Modeling of Hybrid Lumped-Distributed Parameter Mechanical Systems with Multiple Equilibria.

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Abstract: The objective of this effort is to initiate a study in the control of hybrid lumped-distributed parameter mechanical systems with multiple equilibria. To do so, we study an illustrative problem in the form of a vertical flexible beam with a tip mass at one end and fixed to a cart at the other end. The tip mass is considered as a rigid body and the control force is the horizontal actuation on the cart. The objective is to stabilize the beam in the vertically upright position. This paper examines the modeling aspect alone and determines the equilibria of the system.

Keywords: Hybrid lumped-distributed parameter mechanical systems, multiple equilibria, infinite dimensional system, boundary conditions, actuator.

1. INTRODUCTION

A flexible beam with a tip mass, fixed to a moving cart, is representative of a physical system that needs both a partial differential equation (also termed as infinite dimensional system) and an ordinary differential equation (also termed as finite dimensional system) to describe its dynamic behaviour. It is an interconnection of a finite dimensional tip mass and the cart (modeled by ordinary differential equation) with an infinite dimensional flexible beam (modeled using partial differential equations). Such an assembly finds application in structures such as robot arms with flexible links and tip load. For precise control of these systems it is desirable to account for the dynamics of the flexible part with utmost fidelity. Our objective here is to stabilize the system in the vertically upright position.

A preliminary attempt has been made in this direction to stabilize a flexible beam fixed to a moving cart (See Banavar, Dey [2010]). The control objective was to asymptotically stabilize the beam in the vertically upright position under disturbances using the motion of the cart as the control effort. The problem of stabilizing the displacement of an Euler Bernoulli beam using a boundary control approach is discussed in (Osita et al. [2001]). In (Osita et al. [2001]), the tip mass acts as an actuator and its dynamics are incorporated into the beam model. The approach of control by interconnection and energy shaping of the Timoshenko beam can be found in (Macchelli, Melchiorri [2004]) and (Macchelli, Melchiorri [2004]).

1.1 System description

A schematic of the system of our interest is shown in the Fig. 1. It comprises of the following sub systems: a moving cart, a flexible beam fixed at the bottom to the cart and a rigid tip mass attached to the free end of the flexible beam. The cart has a mass $M$ and an actuating force $F_{control}$ applied to the cart causes a motion of the cart. The coupling between the cart and beam causes motion of the beam and the tip-mass. We move on to nomenclature of variables. The beam has a length $L$, width $a$ and thickness $b$. It is symmetric with respect to the body frame $(x_B-y_B-z_B)$ as shown in Fig. 2. Let the horizontal deflection of the beam from the vertically upright position,
the axial deflection due to loading effect of the tip-mass, and the rotation of the cross-section of the beam due to bending at any point \( l \in [0, L] \) be \( w(l, t) \), \( u_0(l, t) \), and \( \phi(l, t) \) respectively (Refer to Fig. 3). The tip-mass is attached to the flexible beam with the centre at \( l = L \). It has a radius \( r \), mass \( m \) and momentum of inertia \( J \). We assume that the motion of the mass is a combination of both translation and rotation. Let \( x_m \), \( y_m \), and \( \theta_m \) represent the horizontal displacement, vertical displacement and rotation of the tip-mass from the vertical axis respectively. \( x(t) \) is the displacement of the cart with respect to fixed frame of reference as shown in Fig. 4. From Fig. 3.c and Fig. 4, it is straightforward to note that \( x_m = w(L, t) + x(t) \), \( y_m = u_0(L, t) \) and \( \theta = \phi(L, t) \).

