Synchronization and Non-Synchronization Properties of Directed Networks with Constant Out-Degree

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Abstract: Networks in which each node is connected to a fixed number of neighbors are raising more and more interest, both for modeling biological and social phenomena, and for improving the performance of distributed algorithms, like consensus problems, distributed sensing, motion coordination. In this paper we highlight some interesting properties about the spectrum of the eigenvalues of Laplacian matrices of networks with fixed number of neighbors, and connect them to the synchronization attitude of the network, through the Master Stability Function approach.

Keywords: Synchronization; Eigenvalue Problems.

1. INTRODUCTION

The interest of the scientific community in synchronization problems has been very high at least for the last two decades. One of the most studied issues in this area is the synchronization of oscillators coupled on networks (Boccaletti [2008], Wu [2007], Osipov et al. [2007]).

The study of synchronization problems has applications in many fields, such as biological and social systems (Strogatz [2003]), coupled lasers (Strogatz [2001]), consensus problems (Li et al. [2010]), coordinated motion of agents (Paley et al. [2007]).

Especially concerning the synchronization of chaotic oscillators, researchers have defined many forms of synchronization, such as complete or identical synchronization, phase synchronization, lag synchronization, generalized synchronization, intermittent lag synchronization, imperfect phase synchronization, and almost synchronization (Boccaletti [2008]). In this paper we refer to the strongest form of synchronization, i.e., complete synchronization, in which a perfect match of the chaotic trajectories of the oscillators occurs, so that the difference between the state variables of any couple of oscillators in the network vanishes asymptotically in time.

Many methods to study the synchronization properties of networks of dynamical systems have been defined in the last 20 years. One of the most known is based on the so-called Master Stability Function and was introduced by Pecora and Carroll [1998]. It is based on the analysis of the transverse modes of the variations around the synchronization manifold, and provides a criterion for the local stability of the synchronization manifold. One key issue of this approach is the localization of the eigenvalues of the Laplacian matrix of the network.

The relation between network structure and its synchronizability properties is a very interesting yet non trivial issue. Some work has been carried out in this direction, which, in our opinion, deserves more in-depth studies. In particular, the synchronizability of a network has been put in relation with the average network distance (Nishikawa et al. [2003]), betweenness centrality (Hong et al. [2004]), degree distribution (Wu [2005]), clustering coefficient (Nishikawa et al. [2003]). In (Duan et al. [2008, 2007]), the authors deal with the issue by considering regular network configurations (and suitable modification starting from them) and studying the related synchronization properties by making use of the complementary graph theory.

In this paper we are interested in the synchronization properties of a particular class of networks: those networks in which each node has the same (out)-degree \( k \), that is to say, each node can communicate exactly with \( k \) neighbors picked around him according to some criterion (randomly, with closest distance, etc.). The communication scheme is asymmetric, in the sense that when a node can retrieve information from a neighbor, this does not necessarily imply the vice versa. As a consequence, the graph which describe the communication scheme is a directed one. In the following, we will refer to this type of networks with...
the term “$k$-neighbors network”. This type of network has recently raised the scientific interest in many fields. At first, interest on these networks has been driven by modeling biological and social phenomena. For example, in (Ballerini et al. [2008]), the authors carried out a field study on the coordinated motion of flocks of starlings and, through a novel approach based on 3D image processing concluded that in those animal groups an individual moves by interacting with a fixed number of neighbors (six, in this case), rather than with all the individuals contained in a given radius of interaction, as conjectured for example by Couzin et al. [2005, 2002]. Ma et al. [2010] show that modeling the pedestrian dynamics with a $k$-nearest neighbors interaction scheme can take into account the phenomenon of self-organized counterflow. A second field of interest concerns technological issues. Problems of coordinated motion of robots, distributed sensing, consensus, and so on, can in fact benefit of the advantages induced by an interaction with a fixed size neighborhood (rather than interacting with all the agents contained in an interaction radius), in term of increased simplicity, robustness, and reduced computational and communication activity (Buscarino et al. [2009]).

