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On Contraction of Piecewise Smooth Dynamical Systems

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Abstract: In this paper we extend to a class of piecewise-smooth dynamical systems, a fundamental property of dynamical systems which has been used in a number of different applications in the case of smooth dynamical systems: contraction theory. We give analytical conditions under which trajectories of discontinuous vector fields, satisfying Caratheodory conditions for the existence and unicity of a solution, converge towards each other. In particular, we prove that if each mode of the vector field is contracting then the dynamical systems of interest is contracting. We apply our results to the problem of synchronizing a network of piecewise linear dynamical systems.

1. INTRODUCTION

Piecewise-smooth dynamical systems are commonly used in Control Theory to model devices of interest and/or synthesize discontinuous control actions e.g., Cortes [2008], di Bernardo et al. [2008]. Despite the large number of available results on their well-posedness and stability, there are very few papers in the literature where the problem of assessing their contraction (or convergence) properties is discussed (Pavlov et al. [2005], Pavlov et al. [2007], Pavlov and van de Wouw [2008], El Rifai and Slotine [2006]).

Contraction theory is a classical tool of smooth dynamical systems theory, which has been used in a wide range of applications. For example, it has been shown that contraction is an extremely useful property to analyze coordination problems in networked control systems such as the emergence of synchronization or consensus (Lohmiller and Slotine [1998], Pham and Slotine [2007], Russo et al. [2010a], Wang and Slotine [2005], Russo and di Bernardo [2009a,b], Russo et al. [2010b]). Indeed, all trajectories of a contracting system can be shown to exponentially converge towards each other asymptotically. Therefore as shown in Wang and Slotine [2005], this property can be effectively exploited to give conditions for the synchronization of a network of dynamical systems of interest.

Historically, ideas closely related to contraction can be traced back to Hartman [1961] and even to Lewis [1949] (see also Pavlov et al. [2004], Angeli [2002], and e.g. Lohmiller and Slotine [2005], Jouffroy [2005] for a more exhaustive list of related references). For autonomous systems and with constant metrics, the basic nonlinear contraction result reduces to Krasovskii's theorem (Slotine and Li [1990]) in the continous-time case, and to the contraction mapping theorem in the discrete-time case (Lohmiller and Slotine [1998], Bertsekas and Tsitsiklis [1989]).

The aim of this paper is to discuss contraction properties of piecewise-smooth dynamical systems. In particular, we focus on systems whose vector fields satisfy the Caratheodory conditions for the existence and uniqueness of an absolutely continuous solution (see Filippov [1988] and Cortes [2008] for further details). We prove that for this class of piecewise-smooth systems, contraction of each individual mode is sufficient to guarantee convergence of all the system trajectories towards each other, i.e. contraction of the overall system of interest. We then present an application to the problem of proving convergence of a PWA system and that of synchronizing a network of piecewise linear systems with time-dependent switchings.

2. MATHEMATICAL PRELIMINARIES

Let x be an n-dimensional vector. We denote with $|x|$ the norm of the vector. Let A be a (real) matrix. Then, $\|A\|$ denotes the norm of A. We recall (see for instance Michel et al. [2007]) that, given a vector norm on Euclidean space ($|\cdot|$), with its induced matrix norm $\|A\|$, the associated *matrix measure* μ is defined as the directional derivative of the matrix norm, that is,

$$\mu(A) := \lim_{\delta \searrow 0} \frac{1}{\delta} \left(\|I + \delta A\| - 1 \right).$$

For example, if $|\cdot|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of A. As we shall see, however, different norms will be useful for our applications. Matrix measures are also known as "*logarithmic norms*", a concept independently introduced by Germund Dahlquist and Sergei Lozinskii in 1959, Dahlquist [1959], Lozinskii [1959]. The limit is known to exist, and the convergence is monotonic, see Dahlquist [1959]. In what follows we report the analytic expression of some matrix measure. We have:

- $\mu_1(A) = \max_j \left(a_{jj} + \sum_{i \neq j} |a_{ij}| \right)$;
- $\mu_2(A) = \frac{1}{2} \left(A + A^T \right)$;

- $\mu_\infty(A) = \max_i \left(a_{ii} + \sum_{j \neq i} |a_{ij}| \right)$.

