On Ulam’s Problem of Path Planning, and “How to Move Heavy Furniture”

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Abstract: Using a combination of geometric insight and the minimum principle, a more concise solution to a problem posed by Ulam is given. This problem concerns the path planning of a rod in such a way that the sum of the distances traveled by both endpoints is minimized. It belongs to the class of scalable optimal control (SOC) problems. As time plays no role it is invariant with respect to time scaling, so that impulsive arguments can be used. The following result is obtained: Any optimal bi-path consists of either a glide, a rotation, a combination of the two, or - at most -, a glide sandwiched between two rotations or a rotation sandwiched between two glides related by permutation of their glide angles.

A glide-ellipse characterizes the solution. One manifestation of this problem is in moving objects against static friction forces, hence the subtitle.

1. INTRODUCTION: ULAM’S PROBLEM

In [Ulam, 1960, p.79] the following problem was posed:

Suppose two segments are given in the plane, each of length one. One is asked to move the first segment continuously, without changing its length to make it coincide at the end of the motion with a second given interval in such a way that the sum of the lengths of the two paths described by the end points should be a minimum. What is the general rule for this minimum motion?

While the Dubins and Reeds-Schepp problem come to mind Dubins [1957], Reeds and Schepp [1990], Verriest [1998], this problem is more intricate. Envision the object as a bicycle-like vehicle, but with two independent steering wheels. The paths traveled by these wheels, the bi-path length, is to be minimized. The problem is relevant for slow motion on sand where static friction is dominant. For instance, problems Courtland [2009] with the Spirit/Opportunity rovers on Mars suggest a potential interest in minimizing the wheel rotation. Likewise, the present setup leads to a solution of the minimum work problem for dragging a heavy armoire without casters over a floor. Indeed, the work done equals the constant friction force (see Feynman et al., 1963, p. 12-4) integrated over the path of the feet.

Under various assumptions on the class of extremals, a solution was given in Gurevich [1976] and Goldberg [1973]. Dubovitskii discussed the problem for the motion of a segment in \( \mathbb{R}^n \), but for which the endpoints lie on prescribed surfaces. His solution, without any a priori assumptions, is based on the integral maximum principle (see Dubovitskii [1985]). A whole chapter is devoted to the construction of an “atlas” of all possible extremals. A new solution based entirely on Cauchy’s surface area formula is given in Icking et al. [1993]. Velocity constrained minimum time paths are also not necessarily the shortest bi-paths.

This paper elaborates on a simpler solution method explored in Verriest [2008]. The objective is to show how a good choice of parameterization together with a combination of the classical maximum principle and high school geometry may lead to a complete analytical characterization and geometric construction. These characterizations and properties add to a growing family of optimal path planning solutions. See also Verriest [2010, 2011].

2. CONFIGURATION SPACE AND REACHABILITY

Refer to the endpoints of the rod as a red (R) and a blue (B) one. Parameterize the problem by the coordinates, \((x, y)\), of one of the end points, say B, of the rod and the orientation of BR with respect to a reference (say the horizontal), \(\theta\). The motion problem is then the transfer:

\[
[(x_b, y_b), \theta] \rightarrow [(x'_b, y'_b), \theta']
\]

in some optimal way.
The rigidity constraint imposes \((x_r - x_b)^2 + (y_r - y_b)^2 = 1\), from which also \(x_b = x\), \(y_b = y\), \(x_r = x + \cos \theta\) and \(y_r = y + \sin \theta\). For this problem, the natural state space is: \(X = \mathbb{R}^2 \times S^1\), and the equations of motion are given by \(\dot{x} = u\), \(\dot{y} = v\), and \(\dot{\theta} = \omega\). The variables \(u\), \(v\) and \(\omega\) are the free control variables.

2.1 Two types of elementary motions

Let us first define two types of motions:

(i) Rotation about \(B\) over an angle \(\theta\): \(\text{Rot}_B(\theta)\). The rotation \(\text{Rot}_B(\theta)\) is similar.

(ii) Glide \(\text{Gli}_{b,r}\): Slide endpoints \(B\) and \(R\) respectively along \(b\) and \(r\). If \(b\) and \(r\) are parallel, at most 1 unit distant from each other, any glide \(\text{Gli}_{b,r}\) results in a new position \(B'R'\) of the rod, parallel to its initial position.

