Flatness Based Control of a Suspension System: A GPI Observer Approach

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Abstract: We propose a robust, flatness based, approach for active regulation of a railway vehicle suspension system externally perturbed by track irregularities. A simplified perturbed input–to–flat-output model for the vehicle model, which is differentially flat, is established. The observer-based control scheme uses the simplified model considering all the additive state-dependent terms and the influence of external disturbance inputs, as a single lumped unknown but bounded disturbance signal. The proposed GPI observer simultaneously estimates the required flat output associated phase variables allowing to complete the linear feedback loop via on-line disturbance cancellation. Simulations reveal the effectiveness of the approach.

Keywords: Disturbance cancelation, robust output feedback control, simplified active control, vehicle suspensions

1. INTRODUCTION

Generalized Proportional Integral (GPI) observers were introduced in Sira-Ramírez and Feliu-Batlle (2010) in the context of sliding mode observers applied to on-line obstacle detection in the operation of flexible robotics systems. The continuous version of these observers is the subject of the work by Cortés-Romero et al. (2010), as applied to chaotic systems synchronization. GPI observers constitute the dual counterpart of GPI controllers, introduced by Fliess et al. (2006). GPI observers are most naturally applicable to the control of perturbed differentially flat nonlinear systems with measurable flat outputs (the concept of flatness was introduced by Fliess et al. (1995)).

The linear GPI observer naturally incorporates, as a self-updating internal model, the polynomial approximation of bounded nonlinear, state dependent, perturbation input signals and it is, simultaneously, capable of accurate on-line estimations of: a) the output related phase variables, b) the state dependent additive perturbation input signal itself and c) the estimation of a certain number of the perturbation input time derivatives. Our use of GPI observers in this control application is deeply inspired in the differential algebraic approach to nonlinear state estimation (see Fliess et al. (2008)).

In this article, we propose a linear flatness-based approach to the robust output feedback controller design task for an active suspension system on a railway vehicle, for which a reference trajectory tracking task is indirectly demanded on the available, minimum phase, flat output for the control of the vertical position variable of the vehicle body. The ride quality of the vehicle is guaranteed by definition of a smooth desired behaviour, which is accounted by the body vehicle vertical acceleration. We provide some results disclosing the benefit of the technique for this engineering problem. A similar performance is achieved for the regulation task uniquely, for which a zero reference is provided instead, although results are no presented here.

This article is organized as follows: Section II contains the formulation of the problem and it depicts the most important properties of the studied suspension system in terms of flatness. Section III presents an application of the proposed approach to the active control of a vehicle suspension system. We include simulation results illustrating the performance of the proposed linear feedback control scheme. Section IV is devoted to the conclusions and suggestions for further work.

2. THE MASS-SPRING-DAMPER SUSPENSION SYSTEM

A simple incremental model for the dynamics of the vehicle in Fig. 1 which is meaningful enough for evaluating the applicability of a control technique for the problem is the following

\[ m_1 \ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = f \]
\[ m_2 \ddot{x}_2 + c_1(\dot{x}_2 - \dot{x}_1) + k_1(x_2 - x_1) + c_2(\ddot{x}_2 - \ddot{\zeta}) + k_2(x_2 - \zeta) = -f \]
\[ y = x_1 \quad (1) \]
where \( x_1 \) and \( x_2 \) stand for the incremental mass displacements from their equilibrium values, i.e. the gravity effects have been removed. The variables \( \zeta \), and its time derivative \( \dot{\zeta} \) represent unknown external disturbance inputs due the track irregularities, known only to be uniformly absolutely bounded. The mass \( m_1 \) is assumed to be larger than \( m_2 \). The output variable that needs active regulation is represented by the upper mass displacement: \( y = x_1 \). The control problem is to design an appropriate control force \( f \), which will result from our approach.

\[ \begin{align*}
  f(t) & \quad \text{(1)} \\
  m_1 & \quad \text{1} \\
  k_1 & \quad \\
  c_1 & \quad \text{2} \\
  x_1(t) & \quad \text{3} \\
  \zeta(t) & \quad \text{4} \\
  m_2 & \quad \text{5} \\
  k_2 & \quad \text{6} \\
  c_2 & \quad \text{7} \\
  x_2(t) & \quad \text{8}
\end{align*} \]

Fig. 1. Mechanical suspension system.