Definition 1. A function $g(t) : [t_0, +\infty] \to \mathbb{R}^n$ is said to be measurable if for any real number a the set $\{t \in [t_0, +\infty] : g(t) > a\}$ is measurable in the sense of Lebesgue.

Definition 2. A function $l(t) : [t_0, +\infty] \to \mathbb{R}$ is summable if the Lebesgue integral of the absolute value of $l(t)$ exists and is finite.

Definition 3. A function $z(t) : [a, b] \to \mathbb{R}^n$ is absolutely continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $[a_1, b_1] \dots [a_n, b_n]$ is a finite collection of disjoint sets in $[a, b]$ then $\sum_k |b_k - a_k| < \delta \implies \sum_k |z(b_k) - z(a_k)| < \varepsilon$..

Let $C \subset \mathbb{R}^n$ be a convex set and consider a dynamical system of the form:

$$\dot{x} = \varphi(x, t), \tag{1}$$

where $x \in C$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$. We say that the vector field $\varphi(x, t)$ fulfills the *Caratheodory conditions* if:

(1) for almost all $t \in [0, \infty]$, the function $\varphi(x, t)$ is continuous for all $x \in C$;
(2) for each $x \in C$, the function $\varphi(x, t)$ is measurable in t;
(3) for all $(x, t) \in C \times [0, +\infty]$, there exist a summable function $m(t)$ such that $|\varphi(x, t)| \leq m(t)$.

Above, we denote with $|\varphi|$ the supremum norm of the function $\varphi(x, t)$:

$$|\varphi| := \sup_{\theta \in (x,t)} |\varphi(\theta)|.$$

Notice that we use the same symbol to denote the norm of a vector and the norm of a function. However, the meaning of the symbol "$|\cdot|$" will be clear from the context.

It is well known (see Filippov [1988]) that, under the above assumptions, an absolutely continuous function $x(t)$ exists that solves (1) so that:

$$x(t) = x(t_0) + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau.$$

Moreover, such a solution is also unique if

$$(x - y)^T (\varphi(x, t) - \varphi(y, t)) \leq l(t)(x - y)^T (x - y), \tag{2}$$

where $l(t)$ is a summable function.

Notice that, as discussed in Filippov [1988], p. 10 and proved in Andreev and Bogdanov [1958], equations that satisfy Caratheodory conditions and those required for the uniqueness of a solution show continuous dependence on initial conditions.

3. CONTRACTION OF PIECEWISE-SMOOTH SYSTEMS

We consider piecewise-smooth dynamical systems of the form

$$\dot{x} = f(x, t, \sigma) := \varphi(x, t), \tag{3}$$

where $\sigma(t) : [0, +\infty) \to \Sigma$ is defined as the *switching signal* with Σ being a finite index set. We assume that for any $\bar{\sigma}$, $f(x, t, \bar{\sigma})$ is defined on the convex set $C \subseteq \mathbb{R}^n$.

In what follows, we will denote as $f_x(x, t, \sigma)$ the Jacobian of the vector field $f(x, t, \sigma)$ and assume that Caratheodory conditions are satisfied by f so that its Jacobian f_x exists almost everywhere in t.

Definition 4. We say that a piecewise smooth system of the form (3) is contracting (in the Caratheodory sense) in a convex set C (termed as contraction region) if there exist a unique matrix measure, μ, such that:

$$\mu(f_x(x, t, \sigma)) \leq -c_\sigma^2, \tag{4}$$

for all $\sigma \in \Sigma$, $x \in C$ and for almost all t, with c_σ being a set of real non-zero scalars. (In what follows, we will define $c^2 := \max_{\sigma \in \Sigma} c_\sigma^2$.)

We say that the piecewise-smooth system is contracting if $C \equiv \mathbb{R}^n$.

