Global Finite-Time Stabilization of a Class of Nonlinear Systems via Bounded Output Feedback Controllers

Haibo Du∗ Chunjiang Qian** Michael T. Frye*** Shihua Li∗

∗ School of Automation, Southeast University, Nanjing, Jiangsu, 210096, P. R. China (e-mail: haibo.du@seu.edu.cn; lsh@seu.edu.cn)
** Department of Electrical and Computer Engineering, University of Texas at San Antonio, TX, 78249, USA (e-mail: chunjian.qian@utsa.edu)
*** Department of Engineering, University of the Incarnate Word, San Antonio, TX, 78209, USA

1. INTRODUCTION

In this paper, we consider the following nonlinear systems
\[ \dot{x}_1 = x_{i+1} + f_i(x_1, \ldots, x_i), \quad i = 1, \ldots, n-1 \]
\[ \dot{x}_n = u + f_n(x_1, \ldots, x_n) \]
\[ y = x_1 \]
(1)

where \( y \in \mathcal{R}, u \in \mathcal{R} \) are system output and control input, respectively. The nonlinear functions \( f_i(x_1, \ldots, x_i), \quad i = 1, \ldots, n \) are \( \mathcal{C}^1 \) functions which are bounded and satisfy \( f_i(0, \ldots, 0) = 0 \). Our objective is to design a bounded output feedback controller to stabilize the above system (1) in a finite time.

Compared to the conventionally asymptotic stabilization via output feedback it is more challenging to design a finite-time stabilizer using output feedback. In literature, there are very few existing results achieved in this direction. In Hong et al. (2001), the problem of output feedback finite-time stabilization of a double integrator system was achieved by coupling a finite-time convergent observer with a finite-time stabilizer. When the planar system has uncontrollable/unobservable linearization, in Qian et al. (2005) an output finite-time feedback controller was constructed. For high-dimensional nonlinear systems, the work of Qian et al. (2006) introduced a homogeneous domination approach to design an output feedback finite-time stabilizer that is homogeneous in nature.

Note that the output feedback stabilizers in the aforementioned results are not bounded. In addition, it was required that the nonlinearities should satisfy a homogeneous growth condition. In this paper, we will show that we are able to design an output feedback controller which is globally bounded and is able to stabilize the system in a finite time. Specifically, first, using the generalized adding a power integrator method (Polendo et al. (2008)), we design a homogeneous control law to globally finite-time stabilize the system (1). Second, we impose a series of nested saturations to the homogeneous controller and obtain a saturated state feedback controller using the techniques introduced in Tsinias et al. (2001) and later generalized in Ding et al. (2010). Third, we design a finite-time convergent observer to estimate the unknown states in a finite time. Replacing the unmeasurable states in the state feedback stabilizer using the estimates from the finite-time observer, we get a bounded output feedback controller which stabilizes the nonlinear system (1) in a finite time.

2. STATE FEEDBACK STABILIZER OF (1)

In this section, a bounded state feedback controller will be constructed to globally finite time stabilize system (1).

First, we define the following change of coordinates \( z_i = x_i/L^{i-1}, \quad i = 1, \ldots, n, \quad v = u/L^n \) under which (1) becomes
\[ \dot{z}_1 = L \dot{z}_{i+1} + f_i(z_1, \ldots, L^{i-1} z_i)/L^{i-1}, \quad i = 1, \ldots, n-1 \]
\[ \dot{z}_n = L v + f_n(z_1, \ldots, L^{n-1} z_n)/L^{n-1}. \]
(2)

To construct a state feedback controller, we define
\[ r_1 = 1, \quad r_{i+1} = r_i + \tau > 0, \quad i = 1, \ldots, n+1 \]
(3)
with a constant \(-1/(n+1) < \tau < 0\).

Lemma 2.1. There exist constants \( \beta_i^* = n, \) functions \( \beta_i^* : \mathcal{R}^{i-1} \rightarrow \mathcal{R}, \quad i = 2, \ldots, n, \) such that for any
constants $\beta_i$ satisfy $\beta_1 \geq \beta_1^*, \beta_2 \geq \beta_2^*(\beta_1), \ldots, \beta_n \geq \beta_n^*(\beta_1, \ldots, \beta_{n-1})$, the following control law
\[ v = -\beta_n(z_1^{1/r_n} - z_n^{1/r_n})^{r_n+1} \]  \hspace{1cm} (4)
with $z_i^* = 0$, $z_i^{1/r_i} = \beta_i(1 - z_i^{1/r_i} - z_{i-1}^{1/r_{i-1}})^{r_i}$, $i = 2, \ldots, n$, locally finite-time stabilizes system (2). In addition, there is a constant $L > 1$ such that the closed-loop system (2)-(4) is globally finite-time stable.

