Sliding Mode State Feedback Control for Uncertain Discrete-Time Markov Jump Systems *

Fengzhi. Huang ∗ Yuanwei. Jing ** Georgi M. Dimirovski ***

∗ College of Information Science and Engineering, Northeastern University, Shenyang, P.R. of China, (e-mail: huangfzh@126.com).
** College of Information Science and Engineering, Northeastern University, Shenyang, P.R. of China, (e-mail: ywjjing@mail.neu.edu.cn)
*** Faculty of Engineering, Computer Engineering Department, Dogus University of Istanbul, TR-347222 Istanbul, Rep. of Turkey, (e-mail: gdimirovski@dogus.edu.tr)

Abstract: This paper considers the development of a state feedback sliding mode controller for a class of uncertain discrete-time Markov jump systems. A sufficient condition is provided to guarantee the existence of the discrete-time sliding modes in an LMI form. And a sliding mode controller is designed to make the system states converge to the designed stochastically stable sliding mode and keep a sliding motion. The controller is designed to tolerate the matched uncertainty, which is considered to be a function of the system states and the control inputs. A numerical example is given to illustrate the proposed method.

Keywords: Sliding mode control, State feedback, Discrete-Time, Markov jump systems, Uncertainty.

1. INTRODUCTION

Markov jump systems, first introduced by Krasovskii (1961), are getting more attention in recent years. These systems, in which the mode-process is a continuous-time discrete-state Markov process taking values in a finite set, could represent a large class of practical systems that suffer random changes which caused by abrupt phenomena as parameter shifting, component and interconnection failures (see Boukas (2005)), as target tracking, manufacturing processes, and fault tolerant control systems for instance.

Over the past decades, a great number of results on the study of Markov jump systems have been obtained. The problem of stability and stabilization of Markov jump systems have been studied in Feng (1992), Mao (1999) and de Souza (2006). Robust \( H_\infty \) control and \( H_\infty \) filtering problems for continue-time and discrete-time linear Markov jump systems have been discussed in de Souza (1993), Shi et al. (1997), Gao (1995), and Xiong (2005).

On another research front line, sliding mode control, which is well-known for its completely robustness for external and internal uncertainty and disturbance in systems, in the past decades, has been widely used to practical systems, such as power systems, robot manipulators, electrical motors, aircrafts. However, sliding mode control for Markov jump systems is new research area, and some important results have been got in Shi (2006), Ma (2009), Wu (2010). Shi (2006) discussed the problem of sliding mode control for linear continue-time Markov jump systems. Ma (2009) studied the sliding mode control problem of the linear continue-time Markov jump systems using a singular control method. Wu (2010) focused on the sliding mode control of continue-time Markov jump singular systems. On the other hand, to our best knowledge, sliding mode control for discrete-time Markov jump systems have not been fully investigated yet. In this paper we develop a sliding mode controller of discrete-time Markov jump systems with matched uncertainty.

Notations. Throughout this paper, \( \mathbb{R}^n \) represents the \( n \)-dimensional Euclidean space. The notation \( \| \cdot \| \) refers to Euclidean norm for vector. \( T \) represents the transpose of a matrix. \( \lambda_{\min}(\cdot) \) represents the minimum eigenvalue of a matrix. \( E\{\cdot\} \) denotes the mathematical expectation. The asterisk \( * \) in a matrix denotes the term that is induced by symmetry. \( \tau \) represents the sampling period of system. Moreover, notation \( (\Omega, \mathcal{F}, P) \) represents a complete probability space, where \( \Omega \) represents the sample space, \( \mathcal{F} \) is the \( \sigma \)-algebra of the sample space, and \( P \) is the probability measure on \( \mathcal{F} \).

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the following discrete formulation:

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where \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^q \) are the state vector, the control input and the measurement output, respectively. \( f(u, x, k) \) represents the uncertainty containing the system nonlinearities and any model uncertainties. \( \{\theta_k, k \geq 0\} \) is a discrete-time homogeneous Markov chain, taking value in a finite set of \( \sigma = \{1, \ldots, N\} \), with a transition probability from mode \( i \) at time \( k \) to mode \( j \) at time \( k+1 \) as

\[
p_{ij} = Pr(\theta_{k+1} = j | \theta_k = i)
\]

where \( p_{ij} \geq 0, \forall i, j \in \sigma, \) and \( \sum_{j=1}^{N} p_{ij} = 1 \).