We will show next that, under certain conditions, trajectories of a contracting piecewise smooth system such as (3) rooted in C globally exponentially converge towards each other, almost everywhere and for almost all $t \in \mathbb{R}^+$.

Our main result can now be stated as follows.

Theorem 1. The trajectories of a time-switching system of the form (3) that:

(1) fulfills the Caratheodory conditions;
(2) satisfies the condition
$$(x - y)^T (f(x, t, \sigma) - f(y, t, \sigma)) \leq l(t)(x - y)^T (x - y); \tag{5}$$
(3) is contracting according to Definition 4,

globally exponentially converge towards each other almost everywhere in C and for almost all $t \in \mathbb{R}^+$.

4. PROOF OF THEOREM 1

Pick any two solutions $x(t)$ and $y(t)$ of (3), two points: $\xi := x(t_0)$, $\zeta := y(t_0)$ and a smooth curve, $\gamma : [0, 1] \to C$ and such that:

$$\gamma(0) = \xi, \ \gamma(1) = \zeta.$$

Let $\psi(t, r) := \phi(t, t_0, \gamma(r))$ be the solution of (3) rooted in $\phi(t_0, r) = \gamma(r)$, with $r \in [0, 1]$. Notice that $\psi(t, r)$ is continuous with respect to r for all t. Also, γ can be chosen so that $\psi(t, r)$ is differentiable with respect to r for almost all the pairs (t, r) (i.e. almost everywhere). Let:

$$w(t, r) := \frac{\partial \psi}{\partial r}, \quad a.e.$$

Thus we have:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial r}\left(f(\psi(t, r), t, \sigma)\right), \quad a.e.$$

Now:

$$\frac{\partial}{\partial r}\left(f(\psi(t, r), t, \sigma)\right) = \frac{\partial}{\partial \psi} f(\psi(t, r), t, \sigma) \frac{\partial \psi(t, r)}{\partial r}, \quad a.e.$$

The above expression can be written as:

$$\frac{\partial}{\partial t} w(t, r) = A(\psi(t, r), t, \sigma) w(t, r), \quad a.e.$$

with A being the Jacobian of $f(x, t, \sigma)$, for almost all $t \in \mathbb{R}^+$, $r \in [0, 1]$. Then, for any $t \in \mathbb{R}^+$ and for any $\bar{r} \in [0, 1]$, solving the above equation for any given $r = \bar{r}$ we have (in what follows integration has to be understood in the sense of Lebesgue):

$$w(t + \delta, \bar{r}) = w(t, \bar{r}) + \int_t^{t+\delta} A(\psi(\tau, \bar{r}), \tau, \sigma) w(\tau, \bar{r}) d\tau =$$
$$= w(t, \bar{r}) + \delta A(\psi(t, \bar{r}), t, \sigma) w(t, \bar{r}) +$$
$$\int_t^{t+\delta} \left(A(\psi(\tau, \bar{r}), \tau, \sigma) w(\tau, \bar{r}) - A(\psi(t, \bar{r}), t, \sigma) w(t, \bar{r})\right) d\tau,$$

almost everywhere. Thus, from the above expression we have:

$$\frac{|w(t+\delta,\bar{r})| - |w(t,\bar{r})|}{\delta} \leq \frac{\|I + \delta A(\psi(t,\bar{r}),t,\sigma)\| - 1}{\delta} +$$
$$\frac{1}{\delta}\int_t^{t+\delta} |A(\psi(\tau,\bar{r}),\tau,\sigma)w(\tau,\bar{r}) - A(\psi(t,\bar{r}),t,\sigma)w(t,\bar{r})|\, d\tau,$$

almost everywhere. Thus, by taking the limit as $\delta \searrow 0$ and considering that from the hypotheses $\mu(A(x,t,\sigma) \leq -c^2$, for all σ, we have:

$$\frac{d}{dt}\,|w(t,\bar{r})| \leq -c^2\,|w(t,\bar{r})|\,,$$

for almost all $t \in \mathbb{R}^+$, $\bar{r} \in [0,1]$ and $\sigma \in \Sigma$.

