Control of Discrete-Time Stochastic Systems with State Equality Constraints

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Abstract: Design conditions for existence of state feedback control stabilizing discrete-time stochastic systems with multiplicative noise where closed-loop state variables satisfy equality constraints in the mean are presented in the paper. Using classical feedback control principle the LMI-based procedure is provided for computation of the gain matrix of state control laws, and influence of equality constraints is explained. The approach is successfully illustrated on example demonstrating validity of the proposed method using an equality constraint tying together all state variables of the system.

Keywords: Stochastic systems, discrete-time control, singular systems, linear matrix inequality.

1. INTRODUCTION

Many real systems operate in a stochastic environment where they are subject to unknown disturbances and in addition, the controller has to rely, in practice, on imperfect measurements. One of the principal reasons for introducing feedback into a system is to obtain relative insensibility to changes in plant parameters and to disturbances.

In the last years many significant results have spurred interest in the problem of determining the control laws for the systems with constraints. For the typical case where a system state reflects a certain physical entities this class of constraints raises because of physical limits and these ones usually keep the system state in a region of the technological conditions. However, this problem can be formulated using techniques dealing with the state constraints directly, where the equations of both, the unconstraint system and the stabilized constraint relations are combined to a coupled system of equations which can be interpreted as a descriptor system (Hahn (1992)). Because a system with state constraints generally does not satisfy the conditions under which the results of descriptor systems can be applicable this approach is very limited.

In principle, it is possible and ever easy to apply a direct design method, namely to design a controller that stabilizes the systems and simultaneously forces the closed-loop systems to satisfy the constraint such that a special form of the constrained problems can be so formulated while the system state variables satisfy equality constraints. This technique has been introduced in Ko and Bitmead (2007), and was used in the state constrained control (Filasová and Krokavec (2010)). Presented principle can be applied especially e.g. in optimization of an active multi-variable combustion process control.

The work presented here is still limited to the assumption presented in Gershon, Shaked and Yaesh (2001), Ko and Bitmead (2007), as well as to extensions applied for uncertain stochastic discrete-time systems with multiplicative noise in Krokavec and Filasová (2008). Thus, reformulated and generalized design conditions are derived, and possible extensions are proven in the paper. To the best of author’s knowledge, the approach presented has not yet been fully investigated in such way in this field.

2. PROBLEM FORMULATION

Through this paper the task will be concerned with state feedback design to control a stochastic uncertain discrete-time linear dynamic system, given by set of equations

\[
q(i + 1) = (F + F_0a(i))q(i) + Cu(i) + Gv(i) \quad (1)
\]

\[
z(i) = Cq(i) + Du(i) \quad (2)
\]

\[
y(i) = C_pq(i) \quad (3)
\]

where \(q(i) \in \mathbb{R}^n\), \(u(i) \in \mathbb{R}^r\), \(y(i) \in \mathbb{R}^p\), \((z(i) \in \mathbb{R}^p)\) is the state, input, output and objective vector, respectively, \(v(i) \in \mathbb{R}^p\) is an exogenous disturbance vector, \(F \in \mathbb{R}^{n \times n}, G \in \mathbb{R}^{r \times p}, C, C_p \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times r}, G_v \in \mathbb{R}^{m \times p}, F_0 \in \mathbb{R}^{n \times n}\) are finite valued, and \(D^T C = 0, D^T D > 0\).

It is assumed that \(a(j), 0 \leq j \leq i\) satisfies condition

\[
E\{a(j)\} = 0, \quad E\{a(h)a(j)\} = \delta_{jh} \quad (4)
\]

where \(E[\cdot]\) denotes expectation and \(\delta_{jh}\) is the Kronecker delta function. Disturbance vector is a non-anticipative process, where \(\{v(i)\} \in l_2(0, \infty); \mathbb{R}^p\).