For each possible value \( \theta_k = i, i \in \sigma \), we will denote \( A(\theta_k), B(\theta_k), C_y(\theta_k) \) associated with mode \( i \) as \( A_i, B_i, C_{yi} \).

System (1) could be written as following:

\[
x(k+1) = A_i x(k) + B_i [u(k) + f(u, x, k)]
\]

\[
y(k) = C_{yi} x(k)
\]

(3)

For the system (3), we assume that the following conditions are satisfied:

A1: The matrices \( A_i, B_i, C_{yi} \) are constant matrices with appropriate dimensions.

A2: The matrices \( B_i, C_{yi} \) are full rank and \( m \leq q < n \).

A3: The uncertain function \( f(u, x, k) \) is bounded as

\[
\|f(u, x, k)\| \leq \alpha \|u\| + \beta(x, k)
\]

where \( \alpha \) is a known scalar such that \( 0 < \alpha < 1 \), and \( \beta(x, k) \) is a known function.

Definition 1. System (1) is said to be stochastically stable for \( u(k) \equiv 0 \) and \( f(u, x, k) \equiv 0 \) and for every initial condition \( x_0 \in \mathbb{R}^n \) and \( \theta_0 \in \sigma \), if the following holds

\[
E\left\{ \sum_{k=0}^{\infty} \|x(k)\|^2 | x_0, \theta_0 \right\} < \infty
\]

(4)

where \( E\{\} \) is mathematical expectation.

Lemma (Costa el al., 2005) When consider \( u(k) \equiv 0 \) and \( f(u, x, k) \equiv 0 \) the unforced system (1) is stochastically stable if and only if there exist a set of symmetric and positive definite matrices \( P_i, i \in \sigma \) satisfying the following inequations

\[
P_i^T Q_i P_i - P_i < 0
\]

(5)

where \( Q_i \doteq \sum_{j \in \sigma} p_{ij} P_j \).

3. SLIDING MODE STATE FEEDBACK CONTROLLER DESIGN

At the beginning of our design, we could change the equation (1) into an appropriate canonical form by choosing a nonsingular matrix \( T(\theta_k) \), for each possible value \( \theta_k = i, i \in \sigma \) denoted as \( T_i \).

Transforming the system state as \( z(k) = T_i x(k) \), system (1) could be changed as following:

\[
z(k+1) = \hat{A}_i z(k) + \hat{B}_i [u(k) + f(u, x, k)]
\]

deleting \( \hat{A}_i, \hat{B}_i \) as \( \hat{A}_1, \hat{B}_1 \) respectively for each possible \( \theta_k = i, i \in \sigma \), then (6) could be written as

\[
z(k+1) = \hat{A}_i z(k) + \hat{B}_i [u(k) + f(u, x, k)]
\]

(7)

where

\[
\hat{A}_i = T_i^{-1} A_i T_i, \hat{B}_i = T_i B_i = \begin{bmatrix} 0 & 0 \\ B_{21} & \bar{C}_{yi} \end{bmatrix} C_{yi} T_i^{-1},
\]

and \( \bar{B}_2 \in \mathbb{R}^m \).

Without loss of generation, we assume that \( z(k), \hat{A}_i \) have the following form

\[
z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}, \hat{A}_i = \begin{bmatrix} \hat{A}_{11i} & \hat{A}_{12i} \\ \hat{A}_{21i} & \hat{A}_{22i} \end{bmatrix}
\]

and \( z_1(k) \in \mathbb{R}^{n_1}, z_2(k) \in \mathbb{R}^{n_2} \), such that (7) could be written as

\[
z_1(k+1) = \hat{A}_{11i} z_1(k) + \hat{A}_{12i} z_2(k)
\]

(8)

\[
z_2(k+1) = \hat{A}_{21i} z_1(k) + \hat{A}_{22i} z_2(k) + \hat{B}_{2i} [u(k) + f(u, x, k)]
\]

(9)

Usually, there are two steps to design a sliding mode controller, one is sliding mode designing, the other is control law designing.

3.1 Sliding Mode Design

In this section a stochastically stable sliding mode will be proposed.

For the transformed system (7), we design sliding modes formed as following:

\[
S = G_i z(k) = 0
\]

(10)

where \( G_i = [G_{1i}, G_{2i}] \), and \( G_{2i} \) is nonsingular, for each \( i \in \sigma \).