1.2 Modelling issues

The beam: We use the Euler-Bernoulli (EB) theory to model the beam. From Fig. 3 it is observed that the flexible displacements vector of the beam is \( u = [w, 0, u_0 - \zeta \phi] \). The gross transverse displacement of a point on the beam at a distance \( l \) from the base with reference to the fixed frame \((x - z)\) is \( v(l, t) = w(l, t) + x(t) \). We note that

\[
v(0, t) = x(t), \]
\[
\frac{\partial v}{\partial t}(l, t) = \frac{\partial w}{\partial l}(l, t) + \dot{x}(t), \quad \frac{\partial v}{\partial z}(l, t) = \frac{\partial w}{\partial z}(l, t).
\]

The gross displacement vector of the flexible beam on a moving cart is thus \( u = [v, 0, u_0 - \zeta \phi] \). The normal strain \( \epsilon \) along the length of the beam (in the \( z \) direction) at a distance \( \zeta \in [-\frac{a}{2}, \frac{a}{2}] \) from the centreline is given by (See VoB et al. [2008])

\[
\epsilon = u_\theta' - \zeta \phi' + \frac{1}{2} v'^2 + \frac{1}{2} (u_\theta' - \zeta \phi')^2 \tag{1}
\]

Note: \( f' = \frac{\partial f}{\partial z} \). EB beam theory assumes that all other strains and shears in the beam are zero. Also the rotation
of the beam from the undeformed position $\phi(z, t)$ is equal to the slope of the transverse displacement of the beam $\frac{\delta u}{\delta z}$. The potential energy of the beam due to strain and gravitation is

$$P_B = \frac{1}{2} \int_0^L \int_A E \varepsilon^2 dA dz - \rho g \left( \frac{L}{2} + w_0 \left( \frac{L}{2} \right) \right)$$

The kinetic energy is given by (See VoB et al. [2008])

$$K_B = \frac{1}{2} \int_0^L \int_V \rho ||\dot{u}||^2 dV$$

where $\rho$, $E$, $G$, $A$, $V$ represent the density per unit volume, Young’s modulus, shear modulus and area of cross-section and volume of the beam respectively. The equations of motion for the beam can be obtained using Hamilton’s principle (See Yu [2006]) which states

$$\delta \int_{t_0}^{t_1} (K_B - P_B) dt = 0$$

along the trajectories of motion between two fixed configurations starting at time $t_0$ and ending at time $t_1$. Evaluating the above integral we have

$$\delta K_B = - \int_0^L \int_A \rho ||\dot{u}||^2 \delta u dA dz$$

$$= - \int_0^L \dot{p}_1 \delta v + \dot{p}_2 \delta \phi + \dot{p}_3 \delta u_0 dz$$

where $I_0 = \int_A \zeta dA = 0$ (since the beam is symmetric in the $x_B - z_B$ plane), $I = \int_A \zeta^2 dA$ and

$$p_1 \triangleq \rho A \dot{u}, \quad p_2 \triangleq \rho I \dot{\phi}, \quad p_3 \triangleq \rho A u_0$$

represent the generalised momenta. Assuming that the gravitational potential energy of the beam is constant, the variational of the potential energy of the beam is

$$\delta P_B = \int_0^L \int_A E \varepsilon dA dz$$

$$= \int_A E \varepsilon' dA |_{t_0}^{t_1} - \int_0^L \frac{\partial}{\partial z} \int_A E \varepsilon(v') \delta v dA dz$$

$$+ \int_0^L \frac{\partial}{\partial z} \int_A E(1 + u_0' - \zeta \phi') \delta u_0 dA |_{t_0}^{t_1} - \int_0^L \frac{\partial}{\partial z} \int_A E(1 + u_0' - \zeta \phi')(\zeta \delta \phi) dA dz$$

Equation (2) must be true for any arbitrary value of $\delta v$, $\delta \phi$, and $\delta u_0$. Further, $\delta v(t_0) = \delta \phi(t_0) = \delta u_0(t_0) = \delta v(t_1) = \delta \phi(t_1) = \delta u_0(t_1) = 0$. With these assumptions, (2) leads to the following equations of motion,

$$\dot{p}_1 = \frac{\partial}{\partial z} \int_A E \varepsilon(v') dA$$

$$\dot{p}_2 = \frac{\partial}{\partial z} \int_A E(1 + u_0' - \zeta \phi')(\zeta) dA$$

$$\dot{p}_3 = \frac{\partial}{\partial z} \int_A E(1 + u_0' - \zeta \phi') dA$$

(3)

We now represent these equations in the form of a state-space model and in terms of the variational derivative of the Hamiltonian.