Problem of the interest is to design an asymptotically stable closed-loop system constrained in state variables by a linear memoryless state feedback controller of the form

\[
u(i) = -Kq(i) \quad (5)
\]

where \(K \in \mathbb{R}^{p \times n}\) is the feedback controller gain matrix, and design constraints takes the equality form

\[
E\{q(i)\} \in \mathcal{N}_L = \{ q : Lq = 0 \}, \quad \text{rank} L = k < r \quad (6)
\]

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3. BASIC PRELIMINARIES

Proposition 1. (Matrix pseudoinverse) Let $\Theta$ is a matrix variable and $A, B, C$ are known non-square matrices of appropriate dimensions such the equality

$$A\Theta B = C$$

(7)

can be set. Then all solution to $\Theta$ means

$$\Theta = A^{\dagger}C B^{\dagger} + \Theta^0 - A^{\dagger}A\Theta B B^{\dagger}$$

(8)

where $A^{\dagger}$ and $B^{\dagger}$ is Moore-Penrose pseudoinverse of $A, B$, respectively, and $\Theta^0$ is an arbitrary matrix of appropriate dimension (e.g. see Boyd et al. (1994)).

Proof. Supposing that the products $AA^T, BB^T$ are regular matrices, then pre-multiplying left-hand side, as well as right-hand side of (7) by the identity matrices gives

$$A\Theta B = AA^T(\Theta^0)^{-1}CB^{T}B^{-1}B^T$$

(9)

and using (11) it yields

$$\Theta = A^T(\Theta^0)^{-1}CB^{T}B^{-1}B^T = A^{\dagger}CB^{\dagger}$$

(10)

where

$$A^{\dagger} = A^T(\Theta^0)^{-1}, B^{\dagger} = (B^TB)^{-1}B^T$$

(11)

Let $\Theta^0$ is a matrix of appropriate dimension such that substituting in (7) it can be written

$$A\Theta^0 B = AA^{\dagger}CB^{\dagger} B = AA^{\dagger}A\Theta^{0} B B^{\dagger}$$

(12)

Thus,

$$A(\Theta^0 - A^{\dagger}A\Theta^0 BB^{\dagger}) B = 0$$

(13)

$$\Theta^0 - A^{\dagger}A\Theta^0 BB^{\dagger} = 0$$

(14)

respectively. Therefore, for an arbitrary $\Theta^0$, (14) implies (8). □

Note, matrix pseudoinverse is generalized for possible singular matrix products $AA^T, BB^T$.

Proposition 2. Let $A \in \mathbb{R}^{m \times n}$ is a real square matrix with non-repeated eigenvalues, satisfying the constraint

$$\mathbf{I}^T A = 0$$

(15)

Then from one of its eigenvalues is zero, and $\mathbf{I}^T$ is the left raw eigenvector of $A$ associated with the zero eigenvalue.

Proof. If $A \in \mathbb{R}^{m \times n}$ is a real square matrix having non-repeated eigenvalues then the eigenvalue decomposition of $A$ takes the form

$$A = U_A \Lambda V_A^T$$

(16)

$$U_A = [u_1 \cdots u_n], \ V_A = [v_1 \cdots v_n]$$

(17)

$$\Lambda = \text{diag} [z_1 \cdots z_n], \ U_A^T V_A = I$$

(18)

where $u_h$, is right eigenvector, and $v_h$ is left eigenvector associated with the eigenvalue $z_h$ of $A$, $h = 1, 2, \ldots n$. Then (15) can be rewritten as

$$d^T [u_1 \cdots u_h \cdots u_n] \text{diag} [z_1 \cdots z_h \cdots z_n] V_A^T = 0$$

(19)

If $\mathbf{I}^T = v_h^T$, then orthogonal property (18) implies

$$[0_1 \cdots 0_h \cdots 0_n] \text{diag} [z_1 \cdots z_h \cdots z_n] V_A^T = 0$$

