Implicit Fractional Model Order Estimation
Using Interacting Multiple Model Kalman Filters

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Abstract: The paper deals with implicit and explicit approaches for fractional nonlinear model order estimation using a benchmark model relating applied angular rate and neuron’s firing intensity within the vestibular system. The implicit approach is based on an interacting multiple models scheme, where several extended Kalman filters with fixed fractional order nonlinear models are running in parallel. An alternative method based on augmented Unscented Kalman filter is proposed where, where the fractional order of the model is estimated explicitly within the filter state. The performance of the estimators is compared with respect to parameters of estimators and fractional derivative approximation.

Keywords: Estimation algorithms, Kalman filters, fractional-order systems.

1. INTRODUCTION

The Fractional Calculus (FC) can be treated as a natural extension of the classical calculus and is based on a firm generalization of the concepts of integral-differential operators to non-integer orders (Oldham and Spanier (1974); Podlubny (1999)). Although considered to be as old as the classical calculus with first ideas tossed in 17th century, the field was for a longer time of merely mathematical interest. Among the reasons of FC-based methods to have few applications before the 2nd half of the 20th century are a conceptually difficult idea of taking derivatives of real order, higher complexity of the involved methods and apparent lack of a clear geometrical and physical interpretation, as well as the success of standard methods in dealing with the most of situations of practical interest (Axtell and Bise (1990); Podlubny (2002)). However in recent decades the ideas of FC have been increasingly applied in almost all the fields of modern science and engineering with an extensive list of references in Debnath (2003) and Machado et al. (2010). By employing FC-based methods one can construct linear models for some of infinite-dimensional physical systems in order to avoid approximations which can result in discrepancy between the behaviour of a real system and its mathematical model. Significant progress in FC-related techniques has been reported in the control area (Barbosa et al. (2003); Chen et al. (2004)) as well as in signal and image processing (Chen et al. (2007); Pu et al. (2008)).

Although dynamical systems with varying (state or time dependent) fractional orders form a promising future direction as a further development of existing FC-based models (Chen et al. (2004)), rather few results on variable-order (VO) operators and models have been reported (Lorenzo and Hartley (2002); Coimbra (2003); Ostalczyk and Rybicki (2008)) with several model order identification methods proposed so far. Some of them (Podlubny (1999)) are based on minimization of the square norm of the difference between the calculated and experimentally obtained frequency response, being rather suitable for batch data processing. Basic model order estimation have been also reported in Sierociuk and Dzielinski (2006), where coupling through the process noise covariance matrix and the Robbins-Monroe scheme were employed to tune the order of the model.

The paper is organized as follows. After explaining the basic concepts of FC, the benchmark model and Kalman filtering (KF) in Section 2, we present the construction of the estimators in Section 3. The performance results for both implicit and explicit methods are discussed in Section 4 with conclusions drawn at the end of the paper.
2. BACKGROUND

2.1 Fractional Calculus

A fundamental FC operator \( aD_\alpha^a \) is a generalization of the concepts of both derivative and integral with several formulations available in the literature. We employ the Grünwald-Letnikov (GL) approach, which is based on the concept of the variable fractional differential of \( x(t) \):

\[
aD_\alpha^a [x(t)] = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor t/jh \rfloor} d_j x(t - jh),
\]

Here \( \alpha(t) \) is the order of the operator at time \( t \), \( a \) is the lower limit, \( h \) is the time increment, usually replaced by the sampling period \( T_s \), \( \Gamma(\cdot) \) is the Euler's Gamma function and \( \lfloor \cdot \rfloor \) is the integer part of the argument. Although other more efficient discrete approximation schemes can be implemented, the choice of computationally heavy GL approach is justified by its simplicity and ease of implementation within the filtering scheme as in Sierociuk and Dzieliński (2006). The important difference from integer-order (IO) derivatives is that the FC operator is a non-local operator and requires an infinite number of terms, whereas IO systems are particular cases for which the memory represented by \( d_j \) is finite. Further details on FC definitions and their important properties can be found in Oldham and Spanier (1974); Podlubny (1999) with recent results on interpretation of the FC operators discussed in Machado (2003); Podlubny (2002).