**Proof.** For simplicity, we define $Z_i = (z_1, \ldots, z_i)$, $i = 1, \ldots, n$, which will be used throughout this paper. Using the result in Ding et al. (2010), we can find functions $\beta_i^* = n$, $\beta_i^*(\beta_1, \beta_2, \ldots, \beta_n^*(\beta_1, \ldots, \beta_{n-1})$ such that for any constants $\beta_i \geq \beta_i^*$, $\beta_i \geq \beta_i^*(\beta_1, \ldots, \beta_{n-1})$ the controller (4) globally stabilizes the following system
\[ \dot{z}_i = L z_{i+1}, i = 1, \ldots, n - 1, \dot{z}_n = L e. \]  \hspace{1cm} (5)
Specifically, there exist a Lyapunov function $V(Z_n)$ which is homogeneous of degree of $2 - \tau$ with respect to (3) and a positive definite function $W(Z_n)$ with a homogeneous degree of 2 such that
\[ \dot{V}(Z_n) \leq -L W(Z_n). \]  \hspace{1cm} (6)
Next we show that there is a constant $L > 1$ such that system (2)-(4) is globally finite-time stable. By (6), the derivative of $V(Z_n)$ along system (2) under the control law (4) is
\[ \dot{V}(Z_n) \leq -L W(Z_n) + \omega_1(Z_n) f_1(c) + \ldots + \omega_n(Z_n) f_n(c)/L^{n-1}, \]  \hspace{1cm} (7)
where $\omega_i(Z_n) = \partial V(Z_n)/\partial z_i$, $i = 1, \ldots, n$. According to homogeneous definition, it is easy to verify $\omega_i(Z_n)$ is homogeneous with a degree of $2 - \tau - r_i$. In addition, if $\|Z_i\| \geq 1$, by the boundedness of $f(\cdot)$, there is a constant $C$ such that
\[ |f_i(z_1, \ldots, L^{-1}z_i)/L^{1-i}| \leq C \sum_{j=1}^i |z_j|^{i-r_j}. \]  \hspace{1cm} (8)
If $\|Z_i\| < 1$, by the $C^1$ property of $f_i(\cdot)$ and $Z_i^{1+r_i}/r_i \leq 1$, there also exists a constant $C$ such that
\[ |f_i(z_1, \ldots, L^{-1}z_i)/L^{1-i}| \leq C \sum_{j=1}^i |z_j| \leq C \sum_{j=1}^i |z_j|^{i-r_j}. \]  \hspace{1cm} (9)
Hence, by homogeneous system theory, we can conclude that there are two constants $c_1$ and $c_2$ such that
\[ \dot{V}(Z_n) \leq -(L c_1 - c_2) V^{\frac{1}{2-r}}(Z_n). \]  \hspace{1cm} (10)
It follows from Theorem 4.2 in Bhat et al. (2000) that the closed-loop system (2)-(4) is globally finite-time stable by choosing $L c_1 > c_2$. □

Next, a bounded control law is designed to solve the global finite-time stabilization problem for system (2). First, by the boundedness of $f_i(\cdot)$, we choose the gain $L$ such that $|f_i(i)|/L^{1-i} \leq 1, i = 1, \ldots, n$. Then, the bounded control law is constructed as following form:
\[ v = v_n(Z_n(t)) = -\beta_n \sigma^{r_n+1}(z_1^{1/r_n} - z_n^{1/r_n})^{r_n+1} \]  \hspace{1cm} (11)
where $v_i(Z_n(t)) = -\beta_i \sigma^{r_i+1}(z_1^{1/r_i} - z_i^{1/r_i})^{r_i+1} (Z_{i-1})$,
\[ i = 1, \ldots, n - 1, \sigma(x) = \begin{cases} \text{sign}(x), & |x| > 1 \\ x, & |x| \leq 1 \end{cases} \]
and the gains $\beta_i$’s will be determined later.

We begin our theorem by introducing an important lemma.

**Lemma 2.2.** For system (2), under control law (11), for every $i = 1, \ldots, n - 1$, there exist functions $\alpha_i(\beta_1, \ldots, \beta_i)$ defined as:
\[ \alpha_1(\beta_1) = \beta_1^{1/r_1}/(2 + \beta_1), \]
\[ \alpha_j(\beta_1, \ldots, \beta_j) = \beta_j^{1/r_j+1}/(r_j)(1 + \beta_{j-1})^{1/r_j} - (2 + \beta_j) \]
\[ + \beta_j^{1/r_j+1} \alpha_{j-1}(\beta_1, \ldots, \beta_{j-1}), \]  \hspace{1cm} (12)
such that the following inequalities hold
\[ |v_i^{1/r_i+1}(Z_i(t)) - v_i^{1/r_i+1}(Z_i(\ell))| \leq \alpha_i(\beta_1, \ldots, \beta_i)(t - \ell), \forall \ell \geq t \]  \hspace{1cm} (13)
provided $|z_j| \leq (1 + \beta_{j-1}), j = i + 1, \ldots, n$.

**Proof.** An inductive argument can be used to show that (13) holds. For the sake of space, we give the first two steps.

