State feedback controller for a class of MIMO non triangular systems

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Abstract: This paper presents a controller design for a class of MIMO nonlinear systems involving some uncertainties. The latter is particularly composed by cascade subsystems and each subsystem is associated to a subset of the system outputs and assumes a triangular dependence on its own state variables and may depend on the state variables of all other subsystems. The main contribution consists in extending the available control results to allow more interconnections between the subsystems. Of fundamental interest, it is shown that the underlying tracking error exponentially converges to zero in the absence of uncertainties, and can be made as small as desired by properly specifying the control design parameter in the presence of uncertainties.

Keywords: Nonlinear systems, MIMO systems, Tracking, State feedback high gain control.

1. INTRODUCTION

In this paper, one aims at addressing the output tracking problem for MIMO uniformly observable systems which dynamical behavior can be described by the following state representation:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + \varphi(u(t), x(t)) + B(u(t) + v(t)) \\
y(t) &= C x(t)
\end{align*}
\] (1)

where

- \( x \) denotes the state of the system and is composed as follows

\[
x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \in \mathbb{R}^n \quad \text{with} \quad x^k = \begin{pmatrix} x_{k,1}^1 \\ \vdots \\ x_{k,n_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}
\]

- \( u \) and \( y \) respectively denotes the input and the output of the system

\[
u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} \in \mathbb{R}^p
\]

where \( u_k = \begin{pmatrix} u_{k,1} \\ \vdots \\ u_{k,p_k} \end{pmatrix}, y_k = \begin{pmatrix} y_{k,1} \\ \vdots \\ y_{k,p_k} \end{pmatrix} \in \mathbb{R}^{p_k} \) for \( k = 1, \ldots, q \) with \( u_{k,i}, y_{k,i} \in \mathbb{R} \) and hence

\[
\sum_{k=1}^q p_k = p
\]

- The matrices \( A \) and \( C \) are respectively given by

\[
A = \text{diag}(A_1, \ldots, A_q), \quad A_k = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}
\]

\[
C = \text{diag}(C_1, \ldots, C_q), \quad C_k = [I_{p_k} 0 \ldots 0]
\]

\[
B = \text{diag}(B_1, \ldots, B_q), \quad B_k = [0 \ldots 0 I_{p_k}]
\]

- \( \varphi(u, x) \) denotes the nonlinear function field which is composed as follows

\[
\varphi(u, x) = \begin{pmatrix} \varphi^1(u, x) \\ \varphi^2(u, x) \\ \vdots \\ \varphi^q(u, x) \end{pmatrix}, \quad \varphi^k(u, x) = \begin{pmatrix} \varphi^1_k(u, x) \\ \varphi^2_k(u, x) \\ \vdots \\ \varphi^q_k(u, x) \end{pmatrix}
\]

with \( \varphi^k(u, x) \in \mathbb{R}^{p_k} \) and each function \( \varphi^k(u, x) \in \mathbb{R}^{p_k} \) is differentiable with respect to \( x \) and assumes the following structural dependence on the state variables.

- for \( 1 \leq i \leq \lambda_k - 1 \):

\[
\varphi^k_i(u, x) = \varphi^k_i(x) = \varphi^k_i(x^1, x^2, \ldots, x^{k-1}, x^k, x_1, x_2, \ldots, x_t)
\]