**Beam equations in a Hamiltonian framework**

Let $\alpha \triangleq (p_1, p_2, p_3, v', \phi', u_0')^T$ represent the state vector. The Hamiltonian of the beam $H_B$ is the sum of kinetic and potential energy,

$$H_B = K_B + P_B = \int_0^L \mathcal{H}_B(\alpha) dz$$

where $\mathcal{H}_B(\alpha)$ represents the Hamiltonian density. The variational derivative of $H_B$ with respect to the state variables is

$$\frac{\delta H_B}{\delta p_1} = \dot{v}, \quad \frac{\delta H_B}{\delta p_2} = \dot{\phi}, \quad \frac{\delta H_B}{\delta p_3} = \dot{u}_0.$$ (4)

$$\frac{\delta H_B}{\delta v} = \int_A E \varepsilon(v') dA$$

$$\frac{\delta H_B}{\delta \phi} = \int_A E(1 + u_0' - \zeta \phi')(\zeta) dA$$

$$\frac{\delta H_B}{\delta u_0} = \int_A E(1 + u_0' - \zeta \phi') dA$$

(5)

The variational derivatives (4) represent the translational velocity, rotational velocity and axial velocity respectively. The equations of motion (3) can now be recast in the Hamiltonian framework as,

$$\frac{\partial \alpha}{\partial t} = \mathcal{J} \frac{\delta \mathcal{H}_B}{\delta \alpha},$$ (6)

where

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & 0 \\ \end{bmatrix}$$

is a 6 × 6 matrix of differential operators, $\frac{\partial}{\partial \alpha}$ represents the vector of efforts and

$$\frac{\delta \mathcal{H}_B}{\delta \alpha} = \begin{bmatrix} \delta H_B/p_1 & \delta H_B/p_2 & \delta H_B/p_3 & \delta H_B/v & \delta H_B/\phi & \delta H_B/u_0 \\ \end{bmatrix}^T$$

represents the vector of flows (See Macchelli et al. [2004]). The rate of change of the Hamiltonian $H_B$ is given by

$$\frac{dH_B}{dt} = \int_0^L \frac{\delta \mathcal{H}_B}{\delta \alpha} \frac{\partial \alpha}{\partial t} dz$$

$$= \int_0^L \frac{\partial}{\partial \alpha} \mathcal{J} \frac{\delta \mathcal{H}_B}{\delta \alpha} dz$$

$$= \int_0^L \frac{\partial}{\partial \alpha} \left( \delta H_B/p_1 \delta v + \delta H_B/p_2 \delta \phi + \delta H_B/p_3 \delta u_0 \right) dz$$

$$= \left( \delta H_B/p_1 \delta v + \delta H_B/p_2 \delta \phi + \delta H_B/p_3 \delta u_0 \right) \bigg|_{t_0}^{t_1}$$

Note that the power balance equation is a quadratic function of the flows evaluated at the boundary of the spatial
domain.

The cart: The cart is described by the 2nd order differential equation \( M \ddot{x} = F_{\text{control}} - F_0 \). Since the cart traverses in the horizontal plane, its total energy (Hamiltonian) \( H_c \) is just the kinetic energy given by \( \frac{1}{2} M \dot{x}^2 \). Defining the momentum \( p_x = M \dot{x} \), \( \dot{H}_c = \frac{1}{2} M \ddot{x}^2 = \frac{1}{2}Mp_x^2 \) and

\[
\dot{x} = \frac{p_x}{M} \quad \ddot{x} = \frac{\partial H_c}{\partial p_x} \quad \dot{p}_x = M \ddot{x} = F_{\text{control}} - F_0
\]

