Digital Output Feedback Control over Signal-to-Noise Ratio Limited Communication Channels

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Abstract: In this paper we address the problem of finding a stabilizing feedback controller for a linear time invariant (LTI) plant model when measurement is performed over a communication channel model subject to channel input logarithmic quantization and a signal-to-noise ratio (SNR) constraint. The location of the communication channel is therefore on the measurement path, that is between the plant and the controller, nevertheless similar results can be obtained for the alternative location over the control path.

In our first approach the communication channel model itself is selected to be an additive white Gaussian noise (AWGN) channel. We then describe the spectral power density of the channel density as a function of the power spectral density of the channel additive white Gaussian noise and the uniform distributed process model arising from a logarithmic quantization. The infimal channel SNR is then obtained through the definition for the channel input power, together with an interpretation of a bound on the logarithmic quantization relative error as a multiplicative modeling error.

In our second approach we extend the proposed analysis to the class of additive colored Gaussian noise (ACGN) channels with memory. The infimal SNR is then obtained again through the definition for the channel input power. Finally, an extension to the case of sampled-data systems is proposed.

1. INTRODUCTION

Stabilizability, performance and robustness in the context of control over networks have been topics of increased interest in recent years. The most general results in the area that consider the problem of stabilizability call for information theoretic arguments to obtain necessary and sufficient lower bounds on the channel transmission data rate Nair and Evans [2004], Nair et al. [2004], Freudenberg et al. [2006], Nair et al. [2007], Charalambous and Farhadi [2008].

Another line of research introduced a framework to study stabilizability of a feedback loop over channels that have a signal to noise ratio (SNR) constraint Braslavsky et al. [2007], with related work in Rantzer [2006], Bassam and Voulgaris [2005]. Braslavsky et al. [2007] obtained the infimal SNR required to stabilize an unstable plant over a memoryless additive white Gaussian noise (AWGN) channel. A distinctive characteristic of the SNR approach is that it is a linear formulation. For the case of linear time invariant (LTI) controllers and minimum phase LTI plant models with no time delay, these infimal SNR bound retrieves the result derived in Nair and Evans [2004] by application of Shannon’s theorem [Cover and Thomas, 1991, §10.3].

We consider here a logarithmic quantizer, see Widrow and Kollár [2008], as post-signal processing involved in the communication link. As a consequence of neglecting most pre- and post- signal processing we do not deal directly with the transmission data rate of the channel, nor are involved with rate distortion theory (see for example [Gallager, 1968, Ch.9]).

Our first contribution in the present paper is to extend on the results of Rojas et al. [2006] by considering a channel output logarithmic quantizer. The quantizer is located at the output of the communication channel under the assumption that the control feedback loop is digital and thus its signals belong to a given alphabet. The channel model first investigated is the memoryless AWGN channel which, due to its additive noise process, requires its output to be quantized in order to guarantee that the received signal is compatible with the digital control feedback loop. We approach the presence of logarithmic quantization as a worst case scenario, and through a robust control argument, consider an upper bound on it as equivalent to a multiplicative modeling error. As a result we obtain a closed-form expression for the infimal LTI SNR subject to logarithmic quantization for a memoryless AWGN channel.

Our second contribution is to quantify the required infimal SNR subject to logarithmic quantization at the channel output for an additive colored Gaussian noise (ACGN) channel with memory. The infimal SNR subject to log-
arithmic quantization is then obtained again through a robust control approach.

As a third contribution we extend the analysis to the case of sampled-data models, where the digital-to-analog converter (DAC) and analog-to-digital converter (ADC) contains each a logarithmic quantizer.

![Diagram of output feedback control stabilization of a discrete-time unstable plant subject to logarithmic quantization error over a discrete-time channel.](image)

Fig. 1. Output feedback control stabilization of a discrete-time unstable plant subject to logarithmic quantization error over a discrete-time channel.