**Step 1.** Noting that $|z_j| \leq (1 + \beta_{j-1}), j = 2, \ldots, n$, we obtain for $t \in [t_k, t_{k+1}]$, $z_1(t)$ is continuous and $z_1(t)$ is bounded. It follows that $z_1(t)$ is uniformly continuous. Therefore, starting from $t_k$ there are time series $t_k, k = 1, \ldots, N$ such that $|z_1^{1/r_1}(t_k)| = 1$,
\[ |v_1^{1/r_2}(z_1(t_k)) - v_1^{1/r_2}(z_1(\ell))| \leq \alpha_1(\beta_1)(t_k - \ell), \forall \ell \geq t_k \]  \hspace{1cm} (14)
with $t_0 = t_1, t_{N+1} = \ell$. If $|z_1^{1/r_1}(t)| \geq 1$ for all $t$ in a time interval $[t_k, t_{k+1}]$, by continuity of the solution $z_1(t_k)$ and $z_1(t_{k+1})$ have the same sign, i.e., $z_1^{1/r_1}(t_k) \geq 1$, $z_1^{1/r_1}(t_{k+1}) \geq 1$ or $z_1^{1/r_1}(t_k) \leq -1$, $z_1^{1/r_1}(t_{k+1}) \leq -1$, which in turn implies $|v_1^{1/r_2}(z_1(t_k)) - v_1^{1/r_2}(z_1(t_{k+1}))| = 0$. Hence, we only need to consider the time interval $[t_k, t_{k+1}]$ where $|z_1^{1/r_1}(t)| \leq 1$, $\forall t \in [t_k, t_{k+1}]$.

Consider one of such intervals $[t^*, t^*]$ where $|z_1^{1/r_1}(t)| \leq 1$. We have
\[ |v_1^{1/r_2}(z_1(t^*)) - v_1^{1/r_2}(z_1(t^*))| = \frac{1}{z_1^{1/r_1}(t^*) - z_1(t^*)}. \]  \hspace{1cm} (15)
For $|z_j| \leq (1 + \beta_{j-1})$, $j = 2, \ldots, n$, we obtain from (2) that
\[ |z_1(t^*) - z_1(t^*)| \leq \int_{t^*}^{t^*} |L z_2(s) + f_1(s)| \, ds \leq \mathcal{L} ((1 + \beta_1) + 1)(t^* - t^*). \]  \hspace{1cm} (16)
By (15) and (16), one obtains when $|z_j| \leq (1 + \beta_{j-1})$, $j = 2, \ldots, n$,
\[ |v_1^{1/r_2}(z_1(t^*)) - v_1^{1/r_2}(z_1(t^*))| \leq \mathcal{L}^{1/r_2}(2 + \beta_i)(t^* - t^*) \]  \hspace{1cm} (17)
Let $\alpha_1(\beta_1) = \beta_1^{1/r_2}(2 + \beta_1)$. Note that $\sum_{k=0}^{N}(t_{k+1} - t_k) = (t - \ell)$. Then by (14) and (17), we have for $|z_j| \leq (1 + \beta_{j-1}), j = 2, \ldots, n$,
Step 2. In this step, we show that (13) holds for $i = 2$ under the following condition
\[ |z_i| \leq (1 + \beta_1 - 1), j = 3, \ldots, n. \] (19)
Similar to what we did in the initial step, for any final point $v_1^{r_1} (Z_2 (t))$ and starting point $v_1^{r_1} (Z_2 (t_1))$, there exist time series $t_k, k = 1, 2, \ldots, N$, such that $|Z_2^{r_1} (t_i) - v_1^{r_1} (Z_1 (t_k))| = 1$ and
\[
\left| v_1^{r_1} (Z_2 (t)) - v_1^{r_1} (Z_2 (t_1)) \right| \\
\leq \sum_{k=0}^{N} \left| v_1^{r_1} (Z_1 (t_k)) - v_1^{r_1} (Z_2 (t_{k+1})) \right| . \] (20)
where $t_0 = t_{N+1} = t$. In this case, if $|Z_2^{r_1} (t) - v_1^{r_1} (Z_1 (t))| \geq 1$ for all $t$ in a time interval $[t_k, t_{k+1}]$, similar to the proof of the initial step, we can also obtain $|Z_2^{r_1} (Z_2 (t)) - v_1^{r_1} (Z_2 (t_1))| = 0$. Hence, we only need to consider the time interval $[t_k, t_{k+1}]$ where $|Z_2^{r_1} (t) - v_1^{r_1} (Z_1 (t))| \leq 1, \forall t \in [t_k, t_{k+1}]$.