In matrix notation, these equations can be written as

\[
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}}_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \partial H_c \\ \partial \mathbf{p}_x \end{bmatrix} + \begin{bmatrix} 0 \\ F_{\text{control}} - F_0 \end{bmatrix}
\]

(7)

The rate of change of Hamiltonian \( H_c \) is

\[
\frac{dH_c}{dt} = \frac{\partial H_c}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial H_c}{\partial \mathbf{p}_x} \dot{\mathbf{p}}_x = \dot{x}(F_{\text{control}} - F_0).
\]

The tip mass: The differential equations governing the translational and rotational motion of the tip-mass are

\[
\begin{align*}
\dot{m} \ddot{x}_m &= R_L \sin \theta - F_L \cos \theta, \\
\dot{m} \ddot{y}_m &= mg - R_L \cos \theta - F_L \sin \theta,
\end{align*}
\]

\( J \ddot{\theta} = \Gamma_L \).

Let \((x_m, p_{x_m} = m \ddot{x}_m), (y_m, p_{y_m} = m \ddot{y}_m), \) and \((\theta, p_{\theta} = J \ddot{\theta})\) represent the canonical coordinate pairs. The Hamiltonian of the tip-mass \( H_m(x_m, y_m, \dot{x}_m, p_{x_m}, p_{y_m}, p_{\theta}) \) is

\[
H_m = \frac{1}{2m}p_{x_m}^2 + \frac{1}{2m}p_{y_m}^2 + \frac{1}{2}Jp_{\theta}^2 - mgym.
\]

In the Hamiltonian framework, the equations of motion (8) are

\[
\begin{bmatrix} x_m \\ y_m \\ \dot{\theta} \\ p_{x_m} \\ p_{y_m} \\ p_{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial H_m \\ \partial x_m \\ \partial y_m \\ \partial p_{x_m} \\ \partial p_{y_m} \\ \partial p_{\theta} \end{bmatrix} + \begin{bmatrix} \delta H_m \\ 0 \\ 0 \end{bmatrix}
\]

(9)

The rate of change of the Hamiltonian is

\[
\frac{dH_m}{dt} = (\dot{x}_m)(R_L \sin \theta - F_L \cos \theta) + (\dot{y}_m)(-R_L \cos \theta - F_L \sin \theta) + \dot{\theta} \Gamma_L.
\]

2. POWER BALANCE AND BOUNDARY CONDITIONS

The total Hamiltonian \( H_T \) of the system is the sum of the Hamiltonians of the beam, cart and the tip-mass. The rate of change of \( H_T \) is given by

\[
\frac{dH_T}{dt} = \frac{dH_B}{dt} + \frac{dH_m}{dt} + \frac{dH_c}{dt}
\]

\[
= \frac{\delta H_B}{\delta p_1}(L) \delta \phi'(L) + \frac{\delta H_B}{\delta p_2}(L) \delta \phi'(L)
\]

\[
+ \frac{\delta H_B}{\delta p_3}(L) - \frac{\delta H_B}{\delta \theta}(0) \frac{\delta \phi}{\delta \theta}(0)
\]

\[
+ \dot{x}_m(R_L \sin \theta - F_L \cos \theta) + \dot{y}_m(-R_L \cos \theta - F_L \sin \theta)
\]

\[
+ \dot{\theta}(\Gamma_L) + \dot{x}(F_{\text{control}} - F_0).
\]

(10)

From (4) we have,

\[
\frac{\delta H_B}{\delta p_1}(L) = \dot{\omega}(L) = \text{translational velocity at } (l = L) = \dot{x}_m
\]

\[
\frac{\delta H_B}{\delta p_2}(L) = \dot{\theta}(L) = \text{rotational velocity at } (l = L) = \theta
\]

\[
\frac{\delta H_B}{\delta p_3}(L) = \dot{\theta}(L) = \text{axial velocity at } (l = L) = \dot{y}_m
\]

\[
\frac{\delta H_B}{\delta \theta}(0) = \dot{\phi}(0) = \text{velocity of the cart} = \dot{x}
\]

\[
\frac{\delta H_B}{\delta \theta}(0) = \dot{\phi}(0) = 0
\]

\[
\frac{\delta H_B}{\delta \theta}(0) = \dot{\theta}(0) = 0
\]