A normalization of the model is carried by defining: 1) the normalized time as: \( \tau = t \sqrt{k_1/m_1} \); 2) the normalized input force: \( u = f/k_1 \). The system constant coefficients are redefined as: \( \epsilon = m_2/m_1 \), \( \gamma_1 = c_1/m_1 \sqrt{k_1/m_1} \), \( \gamma_2 = c_2/m_1 \sqrt{k_1/m_1} \), \( \kappa = k_2/k_1 \). This normalization leads to the following model,

\[
\begin{align*}
  \dot{x}_1 + \gamma_1 (\dot{x}_2 - \dot{x}_2) + (x_1 - x_2) &= u \\
  \epsilon \dot{x}_2 + \gamma_1 (\dot{x}_2 - \dot{x}_1) + (x_2 - x_1) + \gamma_2 (\dot{x}_2 - \dot{\zeta}) + \kappa (x_2 - \zeta) &= -u \\
  y &= x_1 
\end{align*}
\]

where, with an abuse of notation, the “dot” symbol is now used to specify normalized time differentiation.

The corresponding state space description of the normalized perturbed system (2), is given by

\[
\frac{d}{d\tau} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \begin{bmatrix} \dot{\zeta} \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \zeta \\ \zeta \end{bmatrix} + \frac{1}{\epsilon} \begin{bmatrix} u \\ \gamma_2 \end{bmatrix}
\]

for which all the terms associated to the disturbances will be compactly denoted as \( \Delta \) for a simpler but equivalent representation

\[
\frac{d}{d\tau} x_s = A x_s + b u + \Delta.
\]

### 2.1 Flatness of the system

The controllability of the system (3), assessed in absence of external perturbations, is computed as,

\[
C = [b, Ab, A^2 b, A^3 b]
\]

The unperturbed flat output, \( F \), is therefore given by the last row of the inverse of the controllability matrix multiplied by the state vector (see the book by Sira-Ramirez and Agrawal (2004)),

\[
F = [0, 0, 0, 1] C^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}
\]

\[
= \epsilon \frac{\gamma_2}{\kappa} x_2 + \frac{\epsilon}{\kappa} \gamma_2 x_2 - \frac{\epsilon^2 \gamma_2}{\kappa^2} x_2
\]

In fact, using the system state equations, (3), and the definition of the flat output (5), we have that for the unperturbed system,

\[
F = \epsilon \frac{\gamma_2}{\kappa} x_2 + \frac{\epsilon}{\kappa} \gamma_2 x_2 - \frac{\epsilon^2 \gamma_2}{\kappa^2} x_2
\]

whose unperturbed inverse relation is readily computed as:

\[
x_1 = \frac{\kappa}{\epsilon} F + \frac{\gamma_2}{\epsilon} \dot{F} + \dot{\bar{F}}
\]

\[
\dot{x}_1 = \frac{\kappa}{\epsilon} \dot{F} + \frac{\gamma_2}{\epsilon} \ddot{F} + F(3)
\]

\[
x_2 = -\frac{1}{\epsilon} \dot{F}
\]

\[
\dot{x}_2 = -\frac{1}{\epsilon} F(3)
\]

The flat output, \( F \), thus exhibits relative degree equal to 4 with respect to the control input \( u \).

Note that under unperturbed steady state equilibrium conditions: \( \bar{x}_1 = \bar{x}_2 = \bar{F} = 0 \), these imply that \( \bar{x}_2 = 0 \). Hence, \( \bar{F} = \bar{x}_1 = \frac{\gamma_2}{\kappa} \bar{F} \). This latter relation is most useful in establishing the required forced equilibrium for the flat output \( F \) which ideally produces the desired equilibrium for \( y = x_1 \).