Consider one of such intervals $[l^*, r^*]$ where $|Z_2^{r_1} (t) - v_1^{r_1} (Z_1 (t))| \leq 1$. Applying $|d^p - b^q| p \leq \max (d^{p-1}, b^{q-1})$ for a ratio of positive odd integers $p \geq 1$, we have
\[
\left| v_1^{r_1} (Z_2 (l^*)) - v_1^{r_1} (Z_2 (l^*)) \right| \\
\leq \left| v_1^{r_1} (Z_1 (l^*)) - v_1^{r_1} (Z_1 (l^*)) \right| \\
\leq \left| v_1^{r_1} (Z_1 (l^*)) - v_1^{r_1} (Z_1 (l^*)) \right| . \] (21)
where we have used $|z_2 (t)| \leq (1 + \beta_3^{r_1} r^{r_1}) r^{r_1} \leq (1 + \beta_1 r^{r_1}) r^{r_1}$. This is due to the fact that $|Z_2^{r_1} (t) - v_1^{r_1} (Z_1 (t))| \leq 1, \forall t \in [l^*, r^*]$ and inequality $(|x| + |y|) p \leq (|x|^p + |y|^p)$ holds for any $0 < p \leq 1, x, y \in R$. For $|z_2 (t)| \leq (1 + \beta_3^{r_1} r^{r_1}) r^{r_1}$, $j = 3, \ldots, n$, we obtain from (2) that
\[
\left| z_2 (l^*) - z_2 (l^*) \right| \leq \int_{l^*}^{r^*} |Lz_3 (s) + f_2 (s)/L| ds \leq L (1 + \beta_3^{r_1} r^{r_1} + 1) (r^* - l^*) . \] (22)
Note that $\sum_{k=0}^{N} (t_{k+1} - t_k) = (l - l^*)$. Hence, combining (18), (21), and (22), we obtain
\[
\left| v_1^{r_1} (Z_2 (l)) - v_1^{r_1} (Z_2 (l)) \right| \leq L \alpha_2 (l, l^*) . \] (23)
with $\alpha_2 (l, l^*) = \alpha (l, l^*) (1 + \beta_1 r^{r_1} + 1 + \beta_2 r^{r_1} + \alpha_1 (l, l^*)$.

Following the same line shown in the first two steps, we can complete the proof of Lemma 2.2. □

With the help of Lemmas 2.1-2.2, we are ready to prove the first main result of this section.

**Theorem 2.1.** For system (1), there is a constant $L > 1$ such that for any constants
\[
\beta_1 > \max \{ \beta_1, 2r^{r_1} \cdot 3 \}, \\
\beta_2 > \max \{ \beta_2^{r_1} (l), \}
\]
\[ 2^{r_1+1} \{ (1 + \beta_1-1 - \beta_1 - \cdots - \beta_1-1 + 3) \} \] (24)
with $i = 2, \ldots, n$, the bounded control law $u = L \alpha v$ will globally finite-time stabilize system (1), where $v$ is described as (11).

**Proof.** The proof is divided into two steps: First, we show that the control law with coefficients $\beta_3^{r_1}$’s preset in (24) ensures that all states will converge to a region determined by the saturation function $\sigma ()$. Then, the saturated control law (11) reduces to the unsaturated control law (4). As a result, the global finite-time stability for the closed-loop system (1)-(11) can be guaranteed by appropriately choosing the gain $L$.

An inductive method can be used to show that the states will enter a region one by one and stay there forever. For the space limit, we give the first two steps.

**Step 1.** In this step we will prove there exists a time instance $t_1$ such that for $t \geq t_1$
\[
Z_n (t) \in Q_n = \left\{ Z_n : |z_{n-1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| < 1 \right\} . \] (25)
We first use a contradiction argument to prove that there exists a time instance $t_1$ such that $|z_{n-1}^{r_1} (t_1) - v_{n-1}^{r_1} (Z_{n-1} (t_1))| \leq 1/2$. Otherwise, it can be assumed that for all $t \geq 0$ we have $|z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| > 1/2$. We first consider the case when
\[
|z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| > 1/2. \] (25)
Under (25), we have for all $t \geq 0$, \[
|z_{n-1}^{r_1} (t) = -L \alpha_n \sigma_{n-1} (z_{n-1}^{r_1} (t_1) - v_{n-1}^{r_1} (Z_{n-1} (t))) + f_n (\cdot)/L = -L \alpha_n (1/2)^{r_1} + L = L \alpha_n . \]
with $\mu_n = \beta_2 (1/2)^{r_1 - 1} > 0$ determined by (24). It follows that for $t \geq 0, z_n (t) < z_n (0) - L \alpha_n$. As time goes to infinity, $z_n (t) \rightarrow -\infty$, which leads to a contradiction disavowing (25) by noticing the fact $|z_{n-1}^{r_1} (Z_{n-1} (t))| \leq 1/2$. The case $z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t)) < -1/2, \forall t \geq 0$, is also impossible. In conclusion, there must exist a time instance $t_1$ such that $|z_{n-1}^{r_1} (t_1) - v_{n-1}^{r_1} (Z_{n-1} (t_1))| \leq 1/2$. Next, by using a contradiction argument again, we will prove that the following holds
\[
|z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| < 1, \text{ for } t \geq t_1. \] (26)
Suppose (26) is not true, which means there is at least one time instance $t_2$ when $|z_{n-1}^{r_1} (t_2) - v_{n-1}^{r_1} (Z_{n-1} (t_2))| = 1$. Specifically, there are $t_1 < t_1 < t_2 < t_1$ such that either
\[
|z_{n-1}^{r_1} (t_1) - v_{n-1}^{r_1} (Z_{n-1} (t_1))| = 1/2 \] (27)
\[
|z_{n-1}^{r_1} (t_2) - v_{n-1}^{r_1} (Z_{n-1} (t_2))| = 1 \] (28)
\[
1/2 \leq |z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| < 1, t \in [t_1, t_2] \] (29)
in the positive region, or $|z_{n-1}^{r_1} (t_1) - v_{n-1}^{r_1} (Z_{n-1} (t_1))| = -1/2, \|z_{n-1}^{r_1} (t_2) - v_{n-1}^{r_1} (Z_{n-1} (t_2))| = -1, -1 \leq |z_{n-1}^{r_1} (t) - v_{n-1}^{r_1} (Z_{n-1} (t))| \leq -1/2, t \in [t_1, t_2]$ as the negative case.
Next we will first prove that the positive case (27)-(28)-(29) is impossible. To this end, by (29) and the fact that $z_n(t) = -L \beta_n \sigma^{n+1} (z_{n+1}^{1/n} (t) - v_{n-2}^{1/n} (Z_{n-2}(t))) + f_n(t)/L^{n-1}$, then