(20)

and it is evident that (20) can be satisfied only if $z_h = 0$. □

4. LYAPUNOV FUNCTION

Lemma 1. Given the system (1), (2), $r < n$, and the Lyapunov function candidate

$$v(q(i)) = q^T(i)Pq(i)$$

(21)

where $P = P^T > 0, P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix then the mean value of difference of this Lyapunov function is

$$E \{ \Delta v(q(i)) \} = E \{ v(q(i+1)) - v(q(i)) \} =$$

$$= E \left\{ \frac{\|u(i) - u^*(i)\|_2^2}{2} - \|v(i) - v^*(i)\|_2^2 \right\} +$$

$$+ q^T(i)Tq(i) + q^T(i)(F^TPGu(i) +$$

$$+ v^T(i)v(i) - z^T(i)z(i))$$

(22)

where $I_p \in \mathbb{R}^{p \times p}, I \in \mathbb{R}^{n \times n}$ are identity matrices, and

$$U = R + G^T \Sigma$$

(23)

$$H = P(I - \gamma^{-1}G^T \Sigma P)^{-1}$$

(24)

$$V = \gamma I_p - G^T \Sigma G > 0$$

(25)

$$T = F^T \Sigma P + F^T \Sigma P F + C^T C +$$

$$+ K_q^T V K_q - K_q^T \Sigma U \Sigma K_q - P$$

(26)

$$u^*(i) = -K_q(i)$$

(27)

$$v^*(i) = K_u q(i) + K_u u(i)$$

(28)

$$K_u = V^{-1} G^T P G$$

(29)

$$K_q = V^{-1} G^T P F$$

(30)

Hereafter, $\| \xi \|^2_{\Lambda} = \xi^T \Lambda \xi$ denotes square of the Mahalanobis norm of a vector $\xi$ with respect to a symmetric weight matrix $\Lambda$.

Proof. (e.g. compare Krokavec and Filasová (2008)) Using (2) the product $z(i)z(i)$ takes the form

$$z^T(i)z(i) = [q^T(i) \ u^T(i)] \begin{bmatrix} C^T D^T \\ D \end{bmatrix} \begin{bmatrix} q(i) \\ u(i) \end{bmatrix} =$$

$$= [q(i) \ u(i)] \begin{bmatrix} C^T D^T \ C D \end{bmatrix} \begin{bmatrix} q(i) \\ u(i) \end{bmatrix}$$

(31)

Since $CD = 0$, and $D^TD = R > 0$ the difference (22) can be written as

$$\Delta v(q(i)) =$$

$$= q^T(i) \begin{bmatrix} (F + F_o a(i))P(F + F_o a(i)) + \gamma C^T C - P \end{bmatrix} \begin{bmatrix} q(i) \ u(i) \end{bmatrix} +$$