The inclusion of the external perturbation inputs, \( \zeta \) and \( \dot{\zeta} \), into the linear model, leads to the following relations for the perturbed flat output, \( F \), and its time derivatives,

\[
F = \epsilon \gamma_2 x_2 + \frac{\epsilon}{\kappa} \gamma_2 (\dot{\tau} - \frac{\gamma_2}{\kappa^2} x_2)
\]

\[
\dot{F} = \epsilon \gamma_2 x_2 + \frac{\epsilon}{\kappa} \dot{x}_2 - \frac{\epsilon^2 \gamma_2}{\kappa^2} \dot{\zeta}(\tau)
\]

\[
\ddot{F} = -\epsilon x_2 + \epsilon \zeta(\tau) + \frac{\epsilon \gamma_2 (1 - \frac{1}{\kappa}) \zeta(\tau)}{\kappa}
\]

\[\text{For a definition of relative degree, in the general context of nonlinear systems, see the book by Isidori Isidori (2002).}\]
\[ F^{(3)} = -\varepsilon \dot{x}^2 + \varepsilon \dot{\zeta}(\tau) + \varepsilon \gamma^2 \left(1 - \frac{1}{\kappa}\right) \ddot{\zeta}(\tau) - \frac{\varepsilon \gamma^2}{\kappa^2} \zeta^{(3)}(\tau) \] (8)

The fourth order time differentiation of \( F \), which includes the effect of the unknown external perturbations, the control input, and the states of the system, is readily found to be given by the following expression:

\[ F^{(4)} = -x_1 - \gamma_1 \dot{x}_1 + (1 + \kappa)x_2 + (\gamma_1 + \gamma_2) \dot{x}_2 + u - \kappa \zeta(\tau) - \gamma_2 \dot{\zeta}(\tau) + e \zeta(\tau) + \varepsilon \gamma^2 (1 - \frac{1}{\kappa}) \zeta^{(3)}(\tau) - \frac{\varepsilon \gamma^2}{\kappa^2} \zeta^{(4)}(\tau) \] (9)

The key step in our developments is based on the fact that the flat output dynamics (9) may be significantly simplified to the following, non-phenomenological, pure integration model

\[ F^{(4)} = u + \varphi(\tau) \] (10)

where

\[ \varphi(t) = -x_1 - \gamma_1 \dot{x}_1 + (1 + \kappa)x_2 + (\gamma_1 + \gamma_2) \dot{x}_2 - \kappa \zeta(\tau) - \gamma_2 \dot{\zeta}(\tau) + e \zeta(\tau) + \varepsilon \gamma^2 (1 - \frac{1}{\kappa}) \zeta^{(3)}(\tau) - \frac{\varepsilon \gamma^2}{\kappa^2} \zeta^{(4)}(\tau) \] (11)

The disturbance signal, \( \varphi(\tau) \) in the previous equation (11) combines most of the terms from (9), and entitles then: a) state-dependent perturbations with possibly the presence of unknown model parameters, b) the influence of the unknown external perturbations represented in the normalized system model by \( \zeta(\tau) \) and its time derivative \( \dot{\zeta}(\tau) \), c) parametric uncertainties. All other possible un-modeled disturbances, such as Coulomb friction terms, air resistance, etc., are assumed to be included in this single time function \( \varphi(\tau) \). We naturally assume that the possible un-modeled disturbances affecting the flat output dynamics are also unknown, but known to be uniformly absolutely bounded. A key property of the flat output, \( F \), that allows us to comfortably choose this important system simplification is represented by the fact that the flat output is devoid of any zero dynamics (see Isidori (2002) for a study of this important concept, in the general setting of nonlinear systems). Note that, according to the work of Diop and Fliss (1991), the perturbation input \( \varphi(\tau) \) in (10) is observable from the output \( F \), since the quantity \( \varphi(\tau) \) is certainly expressible in terms of the input, \( u \), the output, \( F \), and a finite number of their time derivatives. Indeed, \( \varphi(t) = F^{(4)} - u \).