The present paper is organised as follows: in Section 2 we introduce the preliminaries for the present work, as well as the standard setting for the SNR problem. In Section 3 we adapt and solve the SNR problem subject to channel output logarithmic quantization for the memoryless AWGN channel case. On the other hand, in Section 4 we adapt and solve the SNR problem subject to channel output logarithmic quantization for the ACGN channel with memory. In Section 5 we further discuss the results and their extensions to the sampled-data model scenario and in Section 6 we conclude with final remarks for the present work.

2. PRELIMINARIES

We consider the discrete-time feedback system depicted in Figure 1.

2.1 Assumptions

General assumptions involved in the present discussion, which will be in place unless stated otherwise, are

**Plant model assumptions.** Throughout the present work, if not stated otherwise, it is assumed that the plant model $G(z)$ is a real rational function with the following properties:
- relative degree $n_p = 1$.
- $m$ unstable poles, $|\rho_i| > 1$, with multiplicity $1$, $\forall i = 1, \cdots, m$.
- $q$ non-minimum phase zeros, $|\zeta_i| > 1$, each with multiplicity $1$, $\forall i = 1, \cdots, q$.
- $m$ NMP zeros, that is $\{\eta_1, \cdots, \eta_k, \eta_{m+1}, \cdots, \eta_{m+q}\}$

**Channel model assumptions.** The communication channel is characterized by two parameters: The admissible channel input power, $P$, and the channel additive noise process $n(k)$.

**Channel additive noise process.** The channel additive noise process is labelled $n(k)$ and it is a zero-mean i.i.d. Gaussian white noise process with variance $\sigma_n^2$.

Quantizer assumptions. We adopt the approach presented in Qi et al. [2009], which in terms refers to Elia and Mitter [2001] and Fu and Xie [2005]. Denote the quantized signal by

$$\epsilon(k) = Q(r(k)) = r(k) + \varepsilon(k),$$

where $\varepsilon(k)$ is the quantization error. The quantization density is defined as a factor $\varrho$, with $0 < \varrho < 1$. It now follows that the quantization error can be modeled as $\varepsilon(k) = r(k)|\delta(k)|$, where $\delta(k)$ is the relative quantization error. We assume $\delta(k)$ to be independent from the quantizer block input signal $r(k)$ (and thus independent from the channel additive noise $n(k)$) and to be uniformly distributed over $[-\Lambda, \Lambda]$ with $\Lambda = (1 - \varrho)/(1 + \varrho)$. Notice that this also implies that $\mathbb{E}\{\delta(k)\} = 0$ and $\sigma_{\delta}^2 = \mathbb{E}\{\delta^2(k)\} = \Lambda^2/3$.

2.2 Problem Definition

We now propose what we regard as the standard SNR problem setting. We assume that the controller $C(z)$ is such that the closed-loop system is stable in the sense that, for any distribution of initial conditions, the distribution of all signals in the loop will converge exponentially fast to a stationary distribution. The channel input power, defined by

$$\|s\|_{\text{wor}}^2 \triangleq \lim_{k \to \infty} \mathbb{E}\{s^2(k)\},$$

Under reasonable stationarity assumptions [Åström, 1970, §4.4], the power in the channel input may be computed as

$$\mathbb{E}\{s^2\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{sn}(\omega)\omega^2d\omega,$$

where $\mathbb{E}\{s^2\}$ stands for $\lim_{k \to \infty} \mathbb{E}\{s^2(k)\}$ and it is introduced to easy the notation. The term $S_{sn}(\omega)$ represents the component of the channel input power spectral density that is independent of the noise $n(k)$. The channel input is required to satisfy an imposed power constraint such that $P > \mathbb{E}\{s^2\}$, for some predetermined power level $P$, therefore

$$P \sigma_n^2 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{sn}(\omega)d\omega. \quad (1)$$

From (1) we observe that a fundamental limitation on the channel SNR will then be given by the infimum of the integral expression containing $S_{sn}(\omega)$, which is at the core of the infimal SNR problem definition that follows.

**Problem 1. (Infimal LTI SNR Subject to Logarithmic Quantization).** Find a proper rational stabilizing controller $C(z)$ such that the feedback control loop is stable and achieves the infimum admissible channel SNR in (1).