2.2 Kalman Filtering

Various implementations of KF algorithms differ in the way they represent and transform the probabilities. If the models are linear and the probabilities are Gaussian, the classical Kalman Filter (KF) is an optimal solution in the least square sense with an efficient recursive scheme (Grewal and Andrews (2001); Thrun et al. (2005)). Several options do exist when the process or measurement models are nonlinear. The classical approach is to use an Extended Kalman Filter (EKF), which for the process model at discrete time \( k \): \( x_k = f(x_{k-1}, w_{k-1}, u_{k-1}), x \in \mathbb{R}^n \) and the measurements \( z_k = h(x_k, v_k), z \in \mathbb{R}^m \), process noise \( p(w_k) \sim N(0,Q) \) and measurement noise \( p(v_k) \sim N(0,R) \), results in the prediction step: \( \hat{x}_k^- = f(x_{k-1}^-, u_{k-1}, 0) \) and associated state error covariance \( P_k^- \) calculated using Jacobians of the corresponding partial derivatives. The Jacobians of the nonlinear measurement function are used in the filter correction step to obtain a posteriori state estimate \( \hat{x}_k^+ \) and its error covariance \( P_k^+ \).

An alternative approach, the Unscented Kalman Filter (UKF) eliminates the model linearization step by propagating a set of deterministically chosen points (so-called σ-points) through the actual nonlinear functions. The points are chosen in the way that their mean and covariance match Gaussian random variables. The mean and covariances are recalculated back from the propagated points, yielding more accurate results compared to conventional function linearization routines used within the EKF. Moreover, the filter doesn’t require the analytical Jacobians of the model, which can be non-trivial for some of the applications and potentially can be difficult to implement. In its simplest implementation, the \( n \)-dimensional variable is approximated by \( 2n+1 \) σ-points, chosen symmetrically around the mean to avoid skewness. We employ a slightly more general Augmented UKF (AUKF) framework which allows to treat non-additive process and measurement noises by augmenting at each filter iteration the state vector with the mean values of the noise and covariance with the corresponding noise covariances. The mathematical details of the algorithms are omitted here and in-depth discussion on properties of AUKF, comparison to EKF etc. can be found in extensive literature such as Thrun et al. (2005); Wan and Van Der Merwe (2000) and references therein.

Unfortunately, a single filter cannot alone properly handle all possible situations of interest such as transitions between different models or systems of varying regimes. The interacting multiple-model (IMM) scheme Bar-Shalom et al. (2001); Mazor et al. (1998) is one of the most cost-effective hybrid state-estimation schemes, proposed as an attempt to handle the problem with Markovian switching coefficients in an elegant way with significantly reduced computational load to approximate Bayesian filtering. The IMM consists of a bank of conventional filters and the algorithm to organize the cooperation between the individual filters. The prediction of the state is based on the best suitable model for a particular physical phenomena. The model is one of \( r \) possible models (the system is in one of \( r \) modes) defined in advance: \( M \in \{M_j\}_{j=1}^r \). Then the mixing probability \( \mu_{i,j,k} \) is the probability that mode \( M_i \) was in effect at time step \( k-1 \) under the condition that mode \( M_j \) is in effect at time \( k \). The mixing probabilities are calculated as:

\[
\mu_{i,j,k} = \frac{\bar{c}_i}{\bar{c}_j} \pi_{i,j} \mu_{i,k-1},
\]

where \( \mu_{i,k-1} \) are a prior probability that \( M_i \) was correct (the system was in mode \( i \)) at time \( k-1 \) with the condition for \( \sum_{i=0}^{r} \mu_{i,k-1} = 1 \), the normalizing constants \( \bar{c}_j = \sum_{i=1}^{r} \pi_{i,j} \mu_{i,k-1} \) with \( i,j = 1, \ldots, r \). The transition probabilities \( \pi_{i,j} \) represent the probability for model \( i \)
switching to model \( j \), enabling mixing at the beginning of each filter cycle and are the design parameters for the filter matched to \( j \)th model \( M_{j,k} \) at time \( k \) according to:

\[
\hat{x}_0^{j,k} = \sum_{i=1}^{r} \hat{x}_{i,k-1}^{0} \mu_{i,j,k}, \quad (3)
\]