$$+ u^T(i) (G^T P G + R) u(i) +$$

$$+ v^T(i) G^T P G v(i) +$$

$$+ q^T(i) (F + F_o a(i))^T P G u(i) +$$

$$+ u^T(i) G^T P (F + F_o a(i)) q(i) +$$

$$+ q^T(i) (F + F_o a(i))^T P G v(i) +$$

$$+ v^T(i) G^T P (F + F_o a(i))q(i) +$$

$$+ u^T(i) G^T P G v(i) +$$

$$+ v^T(i) v(i) - z^T(i)z(i)$$

(32)
Since
\[
\begin{aligned}
E_a \left\{ q^T (i) F a(i) P F q(i) + q^T (i) F a(i) P G u(i) + u^T (i) G T P F a(i) q(i) + q^T (i) G a(i) P G u(i) + u^T (i) G T P F a(i) q(i) \right\} = 0
\end{aligned}
\]
it can be obtained
\[
E_a \left\{ \Delta v(q(i)) \right\} = E_a \left\{ \begin{aligned}
q^T (i) \left( F^T P F + F^T a P F a + C^T C - P \right) q(i) + 
+ u^T (i) (G T P G + R) u(i) - 
- v^T (i) (\gamma I - G T P G - R) v(i) + 
+ q^T (i) F^T P G u(i) + u^T (i) G T P F q(i) + 
+ q^T (i) F T P G v(i) + v^T (i) G T P F q(i) + 
+ u^T (i) G T P G v(i) + v^T (i) G T P G u(i) + 
+ \gamma v^T (i) v(i) - z^T (i) z(i)
\end{aligned} \right\} \tag{34}
\]
Using notations (29), (30) with \( V \) as in (25), and completing to squares for \( v(i) \) from (34) results in
\[
- v^T (i) V v(i) + 
+ u^T (i) K^T q v(i) + v^T (i) V K q u(i) + 
+ q^T (i) K^T q v(i) + v^T (i) K q u(i) = 
- \| v(i) - v^*(i) \|^2 V + v^T (i) V v^*(i) = 
\]
with
\[
v^T (i) V v^*(i) = 
\]
\[
\begin{aligned}
= u^T (i) K^T q v(i) + q^T (i) K^T q v(i) + 
+ u^T (i) K^T q v(i) + q^T (i) K^T q v(i)
\end{aligned}
\]
\[
K^T q v = F^T P G V^{-1} G^T P F
\]
\[
K^T q v = F^T P G V^{-1} G^T P F
\]
\[
K^T q v = G^T P G V^{-1} G^T P F
\]
respectively. Let now
\[
W = \begin{aligned}
&= G^T P G + K^T q v = G^T (P + P G V^{-1} G^T P) G
\end{aligned}
\]
then, using Sherman - Morrisson - Woodbury equality
\[
(A + B C B^T)^{-1} = 
= A^{-1} - A^{-1} B (C^{-1} + B^T A^{-1} B)^{-1} B^T A^{-1}
\]
and (25), subsequently it is obtained
\[
H^{-1} = (P + P G V^{-1} G^T P)^{-1} = 
= P^{-1} - G^T G^{-1} G^T P G^{-1} = P^{-1} - \gamma G^T G^{-1} P G^{-1} > 0
\]
respectively, and (40) can be rewritten as
\[
W = G^T H G > 0
\]
Using notations
\[
U = R + W = R + G^T H G \tag{45}
\]
\[
K = U^{-1} K^T q V K q \tag{46}
\]
considering \( u^*(i) \) given in (27), and completing to squares with respect to \( u(i) \) from (34), (36) gives
\[
u^T (i) U u(i) + q^T (i) K^T q v(i) + 
+ u^T (i) K^T q v(i) q(i) = 
= \| u(i) + K q(i) \|^2 U \| u(i) + K q(i) \|^2 
- q^T (i) K^T U K q(i)
\]
Rearranging (32) with (43) results in
\[
E_a \left\{ \Delta v(q(i)) \right\} = E_a \left\{ \begin{aligned}
\| u(i) - u^*(i) \|^2 U - \| v(i) - v^*(i) \|^2 V + 
+ q^T (i) \left( F^T P F + F^T a P F a + C^T C - P \right) q(i) + 
+ q^T (i) F^T P G u(i) + u^T (i) G T P F q(i) + 
+ q^T (i) F T P G v(i) + v^T (i) G T P F q(i) + 
+ u^T (i) G T P G v(i) + v^T (i) G T P G u(i) + 
+ q^T (i) F T P G u(i) + v^T (i) G T P G u(i) + 
+ \gamma v^T (i) v(i) - z^T (i) z(i)
\end{aligned} \right\} = \tag{48}
\]
and with \( T \) given in (26) then (48) implies (22).

5. OPTIMAL STEADY-STATE CONTROL

Theorem 1. For (1), (2), the performance index
\[
J = \sum_{i=0}^{\infty} E_a \left\{ \| z(i) \|^2_2 - \gamma \| v(i) \|^2_2 \right\} < 0 \tag{49}
\]
and \( \gamma > 0, \gamma \in \mathbb{R}, Q^* = Q^T > 0, Q^* \in \mathbb{R}^{n \times n} \) the optimal control law be
\[
\begin{aligned}
&u(i) = -K q(i) \tag{50}
&K(i) = -U^{-1} G^T P F \tag{51}
\end{aligned}
\]
where \( P = P^T > 0 \) is a solution of the equations
\[
\begin{aligned}
P = F^T Y F + F^T a P F a + C^T C \tag{52}
Y = H + P G U^T \tag{53}
\end{aligned}
\]
and \( U = U^T, H = H^T, \) and \( V = V^T > 0 \) are given in (23), (24), and (25), respectively.