(5)
• for i = λk:
  \[ \varphi_{\lambda_k}(u, x) = \varphi^k_{\lambda_k}(u_1, \ldots, u_{k-1}, u^1, x^2, \ldots, x^q) \]  
  \[ \nu(t) = (\nu^1(t), \nu^2(t), \ldots, \nu^q(t)) \in \mathbb{R}^n, \nu^k(t) = (\nu^1_k(t), \nu^2_k(t), \ldots, \nu^q_k(t)) \in \mathbb{R}^{n_k} \]
where the functions \( \nu^k(t) \in \mathbb{R}^{n_k} \) are given by
\[ \nu^k = \begin{cases} 0 & \text{for } k = 1, \ldots, q \text{ and } i = 1, \ldots, \lambda_k - 1 \\ \nu_k(t) & \text{for } k = 1, \ldots, q \text{ and } i = \lambda_k \end{cases} \]
For k = 1, ..., q, let
\[ \varepsilon_k = \hat{v}_k - v_k \]  
Then one assumes that the \( \varepsilon_k \)'s, are unknown bounded functions, i.e.
\[ \exists \beta > 0; \forall t \geq 0; \| \varepsilon_k(t) \| \leq \beta \]  
The control problem to be addressed consists in an admissible asymptotic tracking of an output reference trajectory
that will be noted \( \tilde{y} = \left( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_q \right) \in \mathbb{R}^p \) where \( \tilde{y}_k \in \mathbb{R}^{n_k} \) for
\( k = 1, \ldots, q \) is assumed to be smooth enough, i.e.
\[ \lim_{t \to \infty} \| y(t) - \tilde{y}(t) \| \leq \alpha \]  
where \( \alpha \) is a non negative scalar with is equal to zero if
\[ \lim_{t \to \infty} \nu(t) - \hat{\nu}(t) = 0 \]  
and henceforth \( \beta = 0 \).

The problems of observation and control of nonlinear systems have received a particular attention throughout the last four decades. Considerable efforts were dedicated to the analysis of the structural properties to understand better the concepts of controllability and of observability of nonlinear systems (Hammouri and Farza [2003], Raimani [1998], Isidori [1995], Gauthier and Kupka [1994], Fliss and Kupka [1983]). Several control and observer design methods were developed thanks to the available techniques, namely feedback linearisation, flatness, high gain, variable structure, sliding modes and backstepping (Farza et al. [2005], Agrawal and Sira-Ramirez [2004], Fliss et al. [1999], Sepulchre et al. [1997], Fliss et al. [1995], Isidori [1995], Krstić et al. [1995]). The main difference between these contributions lies in the design model, and henceforth the considered class of systems, and the nature of stability and performance results. A particular attention has been devoted to the design of state feedback control laws incorporating an observer satisfying the separation principle requirements as in the case of linear systems (Mahmoud and Khalil [1996]).

In this paper, one proposes a state feedback controller for a class of non triangular MIMO nonlinear systems described by (1). The following features are worth to be mentioned

• The last equation of each subsystem, where the input intervenes, may depend on the whole state. The system cannot be henceforth considered as a cascade of subsystems that can be sequentially controlled. This is the main issue in the control design.

• The absence of zeros in the considered class of systems is particularly motivated by pedagogical purposes as one seeks to emphasize the rational behind the control design method. Nevertheless, the proposed method can be straightforwardly extended to minimum phase systems (see for instance Praly [2003]).

• For the sake of space limitation, only state feedback control is considered in this paper. Nevertheless, as the system (1) is uniformly observable, the proposed state feedback control law can be combined with an appropriate high gain observer to get an output state feedback controller as in Hajji et al. [2008]. A high gain state observer for a class of systems similar to (1) has been proposed in Farza et al. [2011].

Notice that the authors in Liu et al. [1999] considered a stabilization problem for a class of non uniformly observable systems that has not a triangular form. The proposed method cannot be extended to tackle neither the output feedback stabilization, nor the tracking problem.

The state feedback control design under consideration is particularly suggested from the high gain observer design bearing in mind the control and observation duality. Of particular interest, the controller gain involves a well defined design function which provides a unified framework for the high gain control design, namely several versions of sliding mode controllers are obtained by considering particular expressions of the design function. Furthermore, it is shown that an integral action can be simply incorporated into the control design to carry out a robust compensation of step like disturbances.