\( j = 1, \ldots, r \) and the covariances:

\[
P_0^{j,k} = \sum_{i=1}^{r} \mu_{i,j,k} \left\{ P_{i,k-1}^{j} + \delta x_{i,k} \cdot \delta x_{i,k}^T \right\}, \quad (4)
\]

with \( \delta x_{i,k} = \hat{x}_{i,k-1}^{j} - \hat{x}_{j,k}^{0} \), \( j = 1, \ldots, r \). The estimates of the state and covariance from above are used as an input to the filter matched to \( M_{j,k} \), which uses \( z_k \) to yield \( \hat{x}_{j,k}^{+} \) and \( P_{j,k}^{+} \). Note that in the IMM implementation there is a complete set of conventional KF/EKF equations for each model. The likelihood is calculated as a multidimensional Gaussian probability density function:

\[
\Lambda_{j,k} = \frac{1}{(2\pi)^{m/2} |S_{j,k}|^{1/2}} \exp \left( -\frac{1}{2} (v_{j,k})^T \left( S_{j,k} \right)^{-1} v_{j,k} \right). \quad (5)
\]

Here \( S_{j,k} = H_j P_{j,k}^{+} H_j^T + R \) with measurement noise covariance \( R \), \( j \)th model’s measurement matrix \( H_j \) and an associated a priori error covariance \( P_{j,k}^{-} \). The measurement residual \( v_{j,k} = z_k - H_j \hat{x}_{j,k}^{+} \) is calculated from given measurements \( z_k \in \mathbb{R}^m \). The mode update probability becomes \( \mu_{j,k}^+ = \frac{1}{2} \Lambda_{j,k} c_j \) with normalization constant \( c = \sum_{j=1}^{r} \Lambda_{j,k} c_j \). The final output of the system is a combined state vector of the model conditioned estimates and corresponding covariance, weighted by their model probabilities according to mixture equations \( \hat{x}_{k}^{+} = \sum_{j=1}^{r} \frac{1}{2} \Lambda_{j,k} c_j \hat{x}_{j,k}^{+} \) and

\[
P_{k}^{+} = \sum_{j=1}^{r} \mu_{j,k} \left\{ P_{j,k}^{+} \left[ \hat{x}_{j,k}^{+} - \hat{x}_{k}^{+} \right] \cdot \left[ \hat{x}_{j,k}^{+} - \hat{x}_{k}^{+} \right]^T \right\}. \quad (6)
\]

Note that the combination is only for output purposes and it is not a part of the algorithm recursions. The schematic view of IMM KF for the case of 2 models is presented in Fig. 1. The main benefits of IMM scheme are that it is recursive, modular and has fixed computational requirements per filtering cycle (Mazor et al. (1998)).

### 2.3 Benchmark Neuron’s Firing Intensity Model

The performance of the proposed estimation schemes has been assessed using a fractional model for motion coding in the vestibular system, where the neuron’s spikes are quantified as measurements of the dynamic state of the head. The authors in Paulin et al. (2004) proposed the general dynamical model for the relationship between the head’s angular rate \( \omega (t) \) and the neuron’s firing intensity \( r (t) \) to be as:

\[
r (t) = r_0 + \lambda \frac{d^\alpha \omega (t)}{dt^\alpha}, \quad (7)
\]

where a simplified linear model of (7) can be obtained by removing nonlinearity:

\[
r (t) = r_0 + \lambda \frac{d^\alpha \omega (t)}{dt^\alpha}. \quad (8)
\]

Here \( r_0 \) is a spontaneous firing rate, \( \lambda \) is a sensitivity at a unit frequency, \( \kappa \) is used to adjust the saturation level and \( \omega \) controls the directional symmetry of the nonlinearity with a constraint on \( r (t) \) to be nonnegative in both models. Although being rather simplistic models, they seem to be good candidates for testing as long as their complexity does not obscure the details of the algorithm, filter construction and influence of the approximation parameters. Moreover, both linear and nonlinear models describing essentially the same dynamical phenomena are available.