(11)

Using the above relations, (10) can be written as

\[
\frac{dH_T}{dt} = \dot{x}_m \left( \frac{\delta H_B}{\delta \phi'}(L) + R_L \sin \theta - F_L \cos \theta \right)
\]

\[
+ \dot{y}_m \left( \frac{\delta H_B}{\delta \theta}(L) - \Gamma_L \right)
\]

\[
+ \dot{x} \left( -F_0 - \frac{\delta H_B}{\delta \phi'}(0) \right) + \dot{x}(F_{\text{control}} - F_0).
\]

Since the system is conservative, the rate of change of total energy \( \frac{dH_T}{dt} \) must be equal to the external power supplied \( \frac{dE}{dt} \). This yields the remaining boundary conditions on the beam as

\[
\frac{\delta H_B}{\delta \theta}(L) = -R_L \sin \theta + F_L \cos \theta, \quad \frac{\delta H_B}{\delta \phi'}(L) = -\Gamma_L.
\]

\[
\frac{\delta H_B}{\delta \theta}(L) = R_L \cos \theta + F_L \sin \theta, \quad \frac{\delta H_B}{\delta \phi'}(0) = -F_0.
\]

(12)

2.1 Equilibria of the system

Our interest now lies in examining the equilibria of the system described by (6), (7), and (9) with the boundary conditions (11) and (12). For this purpose we carry out
the following analysis.
At the equilibrium, the cart is stationary. The control force \( F_{control} \) and the net force acting on it is zero, and therefore
\[
\begin{align*}
\dot{x} &= 0 \\
\dot{p}_k &= F_{control} - F_0 = 0 \implies F_0 = 0
\end{align*}
\]
At the equilibrium, the net force and the net torque on the tip mass are zero, and therefore
\[
\begin{align*}
\dot{m}x_m &= R_L \sin \theta - F_{L,\cos \theta} = 0, \\
\dot{m}y_m &= -R_L \cos \theta - F_{L,\sin \theta} + mg = 0, \\
J\dot{\theta} &= \Gamma_\theta = 0.
\end{align*}
\]
Equations (13) and (14) lead to
\[
R_L = mg \cos \theta, \quad F_L = mgsin \theta.
\]
Using the conditions obtained from the cart, tip-mass, and the beam, the boundary conditions (12) are
\[
\begin{align*}
\frac{\delta H_B}{\delta \nu'}(L) &= 0, \quad \frac{\delta H_B}{\delta \phi_0'}(L) = mg, \\
\frac{\delta H_B}{\delta \phi'}(L) &= 0, \quad \frac{\delta H_B}{\delta \phi'}(0) = 0.
\end{align*}
\]
From the differential equations of the beam (6), the equilibrium configurations are the solutions to
\[
\begin{align*}
p_1 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \nu'} = 0 \implies \frac{\delta H_B}{\delta \nu'}(z) = \text{constant}, \\
p_2 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \phi'} = 0 \implies \frac{\delta H_B}{\delta \phi'}(z) = \text{constant}, \\
p_3 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \nu_0'} = 0 \implies \frac{\delta H_B}{\delta \nu_0'}(z) = \text{constant}, \\
p_4 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \phi_1'} = 0 \implies \frac{\delta H_B}{\delta \phi_1'}(z) = \text{constant}, \\
p_5 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \phi_2'} = 0 \implies \frac{\delta H_B}{\delta \phi_2'}(z) = \text{constant}, \\
p_6 &= \frac{\partial}{\partial z} \frac{\delta H_B}{\delta \phi_3'} = 0 \implies \frac{\delta H_B}{\delta \phi_3'}(z) = \text{constant}.
\end{align*}
\]
Using the set of boundary conditions (11) and the equilibrium conditions of the cart and the tip-mass, the solution of the equilibrium equations (20), (21) and (22) gives
\[
\begin{align*}
p_1 &= 0, \quad p_2 = 0, \quad p_3 = 0.
\end{align*}
\]
Using the set of boundary conditions (16) and the expressions for the variational derivatives (4) and (5), the equilibrium equations (17), (18) and (19) give
\[
\begin{align*}
\int_{\alpha} E\nu'(\nu')dA &= 0, \\
\int_{\alpha} E\nu_0'(\nu_0' - \zeta \phi')(\nu_0' - \zeta \phi')dA &= 0, \\
\int_{\alpha} E\nu_0'(\nu_0' - \zeta \phi')dA &= mg
\end{align*}
\]
Using the expressions for strain (1), the above equations take the form
\[
\begin{align*}
EAu_0'\nu' + \frac{EA}{2}(\nu')^3 + \frac{EA}{2}(u_0')^2 \nu' + \frac{EI}{2}(\phi')^2 \nu' &= 0 \\
EIu_0'\phi' + EI(1 + u_0')\phi' + \frac{EI}{2}(\nu')^2 \phi' + \frac{ EI}{2}(u_0')^2 \phi' &= 0 \\
\frac{EI}{2}(u_0')^2 \phi' + EIu_0'\phi'(1 + u_0') + \frac{EI}{2}(\phi')^3 &= 0
\end{align*}
\]
where \( I_2 = \int_{\alpha} \zeta^3 dA \). Also note that \( \int_{\alpha} \zeta^3 dA = 0 \).