This paper is organized as follows. The reference model is introduced in section 2 with an explicit procedure for the determination of its state and input. Some useful preliminaries, notations and definitions are given in section 3 with a technical lemma which is crucial for the controller design. Section 4 is devoted to the state feedback control design with a full convergence analysis of the tracking error. The possibility to incorporate an integral action into the control design is shown in section 5. Simulation results are given in section 6 in order to show the effectiveness of the proposed control design method.

2. REFERENCE MODEL

System (1) can be written under the following more developed form as follows
\[ \begin{cases} \dot{x}^k = A_k x^k + \varphi^k(u, x) + B_k (u_k + \nu_k) \\ y_k = C_k x^k \quad k = 1, \ldots, q \end{cases} \]  
Taking into account the class of systems, it is possible to determine the system state trajectory \( \bar{x} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x_q \end{bmatrix} \in \mathbb{R}^{n_q} \)
\( \mathbb{R}^n \) and the system input sequence \( \bar{u} \in \mathbb{R}^p \) with \( \bar{x}^k \in \mathbb{R}^{n_k} \) and \( \bar{u}_k \in \mathbb{R}^{p_k} \), \( k = 1, \ldots, q \), corresponding to the output trajectory \( \bar{y}(t) \). This allows to define an admissible reference model as follows

\[
\begin{align*}
\begin{cases}
\dot{x}_k^k = A_k \dot{x}_k + \varphi_k^k(\bar{u}, \bar{x}) + B_k(\bar{u}_k + \dot{v}_k) \\
\dot{y}_k = C_k \dot{x}_k \\
\end{cases} \quad \text{for} \quad k = 1, \ldots, q \\
\end{align*}
\tag{11}
\]

where \( \dot{v}_k \) is the available approximation of \( v_k \). The determination of the components of the reference model state, \( \bar{x}_k \) as well as those of its input, \( \bar{u}_k \) is described in the next subsection.

### 2.1 Determination of the state and input of the reference model

One shall firstly detail the determination of \( \bar{x}_1^i, \ i = 1, \ldots, \lambda_1 \). Indeed, one has:

\[
\begin{align*}
\bar{x}_1 &= \bar{y}_1 \\
\bar{x}_i &= \bar{x}_{i-1} - \psi_{i-1}^1(\bar{x}_1, \ldots, \bar{x}_{i-1}) \quad \text{for} \quad i = 2, \ldots, \lambda_1 \\
\end{align*}
\tag{12}
\]

By assuming that the reference trajectory \( \bar{y} \) is smooth enough, one can recursively determine the components of \( \bar{x}^1 \) from \( \bar{y}_1 \) and its \( (\lambda_1 - 1) \) first time derivatives, i.e.

\[
\bar{y}^{(i)} = \frac{d^i \bar{y}}{dt^i}, \ i = 1, \ldots, \lambda_1 - 1,
\]

as follows:

\[
\bar{x}_1^i = \beta_i^1(\bar{y}_1, \bar{y}_{i-1}, \bar{y}_{i-2}), \quad i = 1, \ldots, \lambda_1
\tag{14}
\]

where the functions \( \beta_i^1 \) are defined as follows:

\[
\begin{align*}
\beta_i^1(\bar{y}_1) &= \bar{y}_1 \\
\beta_i^1 &= \sum_{j=0}^{i-2} \frac{\partial \beta_{i-1}^1}{\partial \bar{y}_j}(\bar{y}_1^{(j+1)} - \psi_{i-2}^1(\bar{x}_1, \ldots, \bar{x}_{i-1})) \\
\end{align*}
\tag{15}
\]

Now, one can proceed in a similar manner in order to determine recursively \( \bar{x}_i^k \), for \( k = 2, \ldots, q \) and \( i = 1, \ldots, \lambda_k \), as follows:

\[
\begin{align*}
\bar{x}_i^k &= \beta_i^k(\bar{y}_i, \ldots, \bar{y}_{i-(\lambda_{k-1}-1)}, \bar{y}_{i-\lambda_{k-1}}, \bar{y}_{i-\lambda_{k-2}}, \ldots, \bar{y}_{i-2}, \bar{y}_{i-1}) \\
\end{align*}
\tag{16}
\]

\[
\begin{align*}
\bar{y}_i &= \bar{y}_{i-1} - \psi_{i-1}^k(\bar{x}_1, \ldots, \bar{x}_{i-1}) \\
\end{align*}
\tag{17}
\]