### 3. FILTER DESIGN

The work of Sierociu and Dzieliński (2006) was among the first to propose the construction of a discrete KF for fractional models. Although different approximation schemes for FC operators can be used, we consider one of the simplest approaches, the so-called Short Memory Principle, which could be directly derived from the GL definition (Podlubny (1999)) or as a power series expansion of the generating function corresponding to the Z-transform of the backward difference rule:

\[
a_d^\alpha [x_k] \approx k-L a_d^\alpha [x_k] \approx \frac{\sum_{j=0}^{N(k)} d_{j,\alpha k} x_{k-j}}{h_\alpha k}, \quad (9)
\]

with \( N (k) = \min \{|k|, \lfloor L/h \rfloor \} \), \( L \) being the memory length and the binomial coefficients \( d_{j,\alpha k} \):

\[
d_{j,\alpha k} = (-1)^j \binom{\alpha_k}{j}, \quad (10)
\]

calculated recursively with \( d_{0,\alpha k} = 1 \) and for \( j \geq 1 \):

\[
d_{j,\alpha k} = \left( 1 - 1 + \frac{\omega_k}{j} \right) d_{j-1,\alpha k}. \quad (11)
\]

The filter derivation can be started assuming a constant fractional derivative model \( \Delta_k^\alpha = \text{const} \), similarly as it is done for the case of IO constant velocity model with \( \nu_k = \text{const} \), \( \omega_k = \omega_{k-1} + h \cdot \nu_k \), and \( \omega_k \) being the variable of interest, \( \nu_k = \tilde{\omega}_k \) and \( \Delta_k^\alpha = \nu_k \). Then recall (9):

\[
k-N \Delta_k^\alpha = \frac{1}{h_\alpha k} \sum_{j=0}^{N(k)} d_{j,\alpha k} \omega_{k-j} = \text{const}, \quad (12)
\]

where \( k - N \) refers the finite history of the operator. The dynamical model for \( d_{0,\alpha k} = 1 \) becomes \( \Delta_k^\alpha = \text{const} \):

\[
\omega_k = h_\alpha k \Delta_k^\alpha - \sum_{j=1}^{N} d_{j,\alpha k} \omega_{k-j}. \quad (13)
\]

For \( \alpha_k = 1 \) it reduces to the IO model with \( d_{0,1} = 1 \), \( d_{1,1} = -1 \) and \( d_{j,1} = 0, j \geq 2 \). The state \( \bar{x}_k \) and the process Jacobian \( J_{\bar{x},k} \) for the fractional Kalman filter (FKF) can be written as (omitting indexing \( \alpha_k \) in \( d_{k,\alpha} \)):

\[
\bar{x}_k = \begin{bmatrix} \Delta_k^\alpha \omega_k \omega_{k-1} \ldots \omega_{k-N+1} \\ \Delta_k^\alpha \omega_{k-1} \omega_{k-2} \ldots \omega_{k-N} \end{bmatrix}^T, \quad (14)
\]

\[
\bar{x}_{k-1} = \begin{bmatrix} \Delta_k^\alpha \omega_{k-1} \omega_{k-2} \ldots \omega_{k-N} \end{bmatrix}^T, \quad (15)
\]
\[ J_{A,k} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ h^{\alpha_k} & -d_1 & -d_2 & \cdots & -d_N \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{16} \]