From the above equation we could get the relationship between \( z_1(k) \) and \( z_2(k) \) as:

\[
z_2(k) = -\hat{G}_{2i}^{-1} \hat{G}_{1i} z_1(k)
\]

(11)

Substituting (11) into (8) and connecting with (10), we could obtain the sliding motion equation:

\[
z_1(k+1) = \hat{A}_i z_1(k)
\]

(12)

\[
S = [\hat{G}_{1i} \hat{G}_{2i}] z(k) = 0
\]

(13)

where \( \hat{A}_i = \hat{A}_{11i} - \hat{A}_{12i} K_i, K_i = \hat{G}_{2i}^{-1} \hat{G}_{1i} \).

Remark 1. Through out the above calculation, we could easily get that the stable problem of \( n \)-dimension system (1) could be studied through out studying the \((n-m)\)-dimension system (12), and that is less complex for analyzing.

In the following, a condition for the existence of stochastically stable sliding modes is given in LMI forms.

**Theorem 1.** If there exist a set of positive matrices \( X_i, i \in \sigma \) and matrices \( Y_i, \forall i \in \sigma \) such that the following LMI satisfied

\[
\begin{align*}
\begin{bmatrix}
A_i^T X_i A_i & -X_i & 0 \\
-\beta i & 0 & 0 \\
0 & -\beta i & 0 \\
\end{bmatrix} & > 0
\end{align*}
\]
\[
\begin{cases}
-\chi_j \pi_i (\tilde{A}_{11i}X_i - \tilde{A}_{12i}Y_i) - X_i < 0 \\
\end{cases}
\] (14)

where
\[
\chi_j = \text{diag}\{X_1, X_2, \ldots, X_N\},
\] (15)
\[
\pi_i = [\sqrt{p_{i1}}I, \ldots, \sqrt{p_{iN}}I]^T.
\] (16)

then the reduced system (12) is stochastically stable.

The sliding mode parameters \(K_1\) could be obtained by
\[
K_1 = Y_iX_i^{-1}.
\] (17)

**Proof.** Choose a stochastic Lyapunov function as
\[
V(z(k), k) = z_i^T(k)P_i z_i(k), \ \forall i \in \sigma
\] (18)

Based on the definition 1, for \(\theta_k = i, \theta_{k+1} = j\), we could have
\[
E(\Delta(V)) = E(V(z_i(k + 1), j|z_i(k), i) - V(z_i(k), i))
\]
\[
= z_i(k + 1)^T \left(\sum_{j=1}^{N} p_{ij} P_j z_i(k + 1) - z_i(k)^T P_i z_i(k)\right)
\]
\[
= z_i(k)^T (\hat{A}_i^T Q_i \hat{A}_i - P_i) z_i(k)
\] (19)

If
\[
\hat{A}_i^T Q_i \hat{A}_i - P_i < 0
\] (20)

is satisfied then
\[
E(\Delta(V)) \leq -\lambda_{\min} [-(\hat{A}_i^T Q_i \hat{A}_i - P_i)] z_i(k)^T z_i(k)
\]
\[
\leq -\beta \|z_i(k)\|^2
\]

For \(l \geq 1,
\[
E\{\sum_{k=0}^{l} \|z_i(k)\|^2\}
\]
\[
\leq -\frac{1}{\beta} E[V(z_i(k + 1), (k + 1))] - E[V(z_i(0), 0)]
\]
\[
\leq -\frac{1}{\beta} E[V(z_i(0), 0)] < \infty
\] (21)

According to definition 1, we can get the reduced order system (12) is stochastically stable.

Applying Schur complement to (20), we could get
\[
\begin{bmatrix}
-P_i & 0 & \cdots & 0 & \sqrt{p_{i1}}P_i \hat{A}_i \\
* & -P_2 & \cdots & \sqrt{p_{i2}}P_i \hat{A}_i \\
* & * & \cdots & 0 & \sqrt{p_{iN}}P_i \hat{A}_i \\
* & * & \cdots & -P_N & \sqrt{p_{iN}}P_i \hat{A}_i \\
& & & & -P_i
\end{bmatrix}
< 0
\] (22)

Setting \(X_i^{-1} = P_i\) performing a congruence transformation to (22) by \(\text{diag}\{\chi_j, X_i\}\) and by denoting \(Y_i = K_iX_i\), we could get (14).