Let now $M(t) := -c^2 t$, from the above expression it follows that:

$$\frac{d}{dt}\left(|w(t,\bar{r})|\, e^{-M(t)}\right) \leq 0,$$

almost everywhere. Now, notice that $e^{-M(t)}$ is an increasing function. Therefore, the above inequality implies that:

$$|w(t,\bar{r})| \leq |w(t_0,\bar{r})|\, e^{-c^2 t} \leq K\,|\xi - \zeta|\, e^{-c^2 t}, \quad a.e.$$

Now, since $w(t,\bar{r})$ is defined almost everywhere, we have:

$$\psi(t,1) - \psi(t,0) = \int_0^1 w(t,s)ds.$$

Thus:

$$|x(t) - y(t)| \leq K\,|x(t_0) - y(t_0)|\, e^{-c^2 t}, \quad a.e.$$

This proves the result.

5. STABILITY OF PIECEWISE LINEAR SYSTEMS

Using the concept of contraction for PWS systems, it is possible to give the following result to assess the stability of piecewise linear systems.

Corollary 2. Given a piecewise linear system of the form

$$\dot{x} = A(t,\sigma)x, \tag{6}$$

where $\sigma(t) : [0,+\infty) \to \Sigma$ is the switching signal with Σ being a finite index set, if the matrix $A(t,\sigma)$ is bounded and measurable and there exist some matrix measure such that

$$\mu\left(A(t,\sigma)\right) \leq -c^2, \quad c \neq 0 \quad \forall t \in \mathbb{R}^+, \quad \forall \sigma \in \Sigma, \tag{7}$$

then, all solutions of (6) converge asymptotically towards the origin whatever the switching sequence.

Proof. Under the hypotheses, system (6) satisfies Theorem 1 and therefore is contracting with all of its trajectories converging towards each other. Since, $x(t) = 0$ is a trajectory of (6), the proof immediately follows.

As an example, take the system:

$$\dot{x} = A(\sigma)x, \quad \sigma \in \{1,2\} \tag{8}$$

with

$$A(1) = \begin{pmatrix} -1.0 & 1.5 \\ 0.8 & -3.0 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -3.0 & 1.0 \\ 2.0 & -1.5 \end{pmatrix}.$$

Note that using the matrix measure μ_1 induced by the 1-norm, we have $\mu_1[A(1)] < 0$ and $\mu_1[A(2)] < 0$. Hence, it is immediate to prove asymptotic convergence of all solutions towards each other and onto the origin using Corollary 2.

6. CONVERGENCE OF NETWORKS OF TIME-SWITCHING SYSTEMS

Now, we use the results presented above to derive conditions guaranteeing the synchronization/consensus of a network of diffusively coupled identical piecewise linear dynamical systems with time-dependent switchings of the form:

$$\dot{x}_i = A(\sigma(t))x_i + \Gamma \sum_{j \in N_i} [x_j - x_i]\,, \tag{9}$$

where $x_i \in \mathbb{R}^n$ represents the state vector of node i, N_i denotes the set of the neighbors of the i-th network node. The cardinality of N_i, i.e. the degree of the i-th network node, is denoted with d_i. In the above dynamics, Γ is some coupling matrix also termed as inner-coupling matrix. In what follows the eigenvalues of the network Laplacian matrix (L) are denoted with λ_i with λ_2 being the algebraic connectivity. We assume that $A(\sigma(t))$ is bounded and measurable.

In what follows, we will derive a sufficient condition ensuring that all the solutions of the network globally exponentially converge, almost everywhere, towards the n-dimensional linear subspace $\mathcal{M}_s := \{x_1 = \ldots = x_N\}$ (it is straightforward to check that the subspace is flow invariant for the network dynamics). In what follows, we will denote by $s(t)$ the common asymptotic behavior of all nodes on $\mathcal{M}_s$. Note that $s(t)$ is obviously a solution of each isolated node of (9), i.e. $\dot{s}(t) = A(\sigma(t))s(t)$. We will also say that the network nodes are coordinated (or that the network is coordinated) if

$$|x_i(t) - s(t)| \to 0, \quad a.e.$$

In the special case where $s(t)$ exhibits an oscillatory behavior, we will say that all network nodes are synchronized (or that the network is synchronized).