3. MEMORYLESS AWGN CHANNEL

In this section we consider the channel model to be a memoryless AWGN channel, see Figure 2, and for the proposed relative quantization error we consider the worst case scenario which states that $|\delta(k)| < \Lambda$. We then have that the relationship between the channel input and the channel additive noise is given by

$$T_{sn}(z) = \frac{(1 + \Delta)C(z)G(z)}{1 + (1 + \Delta)C(z)G(z)} = \frac{1 + \Delta T_{sn}(z)}{1 + \Delta T_{sn}(z)}, \quad (2)$$

where $T_{sn}(z)$ is the complementary sensitivity achieving the infimal SNR for $\Delta = 0$ and $\Delta = \pm \Lambda$, that is the worst
Fig. 2. Output feedback control stabilization of a discrete-time unstable plant over a memoryless AWGN channel subject to channel output logarithmic quantization.

case scenario for the quantization effect. From Rojas et al. [2007] we know that $T_o(z)$ to be given by

$$T_o(z) = B_c(z)B_p(z)\sum_{i=1}^m z - \rho_i,$$

(3)

with $r_i = [(z - \rho_i)B_p^{-1}(z)B_c^{-1}(z)]_{z=\rho_i}$, and

$$B_c(z) = \prod_{i=1}^q \frac{z - \zeta_i}{1 - z\zeta_i}, \ B_p(z) = \prod_{i=1}^q \frac{z - \rho_i}{1 - z\rho_i},$$

the Blaschke products for the plant NMP zeros and the plant unstable poles respectively. Notice that from the definition of $T_o(z)$ we can directly obtain the definition for the sensitivity function as $S_o(z) = 1 - T_o(z)$.

**Theorem 2.** (Memoryless AWGN Channel Infimal SNR Subject to Logarithmic Quantization) Consider the discrete-time LTI output feedback represented in Figure 2 and that all the elements of the discrete-time LTI output feedback satisfy the assumptions listed in Section 2 and the present section. The channel SNR is then lower bounded by

$$\frac{P}{\sigma^2} \geq \sum_{i=1}^m \sum_{j=1}^m r_i r_j + \Lambda^2 \left( \sum_{i=1}^m \sum_{j=1}^m t_i t_j - \sum_{i=1}^m \sum_{j=1}^m \frac{g_i g_j}{1 - w_i w_j} \right),$$

(4)

for $\Delta = \Lambda$ and lower bounded by

$$\frac{P}{\sigma^2} \geq \sum_{i=1}^m \sum_{j=1}^m \frac{r_i r_j}{\rho_i \rho_j - 1} + \Lambda^2 \left( \sum_{i=1}^m \sum_{j=1}^m t_i t_j - \sum_{i=1}^m \sum_{j=1}^m \frac{h_i h_j}{1 - w_i w_j} \right),$$

(5)

for $\Delta = -\Lambda$. Furthermore each coefficient is given by

$$r_i = \left[ \left( z - \rho_i \right) B_p^{-1}(z) B_c^{-1}(z) \right]_{z=\rho_i},$$

$$t_i = \left[ \left( z - \eta_i \right) S_o(z) T_o(z)/(1 + \Delta T_o(z)) \right]_{z=\eta_i},$$

$$g_i = \left[ \left( z - \zeta_i \right) S_o(z) T_o(z)/(1 + \Delta T_o(z)) \right]_{z=\zeta_i},$$

$$h_i = \left[ \left( z - w_i \right) S_o(z) T_o(z)/(1 - \Delta T_o(z)) \right]_{z=w_i},$$

and $z_i$ and $w_i$ are the $m + q$ solutions of the following polynomials

$$\prod_{i=1}^q (1 - z_i) \prod_{i=1}^m (1 - z_i \rho_i),$$

$$\prod_{i=1}^q (1 - z_i) \prod_{j=1, j\neq i}^m (1 - z_j \rho_j) \Lambda = 0$$

for $\Delta = \Lambda$ and

$$\prod_{i=1}^q (1 - z_i) \prod_{j=1, j\neq i}^m (1 - z_j \rho_j) \Lambda = 0$$

for $\Delta = -\Lambda$. The proof conclude by directly evaluating in closed-form the resulting squared $H_2$ norm which gives (4) and (5) respectively.

