}
\]

A step function with amplitude 1 is applied to the system as a reference at time \(t = 1s\). The system responses, control signals and error signals of the designed controllers are compared respectively in figures 5 and 6. It is noticed that for SISO systems the designed controllers are dead beat controllers, in the sense that, with adequate initialization of the auxiliary state variables, the formulas for the parameters \(\alpha_k, \beta_k\) and \(\gamma_k\) ensure that the pole of the feedback system is at the origin, for any values of \(b\) and \(c\). This proof is not presented in this paper due to lack of space.

#### 3.2 MIMO Systems

We consider the \(2 \times 2\) MIMO system given as follows

\[
\begin{align*}
x_{k+1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} u_k \\
y_k &= \begin{bmatrix} 4 & 1 \\ 3 & 7 \end{bmatrix} x_k.
\end{align*}
\]

The reference input \(r = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\) is applied to the system at time \(k = 1s\). The performance of the designed controllers are shown in figures 7, 8 and 9. It can be noticed that the plant is stabilized by all the designed controllers and the references are tracked after at most four steps.

In regard to other design methods, we note that the decoupled PID design method proposed in Aström et al. [2002] is not immediately applicable.

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Fig. 5. Step responses of the designed controllers for SISO systems. The PD and PID response are identical for this example.

applicable, because the decoupled transfer matrix (54) is not stable, as required by this method. The decoupled transfer matrix is

$$Q(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}. \quad (54)$$

For several other PID controller designs for (54), stabilization of (53) only occurs for PID parameters very close to zero, resulting in slow step responses with rise time about 200 times greater than the proposed controllers.

Another test is performed applying the references $r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $k = 1s$ and $r_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at $k = 3s$ to (53) with the proposed controllers in unitary feedback configuration. Notice that a change of the first component of the reference acts as a perturbation of the second element of the output and a change of

the second component of the reference acts as a perturbation of the first element of the output. Figure 10 show the responses of the proposed controllers. It can be noticed that the perturbations are rejected and the references tracked. In all cases the PI controller is the one with the most oscillatory response.

4. CONCLUSION

A systematic CLF/LOC design of discrete time-varying and stabilizing PID controllers for SISO/MIMO linear integrator systems is presented. The technique is based on adequate choices of CLF. Each CLF choice, after the LOC step, leads to different formulas for the parameters of the controller. The asymptotic stability of the equilibrium of the closed loop systems is proved. The controller performance, reference tracking and disturbance rejection, are verified with simple SISO and MIMO examples. For the SISO systems, the proposed controller behaves
Fig. 9. Error signal of the designed controllers for MIMO systems. The PD and PID response are identical for this example.

Fig. 10. Step responses of the designed controllers for MIMO systems. In this case the reference inputs are applied at different times ($k = 1s$ and $k = 3s$). The PD and PID response are identical for this example.

like a dead beat controller. For MIMO systems, the proposed controllers show better performance than the decoupled PID Aström et al. [2002] and a comparison with other design techniques is under investigation. Another topic under investigation is an application of the proposed controllers to real world plants, which satisfy the assumptions made above.

REFERENCES