Therefore, if (14) holds, the reduced order system (12) is stochastically stable, and the desired sliding mode parameters could be calculated by equation (17). This completes the proof.

### 3.2 Sliding Mode State Feedback Controller Design

In the above section we designed a sliding mode the states on which are stochastically stable and proved the existence of it. In this section, we will design a sliding mode controller that could drive the system states convergent to the designed sliding mode.

The controller we designed is based on the following reaching law (proposed in Gao (1995)):
\[
S(k + 1) - S(k) = -\epsilon \text{sign}(S(k)) - \tau S(k)
\] (23)

where \(\tau\) is the sample time, \(0 < \epsilon < 1\), \(q > 0\) are constant numbers related with the reaching law.

**Remark 2.** Where designing a sliding mode, two conditions need to be satisfied. One is the existence of the sliding mode, the other is referred to the reachability condition. In the above section we gathered the first condition, and now the reaching law (23) is satisfied the reachability condition, if we choose a proper shot sampling period ensuring \(2 - q\tau > 0\) is satisfied Gao (1995).

For the transformed system (7), we design a controller in the following form:
\[
u = -[G_i \hat{B}_i]^{-1} [Gz(k) + \epsilon \text{sign}(S(k))] - \bar{u}_1
\] (24)
\[
\bar{u}_1 = \begin{cases}
\frac{S(k)}{\|S(k)\|}, & S(k) \neq 0 \\
0, & \text{otherwise.}
\end{cases}
\] (25)

\[
\nu = \frac{1}{1 - \alpha} \left\{ \alpha \|G_i \hat{B}_i\|^{-1} [Gz(k) + \epsilon \text{sign}(S(k))] \right\}
\]
\[
+ \beta (y, k + \gamma)
\] (26)

where \(\hat{G} = G_i \hat{A}_i - (1 - q\tau)G_i\), and \(\gamma > 0\) is a constant number.

Substitute the designed controller to (7), we could get
**Theorem 2.** If there exist position matrix \(P_i, i \in \sigma\) such that with the sliding mode surface calculated by theorem 1, and with proper chosen parameters of \(q, \tau, \epsilon, \gamma\), the following inequalities hold,
\[
(1 - q\tau)^2Q_j - P_i < 0
\] (27)
\[
(\gamma \|G_i \hat{B}_i\| + \epsilon\tau - 2(1 - q\tau)\|S(k)\| < 0
\] (28)
then the designed sliding mode (10) is stochastically stable and the system (7) will converge to (10) and keep a sliding motion by the investigated controller of (24) and (25).

**Proof.** Choose a stochastic Lyapunov function as
\[
V(S(k), k) = S^T(k)P_i S(k)
\] (29)
for \(\theta_k = i, \theta_{k+1} = j\), we could get
\[
E(\Delta V(S(k), k)) = E(V(S(k + 1)) - V(S(k)))
\]
\[
= S^T(k + 1)Q_j S(k + 1) - S^T(k)P_i S(k)
\] (30)

where \(Q_j\) is same to that in (5).

Substituting the sliding mode equation (10) and the control law (24) and (25) to (30), we could get
The state feedback control law is designed in the following theorem1

From the curves in Figs.1-4, it can be seen that the system response under the designed control law is quite satisfactory: the system can track any desired trajectory with appropriate accuracy. In the following, we shall present the main results of the paper.

Analyzing the above equations we could get

$$E(\triangle V(S(k), k)) = [G_i \hat{A}, z(k) + G_i \hat{B}_i [u(k) + f(u, x, k)]]^T$$

$$Q_j [G_i \hat{A}, z(k) + G_i \hat{B}_i [u(k) + f(u, x, k)]] - S^T(k) P_j S(k)$$

$$= [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1) + (1 - \eta) S(k)]^T Q_j [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1) + (1 - \eta) S(k)]$$

$$- 2\epsilon \tau \|S(k)\| Q_j + (\epsilon \tau)^2 \|S(k)\| Q_j = S^T(k)[(1 - \eta)^2 Q_j - P_j] S(k)$$

$$= S^T(k)[(1 - \eta)^2 Q_j - P_j] S(k)$$

$$+ 2(1 - \eta) [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1)]^T Q_j S(k)$$

$$+ [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1)]^T Q_j [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1)].$$