Observe then that from Theorem 2 we obtain the solution to Problem 1 as the maximum bound between the result in (4) and (5). That is the channel SNR is lower bounded by

$$\frac{P}{\sigma^2} \geq \sum_{i=1}^m \sum_{j=1}^m \frac{r_i r_j}{\rho_i \rho_j - 1} + \Lambda^2 \max \left\{ \left\| \frac{S_o T_o}{1 + \Delta T_o} \right\|_2^2, \left\| \frac{S_o T_o}{1 - \Delta T_o} \right\|_2^2 \right\}.$$  

**Example 3.** Consider a minimum phase plant model with one unstable pole $\rho$. The result from Theorem 2 predicts in this case that

$$\frac{P}{\sigma^2} \geq \rho^2 - 1 + \Lambda^2 \left( \rho^2(\rho^2 - 1)/(1 + \Lambda \rho^2) \right)$$

for $\Delta = \Lambda$ and

$$\frac{P}{\sigma^2} \geq \rho^2 - 1 + \Lambda^2 \left( \rho^2(\rho^2 - 1)/(1 - \Lambda \rho^2) \right)$$

for $\Delta = -\Lambda$. We can verify that for robust stability in the first case we have to satisfy $\Lambda \leq 1/|\rho - 1|$, whilst in the second case we have to satisfy $\Lambda \leq 1/|\rho + 1|$. Notice then that for the present example the infimal SNR in this...
proposed worst case scenario is given by the choice of \( \Delta = -\Lambda \), see Figure 3. The fact that \( \Lambda \in [0, 1] \) in Figure 3 is a direct consequence of its definition in Section 2. Also notice, as intuition would have it, that as \( \Lambda \to 0 \) the infimal SNR recovers the known result for plain stabilizability of \( \rho^2 = 1 \).

**Remark 4.** In this section, and in the present paper in general, we address the case of a communication channel located on the measurement path. However, it can be observed that for a memoryless AWGN channel over the control path, the resulting \( T_{sn}(z) \) transfer function relating the channel input to the channel additive noise in closed loop, would be the same as in (2). Therefore, the result from Theorem 2 applies independently from the channel location.

**4. ACGN CHANNEL WITH MEMORY**

![ACGN Channel Diagram](image)

Fig. 4. Output feedback control stabilization of a discrete-time unstable plant over a discrete-time ACGN channel with memory subject to channel output logarithmic quantization.

In the present section we extend the proposed analysis to include the case of an ACGN channel with memory. As for the case of a memoryless AWGN channel, the ACGN channel with memory is also characterized by two parameters: the admissible input power level of the channel, \( P \), and the channel additive noise process \( n(k) \). The difference now is that the ACGN channel with memory also contains two LTI transfer functions \( F(z) \), to model the channel bandwidth, and \( H(z) \), coloring the noise \( n(k) \) (see Figure 4).

Standing assumptions for \( F(z) \) and \( H(z) \) are:

- **Channel bandwidth model assumptions.** The LTI transfer function \( F(z) \) is a stable, biproper, minimum phase transfer function. When required we identify the numerator and denominator of \( F(z) \) as \( F(z) = a_f(z)/b_f(z) \).

- **Channel coloring model assumptions.** The LTI transfer function \( H(z) \) is a stable, biproper, minimum phase LTI transfer function. When required we identify the numerator and denominator of \( H(z) \) as \( H(z) = a_h(z)/b_h(z) \).

**Remark 5.** The biproper requirement is without loss of generality since we can always “add” fast poles at \( z = 0 \) to compensate for any relative degree different from zero in \( F(z) \) and \( H(z) \).