### 2.2 Problem formulation

It is desired to have the flat output, \( F \), tracking a given equilibrium to equilibrium trajectory, as approximately as desired, specified by \( F^{(4)}(\tau) \), so that the corresponding mass displacement, \( y = x_1 \), converges to a small neighbourhood of the final desired corresponding equilibrium, \( \mathbf{x}_f \), after some finite time, in spite of the presence of the unknown but bounded external disturbance inputs, \( \zeta \), \( \dot{\zeta} \), of the unmeasured influence of the states of the system and of the effects of the un-modeled disturbances. The rest-to-rest transfer manoeuvre should take place within the finite interval specified as \( [\tau_0, T] \) in real time, or correspondingly as \( [\tau_0, \tau_{final}] \) in normalized time units.

### 3. FLATNESS BASED CONTROL OF THE MECHANICAL SUSPENSION SYSTEM

The fundamental idea in regulating the given model of the suspension system, or accomplishing a given suitable flat output reference trajectory tracking, is to use the simplified, non-phenomenological model, (10). To accomplish this we primarily estimate, even if in an approximately as desired fashion, the unknown disturbance signal, \( \varphi(\tau) \); secondly, we proceed, based on the estimated disturbance signal, to include a cancellation term of such disturbance effects in the designed controller, \( u \), and then asymptotically impose a linear stable dynamics by appropriately feeding back the estimated time derivatives of the measured flat output \( F \).

We consider a GPI observer including a reasonable, self-updating, time-polynomial model\(^2\) for the unknown, state dependent, disturbance input, \( \varphi(\tau) \). For this internal model, we use an unspecified element of a third degree family of time-polynomials, denoted by \( z_0(\tau) \). Clearly, \( z_0^{(4)}(\tau) = 0 \) represents the internal model of the disturbance input for the proposed observer. Set: \( F_0 = \hat{F} \), \( F_1 = \hat{\dot{F}} \), \( F_2 = \hat{\ddot{F}} \), \( F_3 = \hat{\dddot{F}} \). We have,

\[
\begin{align*}
\hat{F}_0 &= F_1 + \lambda_7 (F_0 - F_0) \\
\hat{F}_1 &= F_2 + \lambda_6 (F_0 - F_0) \\
\hat{F}_2 &= F_3 + \lambda_5 (F_0 - F_0) \\
\hat{F}_3 &= u + \hat{z}_0 + \lambda_4 (F_0 - F_0) \\
\hat{z}_0 &= z_1 + \lambda_3 (F_0 - F_0) \\
\hat{z}_1 &= z_2 + \lambda_2 (F_0 - F_0) \\
\hat{z}_2 &= z_3 + \lambda_1 (F_0 - F_0) \\
\hat{z}_3 &= \lambda_0 (F_0 - F_0)
\end{align*}
\] (12)

The (redundant) estimation error, \( e_F = F - F_0 \), of the flat output \( F \), thus evolves according with the linear perturbed dynamics

\[
\begin{align*}
e_F^{(8)} + \lambda_7 e_F^{(7)} + \cdots + \lambda_1 e_F + \lambda_0 e_F &= \varphi^{(4)}(\tau)
\end{align*}
\] (13)

Clearly, if \( \varphi^{(4)}(\tau) \) is uniformly absolutely bounded, and if the choice of the coefficients: \( \lambda_7, \ldots, \lambda_0 \), is made in such a way that the roots of the dominant characteristic polynomial,

\[
p(s) = s^8 + \lambda_7 s^7 + \cdots + \lambda_1 s + \lambda_0
\] (14)

are located sufficiently far from the imaginary axis, in the left half of the complex plane, then the trajectories of the estimation error, and of its time derivatives, converge to a small neighbourhood of the origin of the phase space of the observer estimation error. The further away the roots are located into the left half of the complex plane, the smaller the radius of the disk representing the