Once all the components of the reference model state \( \bar{x} \) are determined, one can compute the components of the reference model input using system (11) as follows:

\[
\begin{align*}
\bar{u}_1^k &= \varphi_{1}^k(\bar{y}_1, \bar{x}) - \dot{v}_1(t) \\
\bar{u}_k &= \varphi_{k}^k(\bar{y}_1, \ldots, \bar{y}_{k-1}, \bar{x}) - \dot{v}_k(t) \\
\end{align*}
\tag{18}
\]

The tracking problem (9) can be hence turned to a state trajectory tracking problem defined by

\[
\lim_{t \rightarrow +\infty} \| e(t) \| \leq \alpha \quad \text{where} \quad e(t) = x(t) - \bar{x}(t)
\tag{19}
\]

Such a problem can be interpreted as a regulation problem for the tracking error system obtained from the system and model reference state representations (10) and (11), respectively, namely

\[
\begin{align*}
\dot{\bar{x}}_k^k &= A_k \varphi_k^k(u, x) + \varphi_k^k(\bar{u}, \bar{x}) + B_k(\bar{u}_k + \dot{v}_k) \\
\dot{\bar{y}}_k &= C_k \dot{x}_k \\
\end{align*} \\
\tag{20}
\]

### 3. SOME DEFINITIONS AND NOTATIONS

For \( k = 1, \ldots, q \), let \( \Delta_k(\rho) \) be the diagonal matrix defined by:

\[
\Delta_k(\rho) = \text{diag} \left( \frac{1}{\rho_k} I_{p_k}, \ldots, \frac{1}{\rho_k(\lambda_k - 1)} I_{p_k} \right)
\tag{21}
\]

where \( \rho > 0 \) is a real number and one defines \( \delta_k \) which indicates the power of \( \rho \) as follows:

\[
\begin{align*}
\delta_k &= 2^{q-k} \left( \prod_{i=0}^{k-1} \left( \lambda_i - \frac{3}{2} \right) \right) \\
\delta_q &= 1 \\
\end{align*}
\tag{22}
\]

One also defines for \( k = 1, \ldots, q \) and \( i = 1, \ldots, \lambda_k \), the following sequence of scalar numbers:

\[
\sigma_i^k = \sigma_i^1 + (i-1) \delta_k
\]

with \( \sigma_i^k = -(\lambda_k - 1) \delta_k + (\lambda_k - 1) \delta_1 + \eta \left( 1 - \frac{1}{2^{q-k-1}} \right) \)

where \( 0 < \eta \leq 1 \) is an arbitrarily small non negative number.

One can check that \( \sigma_i^k \geq 0 \) for \( k = 1, \ldots, q \) and \( i = 1, \ldots, \lambda_k \).

Similarly to the \( \Delta_k \)'s, one defines for \( k = 1, \ldots, q \), the diagonal matrices \( \Lambda_k \)'s as follows:

\[
\Lambda_k(\rho) = \rho^{-\sigma_i^1} \Delta_k(\rho)
\tag{24}
\]

Notice that, according to the definition of the \( \sigma_i^k \)'s given by (23), one has \( \sigma_{\lambda_k}^k = (\lambda_k - 1) \delta_1 + \eta \left( 1 - \frac{1}{2^{q-k-1}} \right) \) and therefore one has

\[
\sigma_{\lambda_k}^k - \sigma_{\lambda_l}^l = \eta \left( \frac{1}{2^{q-k-1}} - \frac{1}{2^{q-l-1}} \right) \\
\]