For the proposed quantizer model we have that the relationship between the channel input and the channel additive noise in closed loop is given by

\[
T_{sn}(z) = -\frac{1 + \Delta T_r(z)}{1 + \Delta T_r(z)} H(z) F^{-1}(z),
\]

where \( T_r(z) = C(z) G(z) F(z) / (1 + C(z) G(z) F(z)) \) is the complementary sensitivity achieving the infimal SNR for \( \Delta = 0 \). From Rojas et al. [2007] we know that this particular complementary sensitivity is given by

\[
T_r(z) = F(z) H^{-1}(z) B_c(z) B_p(z) \sum_{i=1}^{m} \frac{r_i}{z - \rho_i},
\]

with \( r_i \) adapted to the presence of the channel bandwidth and coloring models, that is

\[
r_i = \left( z - \rho_i \right) B_p^{-1}(z) B_c^{-1}(z) F^{-1}(z) H(z) \left|_{z=\rho_i} \right.
\]

A similar reasoning to the one developed in the previous section can then be applied to the present channel model and results in the next theorem.

**Theorem 6. (ACGN Channel with Memory Infimal SNR Subject to Logarithmic Quantization)** Consider the discrete-time LTI output feedback represented in Figure 4 and that all the elements of the discrete-time LTI output feedback satisfy the assumptions listed in Section 2 and the present section. The channel SNR is then lower bounded by

\[
\frac{P}{\sigma^2} \geq \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{r_i \rho_j}{\eta_i \rho_j} - 1 + \Lambda^2 \left( \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} t_i t_j \right) + \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} \frac{g_i g_j}{\eta_i + \bar{z}_j} + \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} \frac{g_i g_j}{1 - z_i \bar{z}_j} \right)
\]

for \( \Delta = \Lambda \) and lower bound by
\[ P \frac{2}{\sigma^2} \geq \sum_{i=1}^{m} \sum_{j=1}^{m} \rho_i \rho_j - 1 + \Lambda^2 \left( \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} \frac{t_i \bar{t}_j}{\eta_i \bar{\eta}_j} - 1 \right) + \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} \frac{t_i \bar{t}_j}{\eta_i \bar{\eta}_j} + \sum_{i=1}^{m+q} \sum_{j=1}^{m+q} \frac{h_i \bar{h}_j}{\omega_i \bar{\omega}_j} \]  

for \( \Delta = -\Lambda \). Furthermore each coefficient is given by

\[ r_i = \left[ (z - \rho_i) \mathcal{B}_p^{-1}(z) \mathcal{B}_\zeta^{-1}(z) F^{-1}(z) H(z) \right]_{z=\rho_i}, \]

\[ t_i = \left[ (z - \eta_i) S_0(z) T_0(z) F^{-1}(z) H(z) / (1 + \Delta T_0(z)) \right]_{z=\eta_i}, \]

\[ g_i = \left[ (z - z_i) S_0(z) T_0(z) F^{-1}(z) H(z) / (1 + \Lambda T_0(z)) \right]_{z=z_i}, \]

\[ h_i = \left[ (z - w_i) S_0(z) T_0(z) F^{-1}(z) H(z) / (1 - \Lambda T_0(z)) \right]_{z=w_i}, \]

and \( z_i \) and \( w_i \) are the \( m + q \) solutions of the following polynomials

\[ a_f(z) b_k(z) \prod_{i=1}^{m} (1 - z \bar{\zeta}_i) \prod_{j=1}^{m} (1 - z \bar{\rho}_j) \]

\[ + b_f(z) a_k(z) \prod_{i=1}^{m} (z - \zeta_i) \sum_{j=1}^{m} \sum_{j=1}^{m} (z - \rho_j) \Lambda = 0 \]

\[ a_f(z) b_k(z) \prod_{i=1}^{m} (1 - z \bar{\zeta}_i) \prod_{j=1}^{m} (1 - z \bar{\rho}_j) \]

\[ - b_f(z) a_k(z) \prod_{i=1}^{m} (z - \zeta_i) \sum_{j=1}^{m} \sum_{j=1}^{m} (z - \rho_j) \Lambda = 0 \]

**Proof.** The present proof follows the same steps as the proof for Theorem 2 with the addition of the channel bandwidth and coloring models \( F(z) \) and \( H(z) \). \( \square \)