If \(b\) and \(r\) are not parallel, the rigidity imposes limit positions. Moreover, the endpoints do not necessarily move unidirectionally. A point, \(P\), on \(\{P\} = \text{span}\{b\} + \text{span}\{r\}\) is a turning point for \(R\) (if \(\alpha = \angle R'RB\), \(\theta = \angle R'BR\)). The corresponding point \(B_r\) \((r=\alpha)\) reached by combining rotations and glides.

2.2 Reachability

It was shown in Verriest [2008] that the system on \(X = \mathbb{R}^2 \times S^1\) is completely reachable: Given \(BR\), any \(B'R'\) can be reached by combining rotations and glides.

Because of possible turning points, \(BB' + RR'\) may only be a lower bound for the bi-pth length from configuration \(BR\) to \(B'R'\). Moreover, a single glide may not suffice to align \(BR\) with \(B'R'\).

3. PROBLEM FORMULATION

Minimize of the performance index, \(J = \int_0^1 (|dS_B| + |dS_R|)\). In terms of the chosen parameterization, this is given by

\[
J = \int_0^1 \left( \sqrt{u^2 + v^2} + \sqrt{(u - \omega \sin \theta)^2 + (v + \omega \cos \theta)^2} \right) dt.
\]

(1)

Given the initial position, \(BR\), and the final position, \(B'R'\), two special cases are solvable in a purely geometric way Verriest [2008].

(i) If the initial and final position of one endpoint are equal, say, \(R=R'\), then the optimal motion is a rotation about this point \(R\).

(ii) Let the initial and final position of the rod are completely disjoint, i.e., \(\text{let } BR \cap B'R' = \emptyset\). Define the lines, \(r=RR'\) and \(b=BB'\). If \(b\) and \(r\) are parallel, then necessarily \(RB\) // \(B'R'\). The obvious optimal path is the parallel glide \(B \rightarrow B'\), and \(R \rightarrow R'\), as no path beats the Euclidean distance \(BB' + RR'\).

In all other cases, \(r=RR'\) and \(b=BB'\) are crossing, say at \(O\) with an angle \(\gamma \neq 0\). Let \(r\) be directed positively from \(R\) (initial end point) to \(R'\) (final end point). Let \(\alpha\) be the angle \(R'RB\), i.e., \(\alpha\) from \(R'R\) to \(RB\), and likewise \(\alpha'\) the angle from \(r\) to \(R'B'\). Then it is readily seen that \(RB\) can glide along \(r\) and \(b\) to \(R'B'\) without any of the endpoints retracing part of their path if either \(\alpha = \frac{\pi}{2} < \alpha' \leq \frac{\pi}{2} - \gamma\), or \(-\frac{\pi}{2} \leq \alpha' \leq \frac{\pi}{2} - \gamma\). In the first case \(RB\) rotates \(\text{CCW}\) about as \(R\) moves towards \(R'B'\) (Figure 3). In the other case, the rotation is \(\text{CW}\). Thus under these conditions the optimal path is a single glide corresponding to the minimal Euclidean distance. The points associated with these limit cases are the turning points (when \(\alpha = \frac{\pi}{2}\) and complementary turning points defined in Section 2.1).

![Fig. 3. Simple glide](image)

4. OPTIMALITY AND EULER-LAGRANGE EQUATIONS

We apply now the minimum principle of Pontryagin. The Hamiltonian for Ulam’s problem is

\[
H = \sqrt{u^2 + v^2} + \sqrt{(u - \omega \sin \theta)^2 + (v + \omega \cos \theta)^2} + \lambda_x u + \lambda_y v + \lambda_\omega \omega
\]

(2)

Note that \(H\) is convex in \(u\) and \(v\), and differentiable except at the tips \((u=v=0)\) and \((u = \omega \sin \theta, v = -\omega \cos \theta)\). As function of \(u\) and \(v\), \(H\) is minimal at

- one of these tips (non-differentiable cases)
- point satisfying \(\frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0\) (differentiable case).

The adjoint equations (Euler-Lagrange equations) for the problem are

\[
\begin{align*}
\lambda_x &= -\frac{\partial H}{\partial x} = 0 \quad (3) \\
\lambda_y &= -\frac{\partial H}{\partial y} = 0 \quad (4) \\
\lambda_\omega &= -\frac{\partial H}{\partial \theta} = \\
&\quad \frac{(u - \omega \sin \theta)\omega \cos \theta + (v + \omega \cos \theta)\omega \sin \theta}{\sqrt{(u - \omega \sin \theta)^2 + (v + \omega \cos \theta)^2}} \quad (5)
\end{align*}
\]