With this in view, we analyze in this paper the spectral properties of the Laplacian matrices of $k$-neighbors networks. As discussed above, the localization of the eigenvalues of the Laplacian matrix of the network is a critical issue in coping with synchronization problems. We will show that the Laplacian matrices of $k$-neighbors networks exhibit interesting spectral properties, and we highlight the connection of these properties with the synchronization and non synchronization capabilities of a network of chaotic oscillators. The paper is structured as follows. In the next Section we recall the basic facts about the Master Stability Function approach. Then, we discuss the spectral properties of $k$-neighbors networks, by formulating a conjecture on the localization of the eigenvalues of their related Laplacian matrices, based on numerical evidence. Moreover, we give two theorems concerning the non synchronization issues. Finally, our conclusions are drawn and further developments are proposed.

2. THE MASTER STABILITY FUNCTION APPROACH

One well-known and successful approach to study the conditions under which $N$ identical oscillators coupled through an arbitrary network synchronize has been proposed by Pecora and Carroll [1998] by linearizing the network dynamics around the synchronization manifold, provided that: the network nodes have the same dynamics, the coupling function is the same for all nodes and can be approximated by a linear operator around the synchronization manifold, and that such a manifold exists and is invariant.

In (Pecora and Carroll [1998]) the dynamics of each node is modelled as

$$\dot{x}_i = F(x_i) + \sigma \sum_{j=1}^{N} L_{ij} \mathbf{H}(x_j)$$

(1)

where $i = 1, \ldots, N$, $x_i$ is a $m$-dimensional vector of dynamical variables of the $i$th node, $\dot{x}_i = F(x_i)$ represents the dynamics of each isolated node, $\sigma$ is the coupling strength, $\mathbf{H} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the coupling function and $L \in \mathbb{R}^{n \times n}$ is the Laplacian of the network. The dynamics of the whole network will then be given by

$$\dot{x} = \mathbf{F}(x) + \sigma L \otimes \mathbf{H}(x).$$

(2)

Then, the variational equation describing the variations which are transverse to the synchronization manifold is considered, denoting with $\xi$ the collection of the transverse variations:

$$\dot{\xi} = [\mathbb{1}_N \otimes DF + \sigma L \otimes D\mathbf{H}] \xi.$$  

(3)

By diagonalizing $L$, if possible$^1$, one obtains a block-diagonal form of Eq. (3)

$$\dot{\xi}_h = [DF + \sigma \lambda_h \mathbf{D}\mathbf{H}] \xi_h,$$

(4)

where $\lambda_h$ is an eigenvalue of $L$ (we remind that $\lambda_0 = 0$, as $L$ is a zero-row-sum matrix by definition), $DF$ and $D\mathbf{H}$ are the Jacobian matrices of $\mathbf{F}$ and $\mathbf{H}$ computed around the synchronization manifold, and are the same for each block. Therefore, each block differs from any other only by the term $\sigma \lambda_h$. With the aim in mind to study the synchronization properties with respect to different topologies, it is useful to study the variational equation (4) as a function of a generic complex quantity $\alpha + i\beta$. This equation is called the Master Stability Equation:

$$\dot{\xi} = [DF + (\alpha + i\beta) \mathbf{D}\mathbf{H}] \xi.$$  

(5)

The Master Stability Function (MSF) is finally obtained by computing the value of the maximum (conditional) Lyapunov exponent of Eq. (5) as a function of $\alpha$ and $\beta$. To study the stability of the synchronized state for a given topology, one has to compute the eigenvalues $\lambda_h$ (with $h = 2, \ldots, N$) of the Laplacian matrix $L$ and check whether the sign of the MSF (i.e., the sign of the maximum conditional Lyapunov exponent) is negative at each point $\alpha + i\beta = \sigma \lambda_h$. In other words, the network will synchronize if all the products $\sigma \lambda_h$ will fall into the synchronization region $\Omega$, defined as

$$\Omega := \{z \in \mathbb{C} : \text{MSF}(z) < 0\}.$$  

(6)

In this paper we refer to a network of Rössler chaotic oscillators, for which the dynamics of the isolated node is given by

$$\begin{align*}
\dot{x}_1^i &= -(x_2^i + x_3^i) \\
\dot{x}_2^i &= x_1^i + ax_2^i \\
\dot{x}_3^i &= b + x_3^i(x_1^i - c)
\end{align*}$$