For (7) an identical process model is assumed, whereas for the measurement model several modifications have to be introduced. As it has been shown in our previous work Romanovas et al. (2010), it is enough to measure the firing intensity in order to estimate the applied angular rate, which is not the case for \( \alpha \) estimation. Here we assume that both noisy measurements of firing and applied angular rates are available, providing sufficient information to identify the model order uniquely under the rest of the model constraints. The measurement model for (9) becomes:

\[ z_k = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} x_k + \begin{bmatrix} r_0 \\ 0 \end{bmatrix} + v_k, \tag{17} \]

where for (8) the measurement Jacobian \( J_{H,k} \) is calculated as:

\[ J_{H,k} = \begin{bmatrix} \lambda \kappa (1 - \tanh^2 ((\kappa (\Delta_k^{\alpha_k} - \bar{\omega})) 0 & 0 & 1 \times N - 1 \end{bmatrix}, \tag{18} \]

The rest of the filters is a straightforward implementation of KF or EKF with the details omitted here. The works Sierociuk and Dzieliński (2006) and Romanovas et al. (2010) provide some further details on the structure of the basic fractional versions of fixed-order KF and EKF (correspondingly FKF and EFKF). The framework presented here allows a straightforward implementation of the IMM scheme, where several FKF/EFKF are running in parallel, covering the expected range of the dynamical behaviours (model orders \( \alpha \)). IMM is designed to choose interactively the model with \( \alpha \) best fitting given observations by assigning higher probabilities \( \mu \) to a particular model. The assumption regarding Markovian switching is identical to those used to design IO filters and the intrinsically non-local dynamics of each model is approximated by corresponding fixed-order fractional filter within IMM.

It is also possible to estimate directly the order of the model using some nonlinear estimator. Unfortunately, a de-facto standard EKF would require explicit Jacobian calculation for the recursive expression of the binomial coefficients. Instead, we adopt an AUKF framework for fractional process defined above. Then the original state \( x_k \) is augmented with \( \alpha_k \) and associated binomial coefficients \( d_{k,\alpha} \) recalculated with respect to the newly estimated \( \alpha_k \) within each of \( \sigma \)-points. The identity process model is assumed for \( \alpha \) propagation as long as no extra information is available. The rest of the filter is rather straightforward with detailed discussion in Romanovas et al. (2011). Such scheme permits direct model order estimation as continuous variable without any restriction due to pre-defined set of available models.

4. SIMULATION RESULTS

Although the feasibility of the FKF has been shown in Romanovas et al. (2010), the overall performance is strongly dependent on the model order and our general knowledge of the model structure. The mismatch of the assumed and real model order can lead to significant deviation of the estimate from its true value (see Fig. 2) even within a simple state estimation problem. Although joint state and parameter estimation is a fairly established area, the fractional order estimation is a rather emerging problem with few works reported so far. The complexity of the order coupling to the process model and intrinsic approximation difficulties due to non-local nature of FC-based techniques make the task of process order far more challenging compared to parameter estimation in conventional IO models. The performance of the filters was assessed using models (8) and (7) with simulated \( \omega(t) \) and \( r(t) \) measurements, corrupted by a known measurement noise with \( \sigma_\alpha = 1 \) and \( \alpha = 2 \). The system is driven by a saw-like test signal with peak-to-peak amplitude of 40 deg/second, period of 2 sec. and higher driving capabilities compared to the former one in Fig. 2, which had extended regions with 1st order derivative equal to 0, not sufficient for robust model order estimation. The simulated measurements were generated using the GL definition with \( h = T_s = 0.001 \) sec, \( N = 1000 \) samples. The parameters for (8) are \( \alpha = 0.29 \), \( r_0 = 31.9 \), \( \lambda = 1 \) and for (7) are \( \lambda = 30 \), \( \kappa = 0.05 \), \( \bar{\omega} = -5 \). For the filter the process noise covariance \( Q_{\Delta\alpha} = 0.1^2n \) is employed for the fractional derivative term, where \( n \) is the downsampling factor(Romanovas et al. (2010)). With access to both noisy measurements of \( \omega(t) \) and \( r(t) \), the filter was able to track and denoise successfully both signals (not shown). The IMM-EFKF for tracking the system of varying order, consists of 5 fixed-order EFKEF running in parallel with orders \( \alpha_1 = 0.1 \), \( \alpha_2 = 0.3 \), \( \alpha_3 = 0.5 \), \( \alpha_4 = 0.7 \) and \( \alpha_5 = 0.9 \). The number and separation of the assumed models can be adjusted depending on computational resources, required robustness to noisy switching and expected model behavior. In order to evaluate the performance of the framework for intermediate \( \alpha \) values, the real system has been simulated switching sequentially from the order \( \alpha = 0.3 \) to \( \alpha = 0.7 \) with the step \( \Delta\alpha = 0.1 \). The IMM performance in the form of an area plot of estimated probabilities assigned to each model is
of the estimated probabilities. Obviously, the filter is able
to catch $\alpha$ if the correct model is present within the filter
bank. However, for the models of intermediate order it
switches between the bounding models instead of assigning
fixed weighted probabilities. Although the performance is
better for the case of a linear model, this issue still remains.