An intuitive surmise of the possible equilibria is
\((1)\) A vertically upright position of the beam
\((2)\) An inclination to the right
\((3)\) An inclination to the left

We now determine these equilibria through the following exercise. Suppose \( \phi(z,t) = 0 \) is an equilibrium solution

Then
\[
\phi' = 0
\]
With this assumption, the equilibrium equations (24), (25), (26) give
\[
\left( EAu_0' + \frac{EA}{2}(\nu')^2 + \frac{EA}{2}(u_0')^2 \right) \nu' = 0
\]
\[
EAu_0'(1 + u_0') + \frac{EA}{2}(u_0')^2(1 + u_0')
\]
\[
+ \frac{EI}{2}(u_0')^2(1 + u_0') = mg
\]
The solution to these equilibrium equations is
\[
\nu' = 0
\]
\[
u_0' = -1 + \frac{1}{6} \left[ \frac{108c + 12(-12 + 81c^2)^{1/2}}{108c + 12(-12 + 81c^2)^{1/2}} \right]
\]
\[
+ \frac{2}{108c + 12(-12 + 81c^2)^{1/2}}
\]
where \( c = \frac{2mg}{EA} \). From (23), (27), (28) and (29) the equilibrium configuration is obtained as
\[
\alpha^*(z,t) = (0,0,0,0,0,
\]
\[
-1 + \frac{1}{6} \left[ \frac{108c + 12(-12 + 81c^2)^{1/2}}{108c + 12(-12 + 81c^2)^{1/2}} \right]
\]
\[
+ \frac{2}{108c + 12(-12 + 81c^2)^{1/2}}
\]
This equilibrium configuration corresponds to the vertically upright position of the beam.
Suppose \( \phi(z, t) = k_1z \) is another equilibrium solution. Then

\[
\phi' = k_1
\]

With this assumption, the equilibrium equations (24), (25), (26) give

\[
EAu_0' v' + \frac{EA}{2} (v')^3 + \frac{EA}{2} (u_0')^2 v' + \frac{EI}{2} (k_1)^2 v' = 0
\]

is also an equilibrium.

3. CONCLUSION

Our future work will focus on stabilizing the beam at the vertically upright position using the cart motion as an actuator. We are also currently working on

(1) the existence of other equilibria
(2) the nature of the equilibria obtained so far.

REFERENCES