Using such control for \( P < Q^* \) the performance index be negative and takes the value
\[
J = q^T (0) (P - \gamma Q^*) q(0) - 
- \sum_{i=0}^{\infty} E_a \left\{ \| v(i) - v^*(i) \|^2 V \right\} < 0 \tag{54}
\]
Proof. In the system under consideration there the mean value of Lyapunov function along a trajectory of the system under control be computed using (21), (22) as
\[
\sum_{i=0}^{\infty} E \{\Delta V(q(i))\} = E \{-q^T(0)Pq(0)\}
\]  
and using (48) even so, that
\[
\sum_{i=0}^{\infty} E \{\Delta V(q(i))\} = \sum_{i=0}^{\infty} \left\{ \|u(i) - u^*(i)\|^2_U - \|v(i) - v^*(i)\|^2_V + +q^T(i)Tq(i) + +u^T(i)G^TPFq(i) + +q^T(i)F^TPGu(i) + \right\}
\]
Adding (56) to (49) and subtracting (55) from (49) the cost function for the control law (5) can be brought to
\[
J = q^T(0)(P - \gamma Q^T)q(0) + +\sum_{i=0}^{\infty} E \left\{ \|u(i) - u^*(i)\|^2_U - \|v(i) - v^*(i)\|^2_V + +q^T(i)Tq(i) + +u^T(i)G^TPFq(i) + +q^T(i)F^TPGu(i) \right\}
\]
Clearly, the optimal control strategy for \(u(i)\) is given by \(u(i) = u^*(i)\), where \(P\) satisfies
\[
F^TPF + F^TPF + C^TC + K_i^T V K_i - K^T U K - -F^TPGK - K^T G^T PF - P = 0
\]  
Substituting (46) and completing to square for such \(K\) from (58) gives
\[
F^TPGK + K^T G^TPF + K^T UK = = F^TPGUN - K_i^T V K_i + K_i^T V K_i U^{-1} G^T PF + +K_i^T V K_i U^{-1} K_i^T V K_i = S = F^TPGUN - G^T PF
\]  
where
\[
S = = \|G^TPF + K_i^T V K_i\|_U^{-1} = \|G^TPF + UK\|_U^{-1}
\]
Setting
\[
K = -U^{-1} G^T PF
\]  
moves \(S\) to zero, and then, using (59) and (37), the equation (58) takes the form
\[
P = F_i^TPF_i + C^TC + +F^T (P + PG_i V^-1 G_i^-1 G^T P)^{-1} + +F^T P^T G_i V^-1 G_i^-1 G^T P^T F
\]
or, taking into account (42), the next form
\[
P = F^TYF + F_i^TPF_i + C^TC
\]
\[
Y = P + H - P G^-1 G^T P
\]
It is evident that using (61), (63), (64) the performance index (57) takes value (54).

6. LMI-BASED DESIGN

Stabilizing control can be obtained using a solution \(P > 0\) of the discrete algebraic inequality derived from (58) in such way that (52) be negative definite.

Since, using (41), and (23) it can be written
\[
(-P + PGU^{-1}G^T P)^{-1} = = P^{-1} - G(U - G^TPG)^{-1} GT = = P^{-1} - GR^{-1} G^T < 0
\]
and since (43) implies
\[
H^{-1} = P^{-1} - \gamma^{-1} G_v G_v^T
\]
then the inequality derived from (52) can take the next form
\[
F^T(H + P)F + F_i^TPF_i + C^TC - P < 0
\]
which is conditioned by
\[
V = \gamma I_p - G_v G_v^T > 0
\]
\[
H^{-1} = P^{-1} - \gamma^{-1} G_v G_v^T > 0
\]

Theorem 2. For a system given in (1), (2) necessary and sufficient condition for a stabilizing control existence is that for given \(R > 0\), \(\gamma > 0\) there exists a matrix \(X = X^T > 0\), \(X \in R^{n \times n}\) such that
\[
X > 0
\]
\[
Z = \begin{bmatrix} -X & XF^T & XF^T & XG^T \\ -X & 0 & 0 & 0 \\ -X & 0 & 0 & 0 \\ -X & 0 & 0 & 0 \\ * & * & * & -I_m \end{bmatrix} < 0
\]
\[
X_v = X - \gamma^{-1} G_v G_v^T > 0
\]

Therefore, the gain matrix of the control law (66) is
\[
K = -U^{-1} G^T X^{-1} F, \quad U = G^T X^{-1} G + R
\]

Hereafter, * denotes the symmetric item in a symmetric matrix, and 0 denotes a null matrix with consistent dimension.