This means that whatever is the difference between \( \lambda_k \) and \( \lambda_l \), the difference between \( \sigma_{\lambda_k}^k \) and \( \sigma_{\lambda_l}^l \), which are the powers of \( \rho \) on the last rows of \( \Lambda_k(\rho) \) and \( \Lambda_l(\rho) \), respectively, can be made as small as desired by choosing \( \eta \) small enough (very close to zero).
Now, taking into account the structures of $\Lambda_k$, $\Delta_k(\rho)$ and $A_k$, respectively given by (24), (21) and (2), one can show that the following identities hold:

\begin{align}
\cdot \Lambda_k(\rho)A_k^{-1}(\rho) &= \Delta_k(\rho)A_k\Delta_k^{-1}(\rho) = \rho^\delta_k A_k \\
\cdot \rho^{-\delta_k}C_k\Lambda_k^{-1}(\rho) &= C_k\Delta_k^{-1}(\rho) = C_k
\end{align}

Two other set of matrices that shall be used in the controller design are $S_k$ and $P_k$, $k = 1, \ldots, q$ which are defined as follows. Indeed, let consider the following algebraic Lyapunov equation

$$S_k + A_k^T S_k + S_k A_k = C_k^T C_k$$

(27)

where $A_k$ and $C_k$ are defined in (2) and (4), respectively. It has been shown that such this equation has a unique solution $S_k$ that is symmetric positive definite and one has (Gauthier et al. [1992])

$$S_k^{-1}C_k^T = (C_{\lambda_k}^1 I_{p_k}, \ldots, C_{\lambda_k}^l I_{p_k})^T$$

(28)

where $C_{\lambda_k}^i = \frac{\lambda_k!}{i!(\lambda_k - i)!}$ for $i = 1, \ldots, \lambda_k$

Similarly, consider the following algebraic Riccati equation:

$$P_k + A_k^T P_k + P_k A_k = P_k B_k B_k^T P_k$$

(29)

Again, equation (29) admits a unique solution $P_k$ that is symmetric positive definite and it can be expressed as follows:

$$P_k = T_k S_k^{-1} T_k$$

(30)

where $S_k$ is the solution of (27). This can be checked by multiplying the left and right sides of equation (27) by $T_k S_k^{-1}$ and $S_k^{-1} T_k$, respectively and by using the facts that $T_k A_k T_k = A_k$, $T_k T_k = I_k$ and $B_k = T_k C_k^T$. As a result, one has:

$$B_k^T P_k = C_k S_k^{-1} T_k = (C_{\lambda_k}^1 I_{p_k}, \ldots, C_{\lambda_k}^l I_{p_k})$$

(31)

In the sequel, one shall denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and largest eigenvalues of $(\cdot)$, respectively. One also defines the following block diagonal matrix:

$$P = diag(P_1, P_2, \ldots, P_q)$$

(32)

Before ending this section, one shall give a technical lemma, given in [Faral11] needed in the proof of our main result. This lemma allows to provide a sequence of reals that reflects in some sense the interconnections between the blocks nonlinearities.

**Lemma 3.1.** Let

$$k_{l,i}^{i,j} = \begin{cases} 0 & \text{if } \frac{\partial \phi_{i,j}^k}{\partial x_i^j}(u, x) \equiv 0 \\ 1 & \text{otherwise} \end{cases}$$

(33)

for $k, l = 1, \ldots, q$, $i = 1, \ldots, \lambda_k$ and $j = 2, \ldots, \lambda_l$. Then, the sequence of real numbers $\sigma_k^i$ defined by (23) are such that

$$\lambda_{\min}(\sigma_k^i) \geq \lambda_{\min}(\frac{\sigma_k^i}{2} - \frac{\delta_k}{2}) \geq -\frac{\eta}{2^q}$$

(34)