Fig. 6 is a good example of the complexity in interpreting
probability results, as long as the estimator is also affected
by the presence of models located further in $\alpha$ space.
While a number of modifications of a generic IMM scheme
exist (Mazor et al. (1998)), capable of handling some of
the problems by dynamically changing the set of models
as well as proper model separation, the indirect nature
of the approach still results in interpretation difficulties.

For more complex models with several fractional orders,
the IMM-based scheme can become computationally too
demanding due to the rapid growth of number of possible
models to be covered. Although some modifications
are possible such as rudimentary form of IMM estimator
with a mode-set adaptation (Mazor et al. (1998)), where
only a restricted time-varying set of models $M(k)$ has
to be considered at each filter iteration, this introduces
unnecessary heuristics without resolving the main inter-
pretability problem of the approach. This pitfall of the
IMM-based approach can be avoided by using an explicit
estimation scheme where the model order is considered as
a continuous variable and estimated within the filter state
along with the rest of the variables with implementation
details found in Romanovas et al. (2011). The preliminary
results on estimation of $\alpha$ for model (8) are shown in
Fig. 7, where a process noise of $Q_\alpha = 0.0002\mathbf{n}$ has
been chosen experimentally for the model order to provide
enough dynamics to track the changing in time $\alpha$. By
assuming a fixed history length in seconds and increasing
the number of samples $N$ within GL approximation, one
controls the accuracy of the estimated $\alpha$ as a tradeoff with
the computational complexity of the filter.

5. CONCLUSIONS

The work discusses an implicit approach for fractional
order estimation using a benchmark model. The implicit

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Fig. 3. Area plot of the estimated probabilities within
IMM-EFKF with $p_{ii} = 0.995$.

Fig. 4. Area plot of the estimated probabilities within
IMM-EFKF with $p_{ii} = 0.9999$.

Fig. 5. Area plot of the estimated probabilities within
IMM-EFKF with $p_{ii} = 0.9999$ and transition prob-
ability matrix allowing switching to the neighbour
models only.

shown in Fig. 3, where all the model probabilities add to
1. By increasing $p_{ii}$ one reduces the erroneous switching
between the models as confirmed by Fig. 4. The values of
$p_{ii} \approx 1$ can result in a too conservative switching between
the dynamical models and associated tracking errors for
fast changing regimes. Further modifications of IMM are
possible such as constraining the model switching to the
"neighbour" models only (Fig. 5). IMM-based order esti-
mation, although providing reasonably good insight into
the model switching, is an implicit scheme as long as it
is not completely clear how to deal with models of the
intermediate order, the number of models to be included,
the proper model separation as well as the interpretation

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approach is based on the IMM-KF scheme running several
fixed fractional order filters of different orders in parallel.
The former technique has certain drawbacks such as in-
interpretability of the results for intermediate order models,
proper model choice and scalability, although certain com-
putational difficulties are expected in UKF as well, mainly
due to high nonlinearity of the problem, computational feasibility and stability issues. The developed method is
compared to an UKF-based explicit technique for direct
estimation of the model’s order. The preliminary results
on explicit joint state and model order estimation will
be extended in the future work with more efficient FC
approximation schemes, advanced physical models with
several fractional derivatives and better nonlinear estima-
tion schemes.

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