Proof. Pre-multiplying left-hand side, and right-hand side of (67) by \(P^{-1}\) gives
\[
P^{-1} F^T (H + P) F P^{-1} + +P^{-1} F_i^TPF_i P^{-1} + +P^{-1} C^T C P^{-1} - P^{-1} < 0
\]
Setting \(X = P^{-1} > 0\) then (75), (68), and (69) takes the form
\[
X F^T X^{-1} F + X F^T X^{-1} F + +X F_i^T X_i^{-1} F_i + X C^T C X - X < 0
\]
\[
V = \gamma I_p - G_v^T X^{-1} G_v > 0
\]
\[
X_v = H^{-1} = X - \gamma^{-1} G_v G_v^T > 0
\]
7. CONSTRAINED CONTROL
Using control law (5) the steady-state equilibrium control equation takes the form
\[ q(i + 1) = (F + F_a a(i) - G K) q(i) + G_v v(i) \] (79)
Considering the design constraint
\[ E \{ q(i) \} \in \mathcal{N}_L = \{ q : L q = 0 \} \] (80)
\[ L(F - G K) = 0, \quad LF = LGK \] (82)
respectively. Then, (8) implies all solutions of \( K = (LG)^{\ominus 1} LF + (I_m - (LG)^{\ominus 1} LG) K^\circ \) (83)
where \( K^\circ \) is an arbitrary matrix with consistent dimension, and \( (LG)^{\ominus 1} \) is pseudoinverse of \( LG \). This results in
\[ u(i) = -M q(i) + Nu^c(i) \] (84)
\[ M = (LG)^{\ominus 1} LF, \quad N = I_m - (LG)^{\ominus 1} LG \] (85)
\[ u^c(i) = -K^c q(i) \] (86)
(provided \( \varepsilon > 0 \) and \( K^c \) is the tracking desired vector signal, and \( r \) is obtained using the state observer \( \mathcal{V}_w \) designed in such a way that \( \mathcal{V}_w v(i) \) equals the actual \( v(i) \), and \( \mathcal{V}_w \) is defined by \( \mathcal{V}_w = \mathcal{V}_w (\mathcal{V}_w (v(i)) + \varepsilon \bar{v}) \). Theorem 3. For the system given in (1), (2) with equality constraint (6), and gain matrices (85) a solution to constrained control of the form (84), (86) exists if there exists such \( X > 0 \) that inequalities (70)–(73) are satisfied in the same structure but replacing \( F \) and \( G \) by
\[ F^\circ = F - GM, \quad G^\circ = GN \] (87)
respectively. Then the gain matrix \( K^\circ \) is
\[ K^\circ = -U^{-1} G^T X^{-1} F^\circ \] (88)
\[ U = G^T X \gamma G^\circ + R \] (89)
**Proof.** Using equality \( q(i) = q(i) \) and control (84), the system state transformation can be introduced as follows
\[ \begin{bmatrix} q(i) \\ u(i) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -M & N \end{bmatrix} \begin{bmatrix} q^c(i) \\ u^c(i) \end{bmatrix} = T \begin{bmatrix} q^c(i) \\ u^c(i) \end{bmatrix} \] (90)
\[ T^\circ = \begin{bmatrix} I & 0 \\ -M & N \end{bmatrix} \] (91)
to describe modified control law representation for matrices given in (85). Since system (1), (2) is linear in \( q(i) \) with the system state transformation can be introduced as follows
\[ E \{ \Delta v(q(i), u(i)) \} = \begin{bmatrix} q^T(i) & u^T(i) \end{bmatrix} J_V \begin{bmatrix} q(i) \\ u(i) \end{bmatrix} \] (92)
\[ J_V = \begin{bmatrix} F^T P F & -P \\ G^T P F & G^T PG \end{bmatrix} < 0 \] (93)
Using (90), (91) then (92) can be rewritten to the form
\[ E \{ v(q(i), u^c(i)) \} = \begin{bmatrix} q^T(i) & u^cT(i) \end{bmatrix} J^c_V \begin{bmatrix} q(i) \\ u^c(i) \end{bmatrix} \] (94)
where
\[ J^c_V = T^c J_V T^\circ = \left[ (F - GM)^T P (F - GM) - P \\ N^T G^T P (F - GM) \right] < 0 \] (95)
Analogously to (66) it can be defined
\[ F^T (P + H) F^\circ + F_a^T P F_a + C^T C - P < 0 \] (96)
Thus, by following the similar approach as used above in Theorem 2, there are obtained (72), (71) for an equivalent system defined by (87). Note, in this case \( X \) can be singular, and (71) has to be used in a structure \( Z^* = Z^* - \varepsilon I \), where \( 0 < \varepsilon \in \mathbb{R} \) is a design parameter. □
**Remark 1.** Since
\[ F^\circ - G^\circ K^\circ = F - GM - G N K^\circ = \]
\[ = F - GM - (K - M) = F - G K = F_c \] (97)
the eigenvalues spectra of the closed-loop system matrices are equal. As (15) implies the closed-loop system matrix is singular and a singular problem is solved.