**4. STATE FEEDBACK CONTROL**

As it was early mentioned, the proposed state feedback control design is particularly suggested by the duality from the high gain observer design proposed in Farza et al. [2011]. The underlying state feedback control law is then given by

$$v_k(x^e) = \begin{cases} -K_k(\rho^\delta_k B_k^T P_k \Delta_k(\rho)e^k) & \text{if } e^k = x^e - \hat{x}^e \\ \hat{u}_k + v_k(x) & \text{otherwise} \end{cases}$$

(35)

where $e^k = x^e - \hat{x}^e$, $\rho > 0$ is the controller design parameter, $\Delta_k(\rho)$ and $P_k$ are given by (21) and (29), respectively and finally $K_k : \mathbb{R}^{p_k} \to \mathbb{R}^{p_k}$ is a bounded design function satisfying the following property

$$\forall \xi_k \in \Omega_k \text{ one has } \xi_k^T K_k(\xi_k) \geq \frac{1}{2} \xi_k^T \Omega_k \xi_k$$

(36)

where $\Omega_k$ is an arbitrarily (fixed) compact subset of $\mathbb{R}^{p_k}$. Many functions satisfying condition (36) are given in Farza et al. [2005].

Before stating the fundamental result of the paper, one needs the following technical assumption, usually adopted when using high gain approaches, namely

**Assumption 1.** $\phi(u, x)$ is a globally Lipschitz nonlinear function with respect to $x$ uniformly in $u$.

One can then states the following fundamental result

**Theorem 4.1.** Assume that system (1) satisfies assumption (1), then

$$\exists \rho_0 > 0; \forall \rho > \rho_0; \exists \gamma > 0; \exists \mu > 0; \exists \alpha > 0$$

such that for $k \in \{1, \ldots, q\}$

$$\|x^k(t) - \hat{x}^k(t)\| \leq \lambda_{\min}(\phi)\|x(0) - \hat{x}(0)\| + \alpha\rho$$

where $\beta$ is the upper bound of $\|\nu - \hat{\nu}\|$. Moreover, $\lambda_{\min}(\cdot)$ is polynomial in $\rho$, $\lim_{\rho \to \infty} \mu = +\infty$ and $\lim_{\rho \to \infty} \alpha = 0$.

**Sketch of the proof of Theorem 4.1.** Set the tracking error $e(t) = x(t) - \hat{x}(t)$ and let $e^k(t)$ be the $k$'th subcomponent of $e(t)$. One has:

$$\begin{align}
\dot{e}^k &= \Lambda_k e^k + \varphi^k(u, x) - \varphi^k(u, \hat{x}) - B_k K_k(\rho^\delta_k B_k^T P_k \Delta_k(\rho)e^k) - B_k \varepsilon_k \\
&= \Lambda_k e^k - \frac{1}{\rho^2 e^2 + 2(\lambda_k - 1)\delta_k} \rho^\delta_k + \lambda_k A_k \dot{\varepsilon}_k - 2(\lambda_k - 1)\delta_k B_k K_k(\rho^\delta_k + \lambda_k A_k \dot{\varepsilon}_k) B_k^T P_k e^k
\end{align}$$

(37)

where $\varepsilon_k = \hat{u}_k - v_k$ as defined by (7).

For $k = 1, \ldots, q$, set

$$e^k = \Lambda_k(\rho) e^k$$

(38)

where $\Lambda_k(\rho)$ is given by (24).

From equation (37) and using identities (26), one gets

$$\dot{e}^k = \rho^\delta_k A_k e^k + \Lambda_k(\rho) (\varphi^k(u, x) - \varphi^k(u, \hat{x})) - \Lambda_k(\rho) B_k \varepsilon_k$$

$$- \frac{1}{\rho^2 e^2 + 2(\lambda_k - 1)\delta_k} \rho^\delta_k + \lambda_k \dot{\varepsilon}_k - 2(\lambda_k - 1)\delta_k B_k K_k(\rho^\delta_k + \lambda_k \dot{\varepsilon}_k) B_k^T P_k e^k$$