Before blindly solving these equations, we give a geometric interpretation. First, (3) and (4) imply that \(\lambda_x\) and \(\lambda_y\)
remain constant along the optimal path. In terms of $\alpha_r$, the heading of endpoint R, (5) may be reorganized as
\[
\dot{\lambda}_\theta = \omega (\cos \alpha_r \cos \theta + \sin \alpha_r \sin \theta) = \omega \cos(\theta - \alpha_r).
\] (6)
Note that the angle $\alpha_r - \theta$ is the heading of R with respect to the body BR of the rod. Equation (6) implies also that $\dot{\lambda}_\theta = 0$ iff $\omega = 0$ (no rotation) or $\theta - \alpha_r = \pm \frac{\pi}{2}$. Thus motivated, we continue with the details.

4.1 Nondifferentiable cases

Case 1: $u = v = 0$. The Hamiltonian is $H = |\omega| + \lambda_\theta \omega$.
There are 5 subcases for $\omega^*$, the optimal $\omega$:

(i) $\lambda_\theta < -1$, $\omega^* \rightarrow -\infty$
(ii) $\lambda_\theta = -1$, $\omega^* \in [0, \infty)$
(iii) $|\lambda_\theta| < 1$, $\omega^* = 0$
(iv) $\lambda_\theta = 1$, $\omega^* \in (-\infty, 0]$
(v) $\lambda_\theta > 1$, $\omega^* \rightarrow \infty$.

(i) and (v) meaningless, as it corresponds to a path that is discontinuous.

(ii) implies that the rod remains motionless, which is a non-solution.

We may thus conclude that for the two border line cases $\lambda_\theta = \pm 1$, the rate of rotation only being constrained by its sign.

Case 2: $u = \omega \sin \theta, v = -\omega \cos \theta$.
The Hamiltonian is $H = |\omega| + (|\lambda_\theta| + (\lambda_\theta \sin \theta - \lambda_\theta \cos \theta))|\omega|$, which is of Case 1 form, but with $\lambda_\theta$ augmented by $(\lambda_\theta \sin \theta - \lambda_\theta \cos \theta)$. The conclusion is similar.

These two cases are easily interpreted as rotations about B or R respectively.

Conclusion

A turn about B or R or any point between B and R is a potential optimal segment of the solution.

Note that if the value function is the same for both tips, all points on the line segment connecting the tips are also potential optima. It is easily verified that the corresponding motion is a rotation about such a point, however this (singular) control may be decomposed into elementary motions having the same total bi-path length.

4.2 Differentiable Case

If a minimizer exists outside the segment $(0,0)$ to $(\omega \sin \theta, -\omega \cos \theta)$, where $H$ is convex and differentiable, it must be unique. It is found by setting the partial derivatives of $H$ w.r.t. $u, v$ and $\omega$ zero.

First, (3) and (4), substituted in the smooth optimality conditions, imply the constancy of both

\[
\frac{u}{\sqrt{u^2 + v^2}} + \frac{u - \omega \sin \theta}{\sqrt{(u - \omega \sin \theta)^2 + (v + \omega \cos \theta)^2}}
\]
and
\[
\frac{v}{\sqrt{u^2 + v^2}} + \frac{v + \omega \cos \theta}{\sqrt{(u - \omega \sin \theta)^2 + (v + \omega \cos \theta)^2}}.
\]

In terms of the headings, $\alpha_b$ and $\alpha_r$ of B and R, the sums $\cos \alpha_b + \cos \alpha_r = \lambda_x$ and $\sin \alpha_b + \sin \alpha_r = \lambda_y$, remain constant.

Geometric interpretation: Consider thus

\[
\cos \alpha_b + \cos \alpha_r = \xi
\]
\[
\sin \alpha_b + \sin \alpha_r = \eta
\]

Given $\xi$ and $\eta$, these equations have either zero or two solutions (one in the degenerate case). Indeed, circles of radius one, centered at O and $(\xi, \eta)$, either intersect or are disjoint. If $\xi^2 + \eta^2 > 4$, there is no solution. The case $\xi^2 + \eta^2 = 4$ is the degenerate case: giving equal angles. This corresponds with a parallel translation of the rod.

Finally, the case $0 < \xi^2 + \eta^2 < 4$ implies that if $(\alpha^*_b, \alpha^*_r)$ is one solution, its permutation, $(\alpha_{r}^{*}, \alpha_{b}^{*})$ is the second solution (Figure 4). Note that if $\xi = \eta = 0$, the rigidity constraint cannot be satisfied.