(7)

with $x^i(t) = [x_1^i(t) \ x_2^i(t) \ x_3^i(t)]^T$, and $i = 1, 2, \ldots, N$. The following parameters, which set system (7) in a chaotic regime, have been used: $a = 0.2$; $b = 0.2$; $c = 7$. In the considered network, each couple of Rössler systems which interact, according to the position of non-zero entries in the Laplacian matrix $L$, are linearly coupled through their first state variable, that is to say

$$DH = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$  

(8)

$^1$ Non-diagonalizable Laplacians are not dealt with in this paper for the sake of simplicity. Those cases can be coped with in a similar fashion, please refer to Nishikawa et al. [2003], Nishikawa and Motter [2006].
Even though the MSF formalism provides only a necessary manifold, it has been widely adopted as a standard tool to approach is to separate the impact of two key factors on the net work synchronization; the former concerning the dynamics of the isolated node, through the computation of the Laplacian matrix of the net work. Therefore, the localization of the eigenvalues of the Laplacian matrix is a critical issue in synchronization problems. In the following, we will focus our attention along the negative direction of the real axis as long as \( \lambda \) is any eigenvalue of \( L \), with \( 1 \leq i \leq N \), such that \( |\lambda - L_{i,i}| \leq \sum_{j=1}^{N} |L_{i,j}| \). This, for matrices belonging to the set \( \mathcal{L}(k) \), reduces to \( |\lambda + k| \leq k \) for all eigenvalues of \( L \in \mathcal{L}(k) \). Consequently, it follows that

\[
\bigcup_{L \in \mathcal{L}(k)} \Lambda(L) \subseteq \{ z \in \mathbb{C} : |z + k| \leq k \}.
\]

As in our case all the Geršgorin’s circles collapse in one set \( \{ z \in \mathbb{C} : |z + k| \leq k \} \), we will refer in the following of this paper to this circle as being “the” Geršgorin circle.

3.1 Synchronization

Through vast simulation campaigns, we found that, apart from the null eigenvalue, the remainder of the eigenvalues of a generic matrix \( L \in \mathbb{R}^{N \times N} \), \( L \in \mathcal{L}(k) \) are confined in a compact set which is strictly contained in the circle \( \{ z \in \mathbb{C} : |z + k| \leq k \} \) (arousing from the Geršgorin’s theorem). This leads us to formulate the following Conjecture 1. Given a matrix \( L \in \mathbb{R}^{N \times N} \), \( L \in \mathcal{L}(k) \), Let us define the set \( \Lambda_k(L) \) as:

\[
\Lambda_k(L) := \{ \Lambda(L) \setminus \{0\} \}.
\]

Then, there exists a compact set \( \Gamma(N,k) \subset \mathbb{C} \) such that

\[
\bigcup_{L \in \mathbb{R}^{N \times N}, L \in \mathcal{L}(k)} \Lambda_k(L) \subset \Gamma(N,k) \subset \{ z \in \mathbb{C} : |z + k| \leq k \}.
\]

This conjecture has been checked through extensive numerical simulations, even for very large matrices. For each matrix size \( N \) and each neighborhood size \( k \) a large number of random realizations has been computed and the related eigenvalue localization has been plotted, confirming our conjecture. Figs. 2 to 4 show the results obtained for 10⁵ random realizations of Laplacians of networks of 10 nodes, with \( k = 3, 5, 7 \), respectively. Figures also show the related Geršgorin circle.

More in depth, numerical results indicate that, in all the cases examined, the compact set \( \Gamma(N,k) \) in (13) moves along the negative direction of the real axis as long as \( k \) increases. This, along with the possibility of adjusting the coupling constant \( \sigma \) properly, could have strong implications on the synchronization of directed \( k \)-neighbors networks. Therefore, the main idea is that synchronization for a \( k \)-neighbors topology could be obtained by varying...
Fig. 2. Localization of the eigenvalues of $10^5$ random realizations of Laplacians $L$, $L \in \mathbb{R}^{10,10}$, $L \in \mathcal{L}(3)$, and the related Geršgorin circle.