Set

$$V_k(e^k) = e^k B_k^T P_k e^k$$

(39)

where $P_k$ is given by (29) and let $V(e) = \sum_{k=1}^q V_k(e^k)$ be the candidate Lyapunov function.
Proceeding as in Farza et al. [2011], one can show that for \( k = 1, \ldots, q \), one has

\[
\sqrt{V_k(\hat{e}^k(t))} \leq \exp \left( -\frac{\rho(1 - 2n^2\gamma_\mu \rho - \frac{\eta}{2})}{2} t \right) \sqrt{V(e(0))} + \frac{2q\sqrt{\lambda_{\text{max}}(P)}\beta}{(1 - 2n^2\gamma_\mu \rho - \frac{\eta}{2})\rho^{1+(\lambda_1-1)\delta_1}}
\]

which yields to

\[
\|e^k(t)\| \leq \rho(\lambda_1-1)\delta_1 + \mu_P e^{-\frac{\rho(1 - 2n^2\gamma_\mu \rho - \frac{\eta}{2})}{2}} \|e(0)\| + \frac{2\mu_P}{(1 - 2n^2\gamma_\mu \rho - \frac{\eta}{2})\rho^{1-\eta}}\beta
\]

where \( \gamma \) and \( \mu_P \) are positive real constants. This ends the proof of the theorem.

It is worth noticing that the tracking error converges exponentially to zero in the absence of uncertainties and can be made as small as desired in the presence of uncertainties. Indeed, the tracking error remains in a ball with a radius proportional to \( \frac{1}{\rho^{1-\eta}} \) which can be made arbitrarily small for high values of \( \lambda_1 \). Nevertheless, one notices that this radius is also proportional to the estimation error on \( \nu \), namely \( \hat{\nu}(t) - \nu(t) \). Thus, a good estimate \( \hat{\nu}(t) \) of \( \nu(t) \) also allows to obtain an ultimate bound for the tracking error which is relatively small without requiring high values for \( \rho \).

5. INTEGRAL ACTION

One can easily incorporate an integral action, into the proposed state feedback control design for performance enhancement purposes, by simply introducing suitable state variables as follows \( e_0^k = e^k \) where \( e_0^k \) is the integral of the error output associated to the block \( k \). The state feedback gain is then determined from the control design model

\[
\begin{align*}
\dot{e}^k &= A_k e^k + \phi^k(u, x) - \varphi^k(u, \hat{x}) + \hat{B}_k(u_k - \hat{u}_k + \hat{\nu}_k - \nu_k) \\
\dot{\hat{y}}_k &= C_k e^k = e_0^k
\end{align*}
\]

with \( k = 1, \ldots, q \) and

\[
\dot{e}^k = \left( \begin{array}{c}
e_0^k \\ e_0^k \end{array} \right), \quad A_k = \left( \begin{array}{c}
0_{pk} I_{pk} \\
0_{pk} 0_{pk}
\end{array} \right), \quad \hat{B}_k = \left( \begin{array}{c}
0_{pk} \\
0_{pk} \hat{B}_k
\end{array} \right)
\]

\[
C_k = \left( C_k 0_{pk} \right), \quad \varphi^k(u, x) = \left( \begin{array}{c}
\varphi^k(u, x) \end{array} \right)
\]

Indeed, the control design model structure (42) is similar to that of the error system (20) and hence the underlying state feedback control design is the same. The state feedback control law incorporating an integral action is then given by

\[
\begin{align*}
\hat{\nu}_k(\hat{e}^k) &= -K_k \left( \rho^{\lambda(\lambda_1+1)\delta_1} B_k^T \hat{P}_k \Delta_k(\rho) e^k \right) \\
u_k(\hat{e}) &= \hat{u}_k + \hat{\nu}_k(\hat{e}^k)
\end{align*}
\]

with

\[
\Delta_k(\rho) = \text{diag} \left( I_{pk}, \frac{1}{\rho_{pk}}, I_{pk}, \ldots, \frac{1}{\rho_{pk}^{\lambda(\lambda_1+1)\delta_1}}, I_{pk} \right)
\]

where \( \hat{P}_k \) is the unique symmetric positive definite matrix solution of the following Riccati algebraic equation

\[
\hat{P}_k + \hat{P}_k A_k + A_k^T \hat{P}_k = \hat{P}_k \hat{B}_k B_k^T \hat{P}_k
\]

and \( K_k \) is a function satisfying condition (36).