Then we have

$$-2\epsilon \tau (1 - \eta) \|S(k)\| Q_j < 0,$$  \hspace{1cm} (31)

$$(\epsilon \tau)^2 \|S(k)\| Q_j \leq (\epsilon \tau)^2 Q_j,$$  \hspace{1cm} (32)

$$2(1 - \eta) [G_i \hat{B}_i (f(u, x, k) - \bar{u}_1)]^T Q_j S(k)$$

$$\leq -2\gamma (1 - \eta) \|G_i \hat{B}_i\| S(k) Q_j.$$  \hspace{1cm} (33)

As

$$G_i \hat{B}_i \|S(k)\| (f(u, x, k) - \bar{u}_1)$$

$$\leq \|G_i \hat{B}_i\| \|f(u, x, k)\| - \nu < -\gamma \|G_i \hat{B}_i\|,$$

$$-2\epsilon \tau \|G_i \hat{B}_i\| \|f(u, x, k)\| - \nu < -2\epsilon \tau \|G_i \hat{B}_i\| Q_j$$

$$< -2\epsilon \tau - \gamma \|G_i \hat{B}_i\| Q_j.$$  \hspace{1cm} (34)

According to equation (30) and inequalities (31)-(34) we may get if choosing proper value of parameters $\eta$, $\tau$, $\gamma$, $\epsilon$ to guarantee the inequalities in theorem 2 hold, then $E(\triangle V(S(k), k)) < \infty$ is guarantee, based on definition 1, the sliding motion is stochastically stable. That completes the proof.

Remark 3. In discrete-time sliding mode control systems, because of sampling, states can not maintain moving on the sliding surface like its counterpart in continue-time systems, but once driven on the sliding mode, they pass through the sliding surface, and through back at the next sampling point, then pass through again, cycling a sliding band. Systems moving between the band, are completely robust to matched uncertainties and disturbances.

4. NUMERICAL EXAMPLE

Consider the uncertain discrete-time Markov jump system with 2 operation modes and following data

$$A_1 = \begin{bmatrix} 1 & -1.25 \\ 2.5 & 2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & -0.83 \\ 2.5 & 3.5 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad f(u, x, k) = 0.02 u(k) + 0.02 x(k).$$  \hspace{1cm} (35)

The transition probability is consider as

$$p_{11} = 0.5, p_{12} = 0.5, p_{21} = \frac{5}{14}, p_{21} = \frac{9}{14},$$

the sampling period of the system is chosen as $\tau = 0.1 s$.

The parameters of sliding surface we could calculated from theorem 1

$$K_1 = [-0.4241], \quad K_2 = [-3.4125],$$

then the sliding mode surfaces could be chosen as:

$$G_1 = [-0.4241], \quad G_2 = [-3.4125].$$

The state feedback control law is designed in the following

$$U_1 = -[-0.3974 \ 3.3106] z(k) - 0.2 \text{sign}(S(k))$$

$$- \frac{1}{0.2} \{0.2 \} [-0.3974 \ 3.3106] z(k)$$

$$- 0.2 \text{sign}(S(k)) + 0.02 y(k) + 0.002 \} \frac{S(k)}{\|S(k)\|} \frac{S(k)}{\|S(k)\|}$$  \hspace{1cm} (36)

$$U_2 = [-8.3589 \ 3.7803] z(k) - 0.2 \text{sign}(S(k))$$

$$- \frac{1}{0.2} \{0.2 \} [-8.3589 \ 3.7803] z(k)$$

$$- 0.2 \text{sign}(S(k)) + 0.02 y(k) + 0.002 \} \frac{S(k)}{\|S(k)\|} \frac{S(k)}{\|S(k)\|}$$  \hspace{1cm} (37)

with the reaching parameters $\epsilon_1 = \epsilon_2 = 0.2, q = 8$, and the parameter $\gamma = 0.002$.

Fig.3 shows the system response under the designed controller showed in Fig.2 with the jumping modes showed in Fig.1, and Fig.4 shows the designed sliding surface.

From the curves in Figs.1-4, it can be seen that the designed sliding mode controller is feasible and effective for ensuring the closed systems and the sliding motion stable.
A state feedback sliding mode controller has been developed for a class of uncertain discrete-time Markov jump systems. A sufficient condition for the existence of sliding mode, on which the system states are behaving stochastically stable performance, has been established formed in LMI. The underlying system is driven exponentially convergent to the designed surface by the investigated sliding mode controller. The efficiency of the proposed approaches is provided by a numerical example.

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