$$z_n(t) \leq - L \mu_n, \quad t \in [t_1', t_1^*] \tag{30}$$

It follows from (30) that

$$L \mu_n (t_1^* - t_1') \leq z_n(t_1^*) - z_n(t_1') \tag{31}$$

By (27) and the fact that $|v_{n-1}^{1/n} (Z_{n-1})| \leq \beta_n^{1/n}$, the following holds since $\tau_n < 1$

$$z_n(t_1') \leq (1 + \beta_n^{1/n}) \tau_n \leq (1 + \beta_n) \tag{32}$$

Similarly, by (28), we have

$$z_n(t_1) \geq -(1 + \beta_n^{1/n}) \tau_n \geq -(1 + \beta_n) \tag{33}$$

Together with (32) and (33), (31) leads to the following time estimate

$$t_1^* - t_1' \leq z_n(t_1') - z_n(t_1) \leq \frac{2}{L \mu_n} (1 + \beta_n) \tag{34}$$

Furthermore, by (31), we have $z_n(t_1') \leq z_n(t_1)$ which implies

$$z_n^{1/n} (t_1') - v_n^{1/n} (Z_{n-1}(t_1')) \leq z_n^{1/n} (t_1) - v_n^{1/n} (Z_{n-1}(t_1))$$

$$+ v_n^{1/n} (Z_{n-1}(t_1')) - v_n^{1/n} (Z_{n-1}(t_1)) \tag{35}$$

Substituting (27) and (28) into (35) yields

$$1/2 \leq \left| v_n^{1/n} (Z_{n-1}(t_1')) - v_n^{1/n} (Z_{n-1}(t_1)) \right| \tag{36}$$

Using (29) and the fact that $|v_{n-1}^{1/n} (Z_{n-1})| \leq \beta_n^{1/n}$, we have $|z_n(t)| \leq (1 + \beta_n) \tau_n$, $t \in [t_1', t_1^*]$, which enables us to use Lemma 2.2 to estimate (36) as

$$1/2 \leq \left| v_n^{1/n} (Z_{n-1}(t_1')) - v_n^{1/n} (Z_{n-1}(t_1)) \right| \leq \frac{2}{\mu_n} (1 + \beta_n) \tag{37}$$

Substituting (34) into (37) yields

$$1/2 \leq (2/\mu_n) (1 + \beta_n) \alpha_n (1, \beta_n, \cdots, \beta_n) \tag{38}$$

On the other hand, by the definition of $\mu_n$ and the choice of (24) we have

$$\mu_n = (1/2)^{n+1} \beta_n - 1 > 4 (1 + \beta_n) \alpha_n (1, \beta_n, \cdots, \beta_n) \tag{39}$$

Combining (38) and (39) leads to

$$1/2 \leq \frac{2}{\mu_n} (1 + \beta_n) \alpha_n (1, \beta_n, \cdots, \beta_n) < 1/2$$

which obviously is a contradiction. Therefore the case of (27)-(28)-(29) will never happen. Similarly, it can be shown, using an almost same argument as the positive case, that $z_n^{1/n} (t) - v_n^{1/n} (Z_{n-1}(t))$ will never cross $-1$. Hence for $t \geq t_2$ we have

$$|z_n^{1/n} (t) - v_n^{1/n} (Z_{n-1}(t))| < 1 \tag{40}$$

**Step 2.** In this step, we will prove that there is a time $t_2 \geq t_1$ such that

$$Z_{n-1}(t) \in Q_{n-1}(Z_{n-1})$$

$$= \left\{ Z_{n-1} : |z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t))| < 1 \right\} \tag{41}$$

for $t \geq t_2$. Similar to the Step 1, first we will prove that there exists a time instance $t_2 \geq t_1$ such that

$$|z_n^{1/n} (t_2) - v_n^{1/n} (Z_{n-2}(t_2))| \leq 1/2 \tag{42}$$

If there is no such $t_2$, it can be assumed that for $t \geq t_1$

$$z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) > 1/2$$

We first consider the case when

$$z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) > 1/2, \quad t \geq t_1, \tag{43}$$

which consequently leads to the estimate of the controller

$$v_n-1 (Z_{n-1}(t)) = - \beta_n^{1/n} (z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t))) \leq -\beta_n^{1/n} (1/2) \tag{44}$$