**Example 7.** We consider in this example a plant model with an unstable pole at \( \rho = 1.5 \) and a NMP zero at \( \zeta = 10 \). The ACGN channel model with memory is modeled by

\[ F(z) = \frac{5 z - 0.2}{8 z - 0.5}, \quad H(z) = \frac{7 z - 0.1}{9 z - 0.3} \]

The resulting SNR bounds predicted by Theorem 6 can be then observed in Figure 5 as functions of \( \Lambda \). As for the memoryless AWGN channel case, we also observe here that the infimal SNR, in this worst case scenario analysis, is given by the choice of \( \Delta = -\Lambda \), as well as observe that as \( \Lambda \rightarrow 0 \) we regain infimal SNR for stabilizability of \( (p^2 - 1) \mathcal{B}_\zeta^{-1}(p) F^{-1}(p) H(p) \).

**5. SAMPLED-DATA MODELS**

As a first reasonable approximation we consider the holder block, together with the continuous-time plant model \( G(z) \) and the sampler block, with sampling time \( T \), to be an equivalent discrete-time plant model \( G(z) \) (see Figure 7 for more details). Notice that the assumptions presented in Section 2 are still in place. Indeed, as a result of the holding and sampling process, it is known that continuous-time plant models of arbitrary relative degree will be mapped into discrete-time plant models with relative degree \( n_g = 1 \). The change of relative degree is in general due to the introduction of sampling non-minimum phase zeros. Nonetheless, this unwanted characteristics can be avoided if we consider, for example, results such as [Yuz et al. 2004] where these sampling zeros are instead mapped inside the unit circle. Finally, we approximate each quantizer in Figure 7 as explained in Section 2. The resulting transfer function relating the channel input with the channel noise in close loop is then given by

\[ T_{sn}(z) = \frac{(1 + \Delta)^T C(z) G(z)}{1 + (1 + \Delta)^T C(z) G(z)} = \frac{(1 + \Delta_{eq}) T_o(z)}{1 + \Delta_{eq} T_o(z)}, \]

where we have further assumed that each quantizer shares the same value of \( \rho \), its quantization density and that

\[ \Delta_{eq} = 3 \Delta + 3 \Delta^2 + \Delta^3. \]

Again, for this equivalent interpretation of the effect of logarithmic quantization as modeling error in a worst case scenario we observe that
Thus, we observe for example that for sampled-data systems we can adapt directly the result from Theorem 2, keeping in mind though the fact that the positive and negative of the equivalent modeling error $\Delta eq$ are not symmetrical anymore. That is the channel SNR is lower bounded by

$$\frac{P}{\sigma^2} \geq \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{r_i r_j}{\rho_i \rho_j - 1} + \max \left\{ A_n^2 \left\| \frac{S_o T_o}{1 - A_n T_o} \right\|_2^2, A_p^2 \left\| \frac{S_o T_o}{1 + A_p T_o} \right\|_2^2 \right\}.$$ 

6. CONCLUSION

In this paper we have studied the problem of finding a stabilizing feedback controller for a linear time invariant (LTI) plant model when measurement is performed over a communication channel model subject to channel input logarithmic quantization and subject to a signal-to-noise ratio (SNR) constraint.

In our first approach the communication channel model itself was selected to be an additive white Gaussian noise (AWGN) channel for which we obtained the infimal channel SNR subject to channel output logarithmic quantization. To do this we proposed an interpretation of the logarithmic quantization relative error as a source of plant modeling error and thus approached the problem from the point of view of robust control.

In our second approach we extended the proposed analysis to the class of additive colored Gaussian noise (ACGN) channels with memory and obtained the infimal SNR subject to channel output logarithmic quantization. We then introduced the case of sampled-data systems and observed that through a straightforward modification, the robust control approach can also be employed in this setting.

Finally, the two proposed examples suggest that, in a worst case scenario, the lower bound in the relative quantization error is a more demanding imposition on the channel SNR than the upper bound. Nevertheless more research is needed to clarify if this observation is a one-off or a more general behavior.

REFERENCES