6. EXAMPLE

In the following, simulation results are given to show the effectiveness of the proposed controller design using the following MIMO nonlinear system which is under form (1):

\[
\begin{align*}
x_1^k &= x_1^k - x_1^k + s_1 \\
x_2^k &= x_2^k - x_2^k + s_1 \\
x_3^k &= -x_3^k - \frac{x_1^k}{1+\left( x_1^k \right)^2} + x_2^k - \frac{(x_1^k)^2}{1+\left( x_1^k \right)^2} + \nu_1 \\
y_1 &= x_1^k \\
y_2 &= x_2^k - x_1^k - x_2^k + x_2^k + s_1 \\
y_3 &= x_2^k - x_2^k + u_1 + u_2 + \nu_2 \\
y_4 &= y_1 y_2 \quad T = x_1^k (x_1^k)^T
\end{align*}
\]

For the output \( y_1 \), the desired output reference sequence, i.e. \( y_1 \) is generated from a third order generator with unitary static gain and a trapezoidal input sequence, while the desired output reference sequences for the outputs \( y_2 \) and \( y_3 \) are \( y_2 = 0.2 \sin(0.2t) \) and \( y_3 = 0.2 \cos(0.2t) \). Simulation have been carried out by considering the disturbances \( \nu_1, \nu_2, \) and \( \nu_2 \) which are zero until \( t = 200s \) at which they take the following expressions: \( \nu_1 = 0.2 \sin(5t) \), \( \nu_2, 0.3 \sin(10t) \) and \( \nu_2 = 0.3 \sin(5t) \). Moreover, additive step like disturbances, \( s_2,1 \) and \( s_2,2 \), have been applied on the dynamics of \( x_1^k \) and \( x_2^k \) at times \( t = 100s \) and \( t = 135s \), respectively and their respective amplitudes are \( s_2,1 = 0.3 \), \( s_2,2 = 0.3 \). For the first input, a step like disturbance, \( s_1 \), with an amplitude equal to 0.5 has been applied on its dynamics between times \( t_1 = 30s \) and \( t_2 = 60s \). No estimates for the unmodelled disturbances have been considered, i.e. these estimates have been taken equal to zero. The expression of all the design functions \( K_k \) is \( K_k(\xi) = 10 \tanh(\xi) \) and the value of the design parameter \( \rho \) has been fixed to 2. The resulting input/output performances are shown in figure 1 where the outputs are particularly displayed with their desired reference trajectories. The obtained results show the good performance of the proposed controller. A zoom on the tracking errors is shown in figure 2 to emphasize the tracking performance. Notice that the tracking errors go to zero in the absence of unmodelled disturbances. In the presence of such disturbances (from \( t = 200s \)), the tracking errors lie in a ball with a relatively small radius. These performances confirm the theoretical results developed in the previous sections.

7. CONCLUSION

A unified high gain state feedback control design framework has been developed to address an admissible tracking
problem for a class of controllable nonlinear systems. The unifying feature is provided through a suitable design function that allows to rediscover all those well known high gain control methods, namely the sliding modes control. The effectiveness of the proposed feedback control method has been emphasized throughout an illustrative example in simulation.

Notice that an output feedback control can be derived by combining an appropriate observer with the proposed state feedback as the system under consideration is observable for any input. This shall be presented in a forthcoming work.

REFERENCES