With the help of (43), we have

$$z_n(t) = \beta_n^{1/n} (v_n(t) + f_n(t))/L^{n-2} + [L z_n(t) - L v_n(t)] \leq -L \beta_n^{1/n} (1/2) \tau_n + L + [L z_n(t) - L v_n(t)] \tag{45}$$

Next, we estimate the terms in (44). Noting that $0 < \tau_n < 1$, it can be concluded that

$$|z_n - v_n| \leq 2^{1 - \tau_n} |z_n^{1/n} - v_n^{1/n} | \leq 2 \tag{46}$$

Substituting (45) into (44) yields that for $t \geq t_1$

$$z_n(t) \leq -L \mu_n$$

with $\mu_n = \beta_n^{1/n} (1/2) \tau_n - 3$, which is a positive constant. As a matter of fact, by the choice of $\beta_n$ in (24), it can be verified

$$\mu_n > 4 (1 + \beta_n) \alpha_n (1, \beta_n, \cdots, \beta_n) > 0 \tag{47}$$

Integrating (46) in $[t_1, t]$ yields

$$z_n(t) \leq z_n(t_1) - L \mu_n (t - t_1) \tag{48}$$

By (48) and the fact that $|v_n(t)| \leq \beta_n$ we obtain from (42) that

$$\frac{1}{2} \leq z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) \leq \left( z_n^{1/n} (t_1) - 4 \mu_n (t_1 - t_1') \right)^{1/2} + \beta_n^{1/n} \tag{49}$$

which implies that $\frac{1}{2} < -\infty$ as time $t$ goes to infinity. This contradiction shows that the assumption (42) will never hold. For the case $z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) < -\frac{1}{2}$, the proof is similar and is omitted here. Hence it can be concluded that there exists $t_2$ such that

$$z_n^{1/n} (t_2) - v_n^{1/n} (Z_{n-2}(t_2)) \leq 1/2 \tag{50}$$

Following the same line of the proof of the Step 1, next we use a contradiction argument to prove that for $t \geq t_2$

$$z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) < 1 \tag{51}$$

Otherwise, we can assume that there exist $t_2' < +\infty$ and $t_2' < +\infty$ such that we have either the passive case as

$$z_n^{1/n} (t_2') - v_n^{1/n} (Z_{n-2}(t_2')) = 1/2 \tag{52}$$

or the negative case as $z_n^{1/n} (t_2') - v_n^{1/n} (Z_{n-2}(t_2')) = -1/2, \quad z_n^{1/n} (t_2') - v_n^{1/n} (Z_{n-2}(t_2')) = -1, \quad -1 \leq z_n^{1/n} (t) - v_n^{1/n} (Z_{n-2}(t)) \leq -\frac{1}{2}, \quad t \in [t_2', t_2^*]. \tag{53}$$

Next, we focus on the positive case (49)-(50)-(51) to show neither case would happen. First, note that condition (51) implies that (46) holds. Integrating (46) yields

$$L \mu_n (t_2' - t_2) \leq z_n(t_2') - z_n(t_2) \tag{54}$$

By (49) and the fact that $|v_n(t)| \leq \beta_n$ we obtain

$$z_n(t_2') \leq (1 + \beta_n) \tag{55}$$
Similarly, by (50), we have
\[ z_{n-1}(t^*_2) \geq -(1 + \beta_{n-2}). \] (54)
Substituting (53) and (54) into (52) leads to
\[ t^*_2 - t^*_1 \leq \frac{2}{L\rho_{n-1}} (1 + \beta_{n-2}) = \frac{2}{L\rho_{n-1}} (1 + \beta_{n-2}). \] (55)

Note that (46) implies \( z_{n-1}(t^*_2) \leq z_{n-1}(t^*_1) \). Hence, we have
\[
\frac{z_{1/r-1}(t^*_1)}{z_{1/r-1}(t^*_2)} = \frac{z_{1/r-1}(t^*_1)}{z_{1/r-1}(t^*_2)} \leq \frac{z_{1/r-1}(t^*_1)}{z_{1/r-1}(t^*_2)} + v_{n-2}^{-1}(Z_{n-2}(t^*_1)) - v_{n-2}^{-1}(Z_{n-2}(t^*_2)) \]
which leads to the following inequality by using (49)-(50),
\[
1/2 \leq |v_{n-2}^{-1}(Z_{n-2}(t^*_2)) - v_{n-2}^{-1}(Z_{n-2}(t^*_1))| \] (56)

On the other hand, by (51) and the fact that \(|v_{1/r-1}^{1/r-1}| \leq \beta_{1/r-1}/(1 + \beta_{1/r-1})\), it can be concluded that for \( t \in [t^*_2, t^*_1], |z_{n-1}(t)| \leq (1 + \beta_{n-2}) \), which, together with the result of Step 1, implies that for \( t \in [t^*_2, t^*_1], |z_{n-1}(t)| \leq (1 + \beta_{n-2}), j = n - 1, n \). Hence with the help of this inequality, we obtain from Lemma 2.2 that
\[
|v_{n-2}^{-1}(Z_{n-2}(t^*_2)) - v_{n-2}^{-1}(Z_{n-2}(t^*_1))| \leq L\alpha_{n-2}(1, \beta_{n-2}, \beta_{n-2})(t^*_2 - t^*_1) \] (57)