Conclusion

A glide (endpoints traveling in straight line segments) is potentially a segment of the optimal solution.

The double solution hints that abrupt switching between these two angles could be a possibility. However, the optimality condition, $H_\omega = 0$ states that along the optimal path

\[
\dot{\lambda}_\theta = \sin(\theta - \alpha_r).
\] (7)

Compatibility between (6) and (7) implies $\dot{\theta} - \dot{\alpha}_r = \omega$.

Fig. 4. Path Angles cannot be optimal

Hence $\dot{\alpha}_r = 0$. This precludes a zig zag motion, as shown in Figure 5, from being optimal.

Equation (7) implies also that in the nondifferentiable case $\lambda_\theta = 0$ if $\lambda_\theta = \pm 1$, i.e. a rotation initiates or halts when $\lambda_\theta$ reaches $\pm 1$. Since the heading of the endpoint in motion is then perpendicular to the rod, this corresponds to the turning points. We summarize the result of this section as Theorem 1. The solution to Ulam’s problem is comprised of

- pure rotations Rot$_R$ or Rot$_B$ about an endpoint (with $\lambda_\theta = \pm 1$),
- glides Gli$_{(b,r)}$ with $|\lambda_\theta| < 1$ (headings of the endpoints are constant).

If the optimal motion contains more than one slide, then its paths are either equal or complementary: $\alpha_r = \alpha_b$ and $\alpha_b = \alpha_r$. A rotation separates them.
5. GEOMETRY

At this point, the stage is set to proceed with a purely geometric solution to the problem, without any reference to dynamics. To see how this may come about, we use a scaling argument. In the setup of the problem, there were no constraints on the speed of the motion, nor the time required to accomplish it (the integration interval in the performance index).

5.1 Scalable Optimal Control Problem (SOC)

We note the system equations and performance index satisfy the more general SOC-conditions: \( L(x, \mu u) = \mu L(x, u) \), and \( f(x, \mu u) = \mu f(x, u) \) for all \( \mu > 0 \), where \( x \) and \( u \) are respectively the state and control vector.

**Theorem 2.** A finite horizon, SOC problem can be reduced to an impulsive control problem. (i.e., time plays no role except for sequencing of segments)

The usefulness of this result lies on the fact that it suffices to work with the effects of impulsive actions (inasmuch these can be easily computed). One may then completely forget about the dynamics of the system, and do geometry.

**Idea of proof:**

\[
Dx = f(x, u) \Rightarrow DS_p x = f(S_p x, \mu u)
\]

where \( D \) is the differential operator and \( (S_p x)(t) = x(\mu t) \). These operators do not commute: \( DS_p = \alpha S_p D \). In the limit \( \mu \to \infty \), the control \( \mu \) is impulsive, for which only the effect, \( \Delta x \), and the resulting \( \Delta f \) are relevant.

5.2 Key-construction

If the glide paths \( b \) and \( b' \) and \( r \) and \( r' \) were known, let \( \theta \) be the angle \( (b, r) = (b', r') \). We know form the analysis that \( b//r' \) and \( b'/r \) must hold. Let the paths intersect at \( S=b \cap r \) and \( S'=b' \cap r' \). If \( BB' \neq RR' \), then the turning points will be \( T_r \) and \( T_r' \) respectively on \( r \) and \( r' \). The complementary turning points \( C_0 \) and \( C_0' \) coincide with the vertex \( Q = b \cap r \). It follows from the geometry that \( b, r = T_r Q = (b', r') = \theta \). The optimal motion is the glide \( RB \to T_r Q \), rotation \( T_r Q \to T_r' Q \) and the complementary glide \( T_r' Q \to R' B' \).

**Fig. 6. Glide-Rotation-Glide**

Given the initial and final positions \( BR \) and \( B'R' \), the problem thus boils down to determine the glide lines \( b \) and \( r \), and their intersections, \( S \) and \( S' \). Form these the corners \( C \) and \( C' \) and turning points \( T \) and \( T' \) are directly determined.