$k$ and adjusting the coupling constant $\sigma$ properly, so that one can fit all the products $\sigma \lambda_i$ into the synchronization region $\Omega$. The suitability of the proposed procedure clearly depends also on the synchronization region $\Omega$, i.e., on the properties of the node dynamics and coupling function.

Let us now proceed with an example to clarify our statement. Figure 5 shows the synchronization region $\Omega$, for Rössler systems coupled through their first state variable (see Fig. 1), with the cloud of eigenvalues computed through $10^5$ realizations of Laplacian matrices with $N = 10$ and $k = 3$ (thus belonging to $\mathcal{L}(3)$). It is clearly shown that many eigenvalues fall out of the synchronization region. On the other hand, Figure 6 shows the same cloud of eigenvalues, multiplied by a coupling factor $\sigma = 0.7$. According to Conjecture 1, any realization (provided that it is strongly connected) of Laplacians $L \in \mathbb{R}^{10,10}$, $L \in \mathcal{L}(3)$, will allow the network of 10 Rössler systems to synchronize, with $\sigma < 0.7$. To test this, we have carried out a validation test on 100 random realizations of Laplacians belonging to $\mathcal{L}(3)$, different from those for which eigenvalues have been computed to build Figs. 5 and 6, and verified that every realization did synchronize with a coupling factor $\sigma = 0.7$.

We applied this procedure extensively, for several values of $N$ and $k$, always obtaining positive results. Specifically, in Table 1 we report the results for networks with 10...
Let us suppose in fact that, for a given set of $N$ dynamical systems which admits an invariant synchronization manifold, we have found the values $\bar{k}$ and $\bar{\sigma}$ for which the system synchronizes under any $k$-neighbors connection scheme with a coupling constant $\sigma$, according to Conjecture 1 and the procedure illustrated in the previous Section. For the sake of simplicity, let us discuss the case (without loss of generality) with $\bar{\sigma} = 1$, so that we will focus only on the Laplacian eigenvalue location. Consequently, the correspondent synchronization condition is $\Gamma(N, \bar{k}) \subset \Omega$. This analysis obviously holds also in the case $\bar{\sigma} \neq 1$, by suitably scaling the sets $\Gamma$ or $\Omega$ by the coupling factor $\sigma$. Then, we can state the following results.

For $\bar{k} = 1$ the number $N$ of agents does not affect the synchronization properties of the network. In fact, according to (Varga and Rizzo [2010]), the set of the eigenvalues of matrices belonging to $\mathcal{L}(1)$ is dense (with $N$) on the circumference $|z + 1| = 1$ plus the complex plane origin $z = 0$, that is

$$\bigcup_{L \in \mathcal{L}(1)} \Lambda(L) = \{0\} \cup \{z \in \mathbb{C} : |z + 1| = 1\}. \quad (14)$$

This implies that, as long as $N$ increases, the non zero Laplacian eigenvalues will be narrower and narrower among them, and will keep to be confined on the circumference $\{z \in \mathbb{C} : |z + 1| = 1\}$. This means that, if the network synchronizes for some $\bar{N}$, it will synchronize for any $N$ (as the eigenvalues will never go out of the circumference).

On the other hand, an opposite result is obtained for $\bar{k} > 1$, that is to say, the synchronization properties of a $k$-neighbors network are affected by the number of nodes $N$. In (Varga and Rizzo [2010]), in fact, it is proved the following Theorem which states that the set of eigenvalues of matrices belonging to $\mathcal{L}(k)$ is dense (with $N$) in the Gersgöring circle $|z + k| \leq \bar{k}$, that is

$$\bigcup_{L \in \mathcal{L}(k)} \Lambda(L) = \{z \in \mathbb{C} : |z + k| \leq \bar{k}\}. \quad (15)$$