\(^2\) Also known as Taylor Polynomial model
neighbourhood around the origin of the estimation error phase space (this is a direct consequence of the well known, bounded-input, bounded output result in linear systems, see Kailath (1979) and also Cortés-Romero et al. (2010)). In order to avoid, possible, large initial “peaking” of the observer variables responses, a smoothing “clutch” is usually provided to the estimated flat output phase variables: $F_1$, $F_2$ and $F_3$. The “clutch” is defined as a time function smoothly increasing from 0 to 1, during a small time interval $[0, \epsilon]$. The “smoothing” of the observer variables may be carried out in accordance with,

$$F_{j\epsilon} = F_j(\tau)SF(\tau)$$

where, for instance, the smoothing function, $SF(\tau)$, may be defined, for a positive, even, integer $q \geq 2$ as

$$SF(\tau) = \begin{cases} \sin^q\left(\frac{\pi \tau}{2\epsilon}\right) & \text{for } 0 \leq \tau < \epsilon \\ 1 & \text{for } \tau \geq \epsilon \end{cases}$$

In our particular case of an initial equilibrium to a final equilibrium flat output manoeuvre, the value of $\epsilon$ was unnecessary, and it was set to be zero.

It is not difficult to see that, if the estimation error, $e_F$, and its various time derivatives converge to a sufficiently small neighborhood of zero, then the observer variable $z_0(\tau)$ converges asymptotic and exponentially to a small neighborhood around the lumped perturbation input signal, $\varphi(\tau)$. The approximately as desired estimation of the lumped disturbance input, $\varphi(\tau)$, allows one to exercise a cancelling strategy from the designed control input, $u$. Since such a cancelation is performed with a rather close estimate of the disturbance input, the controller may also benefit from a high gain strategy.

The desired trajectory, $F^*(\tau)$, is designed so that the measured flat output, $F$, smoothly increases from a given nominal initial value, $F^*(\tau_0)$, which we take to be zero, towards a final desired value $F^*(\tau_{final})$, within a finite normalized time interval $[\tau_0, \tau_{final}]$.

The GPI observer based flat output feedback controller, $u$, is specified as follows:

$$u = u^* (\tau) - \varphi (\tau) - k_3(F_{3s} - [F^*(t)]^{(3)}) - k_2(F_{2s} - \bar{F}^*(\tau)) - k_1(F_{1s} - \bar{F}^*(\tau))$$

where the nominal control input $u^*(\tau)$, may be computed from the unperturbed simplified model as: $u^*(\tau) = [F^*(\tau)]^{(4)}$. Alternatively, the, strictly speaking, unknown nominal control input may not be included in the controller and its effects will be absorbed by the unknown perturbation input, $\varphi(\tau)$. In our particular case we obtained $u^*(\tau)$ from the unperturbed inversion formula.

The closed loop tracking error, $e = F - F^*(\tau)$, asymptotically, exponentially, tends to be governed by the following predominantly linear dynamics,

$$e^{(4)} + k_3e^{(3)} + k_2\ddot{e} + k_1\dot{e} + k_0e = \varphi(\tau) - \dot{\varphi}(\tau) + \rho(\tau)$$

where, $\rho(\tau)$, depicts the effect of the small flat output phase variables estimation errors, generated by the observer, and the effects of the disturbance signal, on-line,

$$\zeta(\tau),$$

Fig. 2. Block diagram of the feedback control scheme based on a GPI-observer.

estimation errors. It is intuitively clear that the closed loop dynamics in (16) is less severely affected by the uncertainties than the corresponding dynamics of the observer estimation error. This fact results in smaller feedback gains: $\{k_3, k_2, k_1, k_0\}$, than those used for the design of the GPI observer.