Substituting (55) and (57) into (56) yields
\[
1/2 \leq \frac{2}{L\rho_{n-1}} (1 + \beta_{n-2}) \alpha_{n-2}(1, \beta_{n-2}, \beta_{n-2}). \]

By (47), it can be verified\( \frac{2}{L\rho_{n-1}} (1 + \beta_{n-2}) \alpha_{n-2}(1, \beta_{n-2}, \beta_{n-2}) < \frac{1}{2} \) which leads to a contradiction. Therefore the positive case (49)-(50)-(51) will never happen. For the negative case \(-1 < \frac{z_{1/r-1}(t)}{v_{n-2}^{-1}(Z_{n-2}(t))} < -\frac{1}{2}\), the proof is similar to the positive case and thus is omitted here. Hence for \( t \geq t^*_2 \) we can conclude that \(|v_{1/r-1}^{1/r-1}(t) - v_{n-2}^{-1}(Z_{n-2}(t))| < 1\) which implies for \( t \geq t^*_2, \) we have \( Z_{n-1}(t) \in Q_{n-1} \)
\[
\{ Z_{n-1}; |z_{1/r-1}(t)| - v_{n-2}^{-1}(Z_{n-2}(t)) < 1 \}. \]

Proceeding by this way, we can obtain there exists a time instant \( t^n \), after \( t_{n-1} \), \( Z_{n}(t) \) will enter and stay in the set
\[
Q = \{ Z_{n}; z_{1/r}^{1/r}(t)| < 1, \ |z_{1/r}^{1/r}(t) - v_{n-2}^{-1}(z_{1/r}(t))| < 1, \ |z_{n}^{1/r}(t) - v_{n-1}^{-1}(Z_{n-1}(t))| < 1 \}. \]

When \( t \geq t^n \), the saturated control law (11) becomes the unsaturated control law (4). By Lemma 2.1, there is a constant \( L > 1 \) such that the control law (4) globally finite-time stabilizes system (2). As a result, the closed-loop system (1)-(11) is globally finite-time stable. \( \square \)

Remark 2.1. It should be pointed out that the selection of gain \( L \) is determined by the gains \( \beta_{i}'s \). That is to say that we first preset the controller parameters \( \beta_{i}'s \) satisfying condition (24). Then, by Lemma 2.1, we can find a gain condition for \( L \).

3. GLOBAL FINITE-TIME STABILIZATION OF (1)

VIA BOUNDED OUTPUT FEEDBACK CONTROLLER

Since the states \( x_2, \cdots, x_n \) are not available for feedback, the controller (11) is not implementable. Hence, in this section, we will construct an observer whose states will converge to the real values in a finite time. Using the approach in Li et al. (2009), we can design for (1) a homogeneous output feedback stabilizer.

Lemma 3.1. For any constant \( \tau \in (-1/n, 0) \), there exist constants \( a_i, K > 1, i = 1, \cdots, n \), such that the states \( (\hat{x}_1, \cdots, \hat{x}_n) \) of following observer
\[
\dot{x}_i = \hat{x}_i + f_i(\hat{x}_1, \cdots, \hat{x}_i) + K^i a_i (x_i - \hat{x}_i)_{\tau+i}^{\tau},
\]
for \( i = 1, \cdots, n - 1, \)
\[
\dot{\hat{x}}_n = u + f_n(\hat{x}_1, \cdots, \hat{x}_n) + K^n a_n (x_1 - \hat{x}_1)_{\tau+i}^{\tau}
\] (58)
with the constants \( r_i \)'s defined as (3) will converge the the real states \( (x_1, \cdots, x_n) \) in a finite time.

Proof. First, we obtain the error dynamics of (1) and (58) by defining \( e_i = (x_i - \hat{x}_i)/K^{i-1}, i = 1, \cdots, n \)
\[
\dot{e}_i = K^{i-1} f_i(x_1, \cdots, x_i) - f_i(\hat{x}_1, \cdots, \hat{x}_i) - K^{i-1} a_i e_i,
\]
for \( i = 1, \cdots, n - 1, \)
\[
\dot{e}_n = K^{n-1} f_n(x_1, \cdots, x_n) - f_n(\hat{x}_1, \cdots, \hat{x}_n) - K^n a_n e_n
\] (59)

If \( \sum_{k=1}^{n} |x_k - \hat{x}_k| \geq 1 \), by the boundedness of \( f_i \), there is a constant \( c > 0 \) such that
\[
|f_i(x_1, \cdots, x_i) - f_i(\hat{x}_1, \cdots, \hat{x}_i)| \leq c \sum_{k=1}^{n} |x_k - \hat{x}_k|^{\tau+i}/K^{i-1}.
\]