If Conjecture 1 holds, then for a given $\bar{k}$ and $\bar{N}$ the eigenvalues of any Laplacian related to the $k$-neighbors connection scheme will fall in the compact set $\Gamma(N, \bar{k})$, strictly contained in the circle $\{z \in \mathbb{C} : |z + \bar{k}| \leq \bar{k}\}$. Moreover, if $\bar{k}$ is found such that $\Gamma(N, \bar{k}) \subset \Omega$, this will imply that the network will synchronize under any $k$-neighbors connection scheme. Now, Theorem 3 implies that, increasing $N$, the compact set $\Gamma(N, \bar{k})$ will eventually cover the whole Gersgöring circle $|z + k| \leq \bar{k}$. As stated at the beginning of this section, as the Gersgöring circle $|z + \bar{k}| \leq \bar{k}$ is not strictly contained in the synchronization region, this will eventually lead to some realizations of the $k$-neighbors network to have eigenvalues falling out of the synchronization region, and consequently to have some $k$-neighbors realizations which will not synchronize.

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2 The notation $\overline{S}$ indicates the topological closure of the set $S$. 

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### Table 1. Coupling factor needed to synchronize a network of 10 Rössler systems, coupled through a linear coupling on their first state variable, connected with a $k$-neighbors scheme.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>&lt; 0.02</td>
</tr>
<tr>
<td>3</td>
<td>&lt; 0.7</td>
</tr>
<tr>
<td>4</td>
<td>&lt; 0.5</td>
</tr>
<tr>
<td>5</td>
<td>&lt; 0.5</td>
</tr>
<tr>
<td>6</td>
<td>&lt; 0.45</td>
</tr>
<tr>
<td>7</td>
<td>&lt; 0.4</td>
</tr>
<tr>
<td>8</td>
<td>&lt; 0.4</td>
</tr>
</tbody>
</table>

Nodes and variable $k$. Note that we exclude the trivial cases $k = 1$ and $k = 9$, for which we have analytical results. In fact, in the former case the only strongly connected topology is the ring, for which the eigenvalues are distributed on the unitary circle $|z + 1| = 1$, equally spaced in angle (Varga and Rizzo [2010]); whereas in the latter case, which corresponds to an all-to-all connection scheme, there is one null eigenvalue, and the remaining all coincident in $-10$.

#### 3.2 Non-Synchronization

Let us now consider a set of $N$ identical dynamical systems which admits an invariant synchronization manifold. The results of Section 3.1 state that, under certain conditions, we can find $\bar{k}$ and $\bar{\sigma}$ for which the network synchronizes under any $k$-neighbors scheme and the coupling factor $\bar{\sigma}$. Let us also suppose that the circle $|z + k/\bar{\sigma}| \leq k/\bar{\sigma}$ is not strictly contained into the synchronization region $\Omega$ (on the contrary, the result is trivial as the network will always synchronize). We want to answer this question: “once we have found $\bar{k}$ and $\bar{\sigma}$ for a given number of nodes $\bar{N}$ such that any $k$-neighbors configuration synchronize, is this result independent from the number of agents?” The answer is affirmative for $k = 1$ and negative for $k > 1$. The proof of this statement resides in recent results found by two of the authors of this paper and is reported in (Varga and Rizzo [2010]).
4. CONCLUSION

In this paper we have shown some results about the synchronizability of networks in which each node is connected with a fixed number of nodes, that is what we have called here $k$-neighbors networks. These results are derived by studying the spectral properties of the Laplacian matrices of this kind of networks, and relating them with the Master Stability Function Theory. In particular, we have highlighted the role of both the number of neighbors and of the coupling factor with respect to the synchronization attitude of the network. Moreover, we have shown that the number of agents has a key role in synchronization. In fact, we proved that the eigenvalue set of Laplacian matrices of $k$-neighbors networks is dense on the unitary circumference for $k = 1$, and in the Gershgorin circle for $k > 1$. This implies that the synchronization properties of $1$-neighbor networks do not depend on the number of agents, whereas $k$-neighbors networks (with $k > 1$) denote synchronization properties which are dependent on the number of agents. In particular, an increase of the number of agents tends to push the system towards de-synchronization. Further research is now focused on proving Conjecture 1 of this paper, concerning the bounds of the non null eigenvalues of Laplacians of $k$-neighbors networks.

REFERENCES