The block diagram in Fig. 2 assembles the definition of this flat output feedback control scheme based on a GPI-observer design. The use of an ideal actuator was considered; the coefficients in the linear combination of the displacements and velocities of the masses defining $F$ were named $\alpha_i$. Also, the physical measurement of the flat output is considered in this approach to be ideal. However, under a noisy sensor measurement condition, in order to compensate the adverse effects of the high gains of the observer, the “pre-addition” of an extra integrator to the structure in (12) would be required for the particular case of Gaussian type noises with media zero, for example, so that the noise is pre-filtered by the observer itself.

3.1 Simulation results

We tested our robust flat output, GPI observer based, active feedback control scheme of the mechanical suspension system model, (1), using the following physical parameters for a railway vehicle:

$$m_1 = 9500 \text{ [kg]}, \quad m_2 = 2500 \text{ [kg]},$$

$$k_2 = 2.5 \times 10^6 \text{ [N/m]}, \quad c_2 = 1.79 \times 10^4 \text{ [N \cdot s/m]},$$

$$k_1 = 0.5 \times 10^6 \text{ [N/m]}, \quad c_1 = 100 \text{ [N \cdot s/m]}$$

yielding the flat output: $F = 0.05263x_1 + 0.01314x_2 - 0.00273x_1 - 0.00072x_2$.

Here, we desire that the final equilibrium value for $x_1$ be given by $x_1 = 0.01$ [m]. This means that the corresponding equilibrium value for the flat output, $F$, according to its relationship with the states, is given by $F = 5.2631 \times 10^{-4}$ [m]. We set out a rest to rest smooth trajectory for $F^*(\tau)$, starting at the value of zero at time $\tau_0 = 0$, and exhibiting the final value $F^*(\tau_{final}) = 5.2631 \times 10^{-4}$ at time, $\tau_{final} = 40$ (i.e. $T = 5.51$ [s], in real time), in normalized time units.

The perturbation input to the model, $\zeta(\tau)$, was set to be a stochastic function with mean PSD (Power Spectral Density)
In this article, we have described the design of an observer-based robust linear output feedback controller for the ac-

Fig. 3. Performance of the observer based GPI controller for a flatness based trajectory tracking task in a perturbed mechanical system.

\[
S(f_t) = \frac{(A_v e)}{(2\pi f_t^2)}
\]  

(17)

with \( A_v \) being the roughness factor of a railway track given as \( A_v = 3.6 \times 10^{-6} \text{[m]/cycle} \), \( v = 55 \text{[m/s]} \) the speed of the railway vehicle, and \( f_t \) the temporal frequency. The signal was generated by colouring a white noise with sample rate of 4.5 [ns], using a first order low-pass filter with cut-frequency of 1.73 [rad/s].

The observer gains, \( \{\lambda_1, ..., \lambda_0\} \), were chosen by identifying, term by term, the coefficients of the polynomial, (14), with those of a desired Hurwitz polynomial given by \( (s^2 + 2 \zeta \omega_n s + \omega_n^2)^4 \). In our simulations, we set, \( \zeta = 1.2, \omega_n = 10 \) for the observer based controlled normalized system. The controller gains, \( \{k_3, k_2, k_1, k_0\} \), governing the dominant linear dynamics, (16), were set by also identifying, term by term, the coefficients of the underlying 4-th degree closed loop characteristic polynomial, with those of the desired Hurwitz polynomial: \( (s^2 + 2 \zeta \omega_n s + \omega_n^2)^2 \). In this instance, we set \( \zeta = 1, \omega_n = 0.75 \). The coefficients of the desired Hurwitz polynomial for both the observer and the controller were chosen to meet a desirable and convenient fast asymptotic and exponentially convergent dynamic to a neighborhood of zero for the error of estimation and the tracking error, respectively.

Fig. 4. Vertical acceleration of the upper mass, and secondary suspension deflection \( \ddot{x}_1(t) \), and \( x_1(t) - x_2(t) \) (defl.), respectively.

Fig. 5. Convergence of the trajectories of the observer estimation error, and of its time derivatives.