By the C^1 property of \( f_i(\cdot) \) and \( 2^{\tau+i}/K^{i-1} \leq 1, \) in the case when \( \sum_{k=1}^{n} |x_k - \hat{x}_k| \leq 1, \)
\[
|f_i(x_1, \cdots, x_i) - f_i(\hat{x}_1, \cdots, \hat{x}_i)| \leq \dot{c} \sum_{k=1}^{n} |x_k - \hat{x}_k|
\]

\[
\leq \dot{c} \sum_{k=1}^{n} |x_k - \hat{x}_k|^{\tau+i}/K^{i-1}.
\]

Hence,
\[
|f_i(x_1, \cdots, x_i) - f_i(\hat{x}_1, \cdots, \hat{x}_i)| \leq \dot{c} \sum_{k=1}^{n} |x_k - \hat{x}_k|^{\tau+i}/K^{i-1}
\]
\[
\leq \dot{c} \sum_{k=1}^{n} |e_k|^{\tau+i}/K^{i-1}.
\] (60)

As shown in Li et al. (2009), for any \( K > 1 \), there are constants \( a_i, i = 1, \cdots, n \) such that the homogeneous system is globally finite-time stable
\[
\dot{e}_i = K^{i-1} e_i - K^{i-1} a_i e_i, i = 1, \cdots, n - 1, \]
\[
\dot{e}_n = -K^n a_n e_n.
\] (61)

Using the results of Proposition 2.2 in Li et al. (2009), there exists a homogeneous Lyapunov function \( V(e) \) which is homogeneous of degree 2 with the dilation \( (r_1, \cdots, r_n) \) such that
\[
\dot{V}(e)|(61) \leq -K H(e)
\] (62)
where \( H(e) \) is a homogeneous function of degree \( 2 + \tau \) and is a positive definite function. Then by (60) and (62), the derivative of \( V(e) \) along system (59) is
\[
\dot{V}(e)|(59) \leq -K H(e) + c \sum_{i=1}^{n} \left( \frac{\partial V(e)}{\partial e_i} \right) \sum_{k=1}^{n} |e_k|^{\tau+i}/K^{i-1}.
\]
According to homogeneous definition, it is easy to verify \(|V(\epsilon)/\partial \epsilon| \sum_{i=1}^{n} \frac{\tau}{\epsilon_{i}}^{n} \) is homogeneous with a degree of \(2 + \tau\). Hence, by homogeneous system theory, we can conclude that there are two constants \(k_{1}\) and \(k_{2}\) such that \(V(\epsilon)(0) \leq -(Kk_{1} - k_{2})V(\epsilon)/\partial \epsilon\). It follows from Theorem 4.2 in Bhat et al. (2000) that the error dynamic (59) is globally finite-time stable for a large enough \(K\).

**Remark 3.1.** In Perruquetti et al. (2008); Shen et al. (2008, 2009); Menard et al. (2010), the problem of designing finite-time convergent observer was also discussed. However, using these observers only guarantee that there exists a \(\tau \in (-1/n, 0)\) such that the observer error system (61) is finite-time convergent. Using the method of finite-time observer proposed in Li et al. (2009), it can be guaranteed that there exist observer gains such that the observer error system (61) is finite-time convergent for any \(\tau \in (-1/n, 0)\).

Lemma 3.1 shows that the estimated states \((\hat{x}_{1}, \cdots, \hat{x}_{n})\) will converge to the real states \((x_{1}, \cdots, x_{n})\) in a finite time.

By Lemma 3.1, we have the following theorem.

**Theorem 3.1.** The output feedback controller

\[ u(\hat{x}) = L^{n}v_{n}(\hat{Z}_{n}) \tag{63} \]

where \(\hat{Z}_{n} = (\hat{x}_{1}, \hat{x}_{2}/L, \cdots, \hat{x}_{n}/L^{n-1})\) and \(v_{n}(\cdot)\) is defined as (11), together with the finite-time convergent observer (58) render the system (1) globally finite-time stable.

**Example 3.1.** Consider the following nonlinear system:

\[ \dot{x}_{1} = x_{2}, \quad \dot{x}_{2} = x_{3} + \sin(x_{2}), \quad \dot{x}_{3} = u, \quad y = x_{1}. \tag{64} \]

According to Theorem 3.1, a bounded output feedback controller can be designed to globally finite-time stabilize the system. For instance, if we choose \(\tau = -1/5\), then the bounded finite-time output feedback controller can be constructed as

\[ u = -L^{3}\beta_{3}\sigma^{2}\left((\dot{x}_{2}/L)^{2} + \beta_{2}\sigma((\dot{x}_{2}/L)^{2} + \beta_{1}\sigma(\dot{x}_{1}))\right) \]

\[ \dot{\hat{x}}_{1} = \hat{x}_{2} + Ka_{1}(x_{1} - \hat{x}_{1})^{2}, \]

\[ \dot{\hat{x}}_{2} = \hat{x}_{3} + \sin(\hat{x}_{2}) + Ka_{2}(x_{1} - \hat{x}_{1})^{2}, \]

\[ \dot{\hat{x}}_{3} = u + Ka_{3}(x_{1} - \hat{x}_{1})^{2}, \]

with appropriate gains \(L, \beta_{1,2,3}, a_{1,2,3}\) and \(K\). Simulation results are shown in Fig. 1.

---

**REFERENCES**