7.1 Constrained tracking problem
Considering a tracking problem defined by
\[ u(i) = -K q(i) + W_a w(i) \] (98)
where \( w(i) \in \mathbb{R}^r \) is the tracking desired vector signal, and \( W \in \mathbb{R}^{r \times r} \) is the gain matrix of tracking signal. A forced motion of the system (1), (2) can be written in the form
\[ \begin{bmatrix} q(i + 1) \\ q_a(i) \end{bmatrix} = \]
\[ \begin{bmatrix} F + F_a a(i) - G K \\ F_c \end{bmatrix} q(i) + G_w w(i) + G_v v(i) \] (99)
Theorem 4. Using the state control (98) satisfying the equality constrain (80) then the state constraints given on the system state variables attains the value
\[ q_a(i) = LG_w w(i) + LG_v v(i) \] (100)
**Proof.** Applying (81) then (99) implies
\[ E \{ q(i + 1) \} = G_v v(i) + LG_w w(i) + \]
\[ + L(F + F_a a(i) - G K) E \{ q(i) \} \] (101)
and it is evident that (101) implies (100)

7.2 Observer state feedback
Considering the control law
\[ u(i) = -K q_e(i) \] (102)
where \( q_e(i) \in \mathbb{R}^m \) is obtained using the state observer
\[ q_e(i + 1) = F q_e(i) + G u(i) + J (y(i) - y_e(i)) \] (103)
\[ y_e(i) = C q_e(i) \] (104)
with \( y_e(i) \in \mathbb{R}^m \), and with \( J \in \mathbb{R}^{n \times m} \) designed in such way that \( F - J C \) is stable.

**Theorem 5.** The observer state feedback control (102) of the system (1), (3) with the state observer (103), (104) results in the closed-loop system satisfying the constraint (5) in an observer steady-state regime.
Proof. Assembling the system state equation (1), (2) and the estimator state equation (103), (104) gives
\[\begin{bmatrix} q(i+1) \\ q_e(i+1) \end{bmatrix} = \begin{bmatrix} G_v \\ 0 \end{bmatrix} v(i) + \begin{bmatrix} F + F_a a(i) \\ 0 \end{bmatrix} q(i) + \begin{bmatrix} -GK \\ -JC \end{bmatrix} e(i) \]