\( \pi_1 = 0.01 \text{[m]}, \) with a r.m.s. of the error of \( 8.6 \times 10^{-4} \text{[m]} \), while the vertical acceleration, \( \ddot{x}_1 \), of the upper mass evolve within reasonable bounds, as shown in Fig. 4. The latter, for instance, has a r.m.s value of 0.72 [\%g] (percentage of the gravity of Earth value, with \( g \) approximately 9.81 [m/s^2]), which represents a high quality of ride for vehicle passengers. Moreover, the suspension deflection, for which physical constraints are always imposed, is also acceptable.

Finally, Fig. 5 depicts the convergence of the trajectories of the GPI observer estimation error, and of its time derivatives, to a small neighborhood of the phase space origin. For illustrative purposes, we divided the phase space in two bidimensional subspaces, \( \{F - \dot{F}, \dot{F} - \ddot{F}\} \), and \( \{\dot{F} - \ddot{F}, F^{(3)} - F^{(3)}\} \), and abused in the units notation by using [mag] to denote the magnitude of the adimensional variables \( F \) and \( \dot{F} \), as well for their time derivatives. We found as expected, the ratio of the encirclement of such convergence of the estimation in the phase space remains acceptable for parameters uncertainties causing a consequent uncertainty in the value of the gain relating the control input with the flat output in the pure integration model in (10). For this particular case, we obtained that uncertainty in the gain -exactly known in (10) to be unitary-, can be of \( \pm 30\% \) without causing the system to be unstable and still having an acceptable performance. Particularly, an uncertainty of \( \pm 20\% \), produces a maximum r.m.s. of the error of tracking for \( x_1 \) of \( 1.30 \times 10^{-3} \text{[m]} \), and for the flat output \( F \)-variable on which the feedback control is based-, of \( 1.06 \times 10^{-5} \text{[m]} \).

4. CONCLUSION

In this article, we have described the design of an observer-based robust linear output feedback controller for the ac-

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tive control of a trajectory tracking problem in a mechanical suspension system. The system is shown to be flat with the flat output given by a suitable linear combination of all state variables. The resulting input-to-flat-output dynamic system is radically simplified by assuming that important additive state-dependent terms, and unknown external perturbations, may be lumped into an uniformly absolutely bounded signal, treated as a disturbance, or perturbation, input $^3$. This additive disturbance may be on-line, approximately, estimated by means of a high-gain linear observer with a sufficiently rich internal model for the disturbance input. The input-to-flat-output description of the plant is thus modeled as a pure integration system influenced by additive absolutely bounded, yet observable, perturbation input signals. Our basic assumption is that such state-dependent disturbance inputs can be treated as absolutely, uniformly, bounded time signals and, thus, they can be locally approximated by an arbitrary representative of a fixed-degree family of Taylor polynomials adopted as self-updating internal models in the GPI observer. The proposed linear control approach, most naturally, applies then, even, to the class of nonlinear systems.

Although the formulation described is different to a normal ground vehicle suspension, which has to follow low frequency disturbances (intended factors of the guiding surface), but isolate the vehicle body from the high frequency irregularities, some kind of preview or database system is available to define the intended inputs as reported in Zhou et al. (2010), and Hong et al. (2002). Thus, the calculation of $F^*$ in real-time and the flatness based active suspension design is then compatible with the real engineering problem. However, because the focus is upon the upper mass’ dynamic behaviour (always considering the associated physical constraints), it would be more convenient if the flat output would have a stronger dependence on the state $x_1(t)$, rather than on the other states, so that the reference definition process would be more robust to parameters variations. Therefore, it will be a consideration for a further design, either the inclusion of additional mechanical elements, or the definition of an affine flat output upon existence (Waldherr and Zeitz (2008) for reference).

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$^3$ That state dependent uncertainties may be treated as unstructured, bounded, time signals, is usually the case in control engineering practise. See Gao (2006), Gao and Rhinehart (2004), Han (2009) and Fliess and Join (2006)