It follows from the analysis of the optimal problem that \( b \) and \( r \), together with \( b' \) and \( r' \) must satisfy:

\[
b//b' \text{ and } d(b, b') = 1, \quad B \in b, \quad b' \in b' \\
r//r' \text{ and } d(r, r') = 1, \quad R \in r, \quad r' \in r'.
\]

Hence a key geometric construction is:

*Given arbitrary points \( A \) and \( B \) in the plane, draw parallel lines \( a \) and \( b \) with \( A \in a \) and \( B \in b \) a unit distance apart. This is a standard type construction from Euclidean geometry. Draw the line \( AB \), and the circle \( \text{centered at } B \). Hence there are two solutions to problem: The line \( b \) determined by \( UB \) is parallel is \( AL'=a \). The second solution is \( b'=LB \), which is parallel to \( a=AU' \).*

**Fig. 7. Key construction**

It is clear that the construction hinges on the condition \( AB \geq 1 \), for the points \( U, U' \) and \( L, L' \) to exist, and hence the problem to be solvable. In the limit case, \( \overline{AB} = 1 \), the points \( U \) and \( L \) coincide with \( B \), so that then is a unique solution with \( a \) and \( b \) \( \perp AB \).

Given the coordinates of the points \( A \) and \( B \), some analytic geometry leads to an expression for the slopes of \( a \) and \( b \).

\[
\alpha_{\pm} = \frac{\Delta x y_{\pm} - \Delta y x_{\pm}}{\Delta x^2 - 1} = \frac{\sqrt{\Delta x^2 + \Delta y^2 - 1}}{\Delta x^2 - 1}.
\]

where \( \Delta x = x_A - x_B \), \( \Delta y = y_A - y_B \). Hence, given the points \( A \) and \( B \), an analytic expression for the parallels \( a \) through \( A \) and \( b \) through \( B \) with unit distance is obtained (two solutions if \( d(A, B) > 1 \), one for \( d(A, B) = 1 \) and none if \( d(A, B) < 1 \)). The coordinates of \( S=a \cap b \) are now easily obtained.

6. GLIDE ELLIPSE

Given the lines \( y = \pm ax \), with \( 0 < a < 1 \) is known. (The case \( a > 1 \) is similar.) Let a slope \( a \) in the interval \((a, \infty)\) be given, (the other cases proceed similarly). The midpoint, \( M \), of the line with slope \( \alpha \) which intersects \( y = ax \) and \( y = -ax \) respectively in \( P \) and \( T \), lying to the right of the origin (the other solution is symmetric w.r.t. \( O \)), and such that \( TP = 1 \) has coordinates \( x_M = \frac{a}{2a\sqrt{1+a^2}} \).
\( y_M = \frac{a}{2\sqrt{1+\alpha^2}} \). Eliminating \( \alpha \) we see that the midpoint lies on the glide-ellipse

\[
\left( \frac{x_M}{a} \right)^2 + \left( \frac{y_M}{\frac{a}{2}} \right)^2 = 1.
\]

This result is a special case of Van Schooten’s problem, solved in the 17th century Honsberger [1970]. The restriction \( a < 1 \) implies that the major and minor axes of this ellipse are respectively \( 2A = \frac{a}{\alpha} \) and \( 2B = a \).

First we see that the area of the glide ellipse \( \Sigma = \pi AB = \frac{\pi}{2} \), and hence does not depend on the angle between the glide lines \( \theta = 2\arctan \alpha \).

The turning and complementary turning points are determined by letting the slope be \( \alpha = \frac{1}{\sqrt{2}} \). The point on the glide ellipse corresponding with this condition has coordinates \( \left( \frac{a}{2\sqrt{1+\alpha^2}}, \frac{a^2}{2\sqrt{1+\alpha^2}} \right) \). Note that the slope of the glide-ellipse at this point is \( -a \). Hence the glide ellipse is constructible given \( BR \) and \( B'R' \). The significance if the glide-ellipse is that during a slide of end point \( B \) along \( b \equiv y = ax \) and endpoint \( R \) along \( r \equiv y = -ax \), between \( O \) and \( C_b \) or \( T_r \), respectively gives for the path lengths \( \int |dS_M| = \frac{1}{2} \int (|dS_R| + |dS_B|) \). Hence the bi-path length is twice the path length that the midpoint \( M \) travels on the glide-ellipse. Conversely, this construction leads to the rectification of the arc length of an ellipse, which may be of interest in its own right.

7. CONCLUSIONS AND EXTENSIONS

We solved a problem posed by Ulam, which belongs to the class of SOC problems, and is solved by combining Euclidean geometry with the maximum principle. Any optimal bi-path consists of either a glide, a rotation, a combination of the two, or - at most - , a glide sandwiched between two rotations or a rotation sandwiched between two glides related by permutation of their glide angles. A glide-ellipse characterizes the solution. The problem of the optimal bi-path to a given line, (parallel parking) is solved by constructing the path for the initial position and its mirror image w.r.t. the wall.

REFERENCES