Thus, using the state transformation
\[\begin{bmatrix} q(i) \\ e(i) \end{bmatrix} = T_o \begin{bmatrix} q(i) \\ q_e(i) \end{bmatrix}, \quad T_o = T_o^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \]

where \(e(i) = q(i) - q_e(i)\) is the the state reconstruction error, then both systems are governed by the equation
\[\begin{bmatrix} q(i+1) \\ e(i+1) \end{bmatrix} = \begin{bmatrix} L(G_v + F_a a(i)) \\ 0 \end{bmatrix} q(i) + \begin{bmatrix} G_v \\ 0 \end{bmatrix} v(i) + \begin{bmatrix} L(GK) \\ L(JC) \end{bmatrix} e(i) \]

which implies the separation principle. Defining the congruence transform matrix \(T_g\) as follows
\[T_g = \text{diag} \{ L \ I \} \]

then multiplying left-hand side of (107) by \(T_g\) it is obtained
\[\begin{bmatrix} Lq(i+1) \\ Le(i+1) \end{bmatrix} = \begin{bmatrix} LG_v \\ G_v \end{bmatrix} v(i) + \begin{bmatrix} L(JC) \\ L(GK) \end{bmatrix} e(i) \]

Applying constraint states condition to (109) states
\[E_o \begin{bmatrix} Lq(i+1) \\ Le(i+1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
and it is evident that with a stable \(F - JC\), and in an estimator steady-state regime, i.e. when \(e(i+1) = e(i) = 0\), the observer state feedback control satisfies (81).

8. ILLUSTRATIVE EXAMPLE

To demonstrate properties the system described by state-space equations (1), (2) was considered, where \(R = 0.01 I_2\), \(\gamma = 8\), \(t_s = 0.1\) s, and

\[F = \begin{bmatrix} 0.9993 & 0.0987 & 0.0442 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \quad G = \begin{bmatrix} 0.0051 & 0.0050 \\ 0.1029 & 0.0987 \\ 0.0387 & -0.0388 \end{bmatrix} \]

\[G_v = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_a = \begin{bmatrix} 0 & 0 & 0.0004 \\ 0 & 0 & 0 \\ 0.0388 & 0 & 0 \end{bmatrix} \]

Assuming the matrix equality constraint
\[2q_1(t) + q_2(t) = 1 \Rightarrow L = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \]

there were obtained the feedback gain matrix parameters
\[M = \begin{bmatrix} -4.5975 & 4.1756 & -1.4212 \\ -10.8552 & 9.8590 & -3.3557 \end{bmatrix} \quad N = \begin{bmatrix} 0.8479 & -0.3591 \\ -0.3591 & 0.1521 \end{bmatrix} \]

and the new design parameters were recomposed as follows
\[F^o = \begin{bmatrix} 1.0770 & 0.0281 & 0.0282 \\ 1.5233 & -0.4415 & 0.5550 \\ -0.6308 & -0.4978 & 0.4985 \end{bmatrix}, \quad G^o = \begin{bmatrix} 0.0025 & -0.0011 \\ 0.0518 & -0.0219 \\ 0.0467 & -0.0198 \end{bmatrix} \]

Fig. 1. State variables step response \(q_d(i) = Lq(i)\)

Solving (70)–(74) with respect to the LMI variable \(X\), and with parameters \(F^o, G^o\) using Self-Dual-Minimization (SeDuMi) package for MATLAB (Peaucelle et al. (2002)) problem was feasible for \(\varepsilon = 0.005\), and yields
\[K^oT = \begin{bmatrix} -101.1183 & 42.8266 \\ 2.2993 & -0.9738 \\ -7.9346 & 3.3605 \end{bmatrix}, \quad \rho(F - GK) = 0.5107\]

It can be seen there the stable closed-loop system matrix.

The simulation result of the forced closed-loop with \(q(0) \neq 0, v(i+1) = 0.02, w^2(i) = [-0.1 \ 0.5]\) is shown in Fig. 1.

9. CONCLUDING REMARKS

The paper presents the control design principle for discrete-time linear stochastic multi-variable dynamic systems with multiplicative noise and state variable constraints in the form of matrix equalities. Presented applications can be considered as a task concerned the class of \(H_\infty\) stabilization control problems where the stabilizing solutions were new formulated.

REFERENCES