Theorem 3. The trajectories of all nodes in the network (9) exponentially converge towards each other almost everywhere (i.e., the network is coordinated a.e.) if (i) the topology of the network is connected and (ii) there exist some matrix measure, μ, such that:

$$\mu\left(A(\sigma(t)) - \lambda_2\Gamma\right) \leq -c^2, \quad c \neq 0,$$

for all $\sigma \in \Sigma$ and for almost all t.

Before starting with the proof of the Theorem, we report here two useful results, Arcak

Lemma 4. Let $\otimes$ denote the Kronecker product. The following properties hold:

- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;
- if A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

Lemma 5. For any $n \times n$ real symmetric matrix, A, there exist an orthogonal $n \times n$ matrix, Q, such that

$$Q^T A Q = U, \tag{10}$$

where U is an $n \times n$ diagonal matrix.

Proof of Theorem 3 Define:

$$X := \left[x_1^T, \ldots, x_N^T\right]^T, \; S := 1_N \otimes s, \; E := X - S,$$

where 1_N denotes the N-dimensional vector consisting of all ones. (Notice that such a vector spans $\mathcal{M}_s$.) The network dynamics can then be written as:

$$\dot{X} = (I_N \otimes A(\sigma(t)))X - (L \otimes \Gamma)X.$$

Thus, we have:

$$\dot{E} = (I_N \otimes A(\sigma(t)))E - (L \otimes \Gamma)X \tag{11}$$

Notice that coordination is attained for the network if all the dynamics of (11) transversal to $\mathcal{M}_s$ are contracting. Furthermore, notice that

$$\begin{aligned}&(L \otimes \Gamma)X = \\ &(L \otimes \Gamma)(E + S) = \\ &(L \otimes \Gamma)E + (L \otimes \Gamma)S = \\ &(L \otimes \Gamma)E + (L \otimes \Gamma)(1_N \otimes s) = \\ &(L \otimes \Gamma)E,\end{aligned}$$

where the last equality follows from Lemma 4 and from the fact that $L \cdot 1_N = 0$, since the network is connected by hypotheses. Thus:

$$\dot{E} = (I_N \otimes A(\sigma(t)))E - (L \otimes \Gamma)E. \tag{12}$$

Since L is symmetric, by means of Lemma 5 we have that there exist an $N \times N$ orthogonal matrix Q ($Q^T Q = I_N$) such that:

$$\Lambda = Q^T L Q,$$

where Λ is the $N \times N$ diagonal matrix, having on its main diagonal the eigenvalues of L.

Define the following coordinate transformation:

$$Z = (Q \otimes I_n)^{-1} E.$$

In the new coordinates (12) becomes

$$\dot{Z} = (Q \otimes I_n)^{-1} \left[(I_N \otimes A(\sigma(t))) - (L \otimes \Gamma)\right](Q \otimes I_n)Z.$$

Then, using Lemma 4, we have:

$$\begin{aligned}&(Q \otimes I_n)^{-1}(I_N \otimes A(\sigma(t)))(Q \otimes I_n) = \\ &\left(Q^{-1} \otimes I_n\right)(I_N \otimes A(\sigma(t)))(Q \otimes I_n) = \\ &\left(Q^{-1} \otimes A(\sigma(t))\right)(Q \otimes I_n) = \\ &(I_N \otimes A(\sigma(t))).\end{aligned}$$

Analogously:

$$\begin{aligned}&(Q \otimes I_n)^{-1}(L \otimes \Gamma)(Q \otimes I_n) = \\ &\left(Q^{-1} \otimes I_n\right)(L \otimes \Gamma)(Q \otimes I_n) = \\ &\left(Q^{-1} L \otimes \Gamma\right)(Q \otimes I_n) = \\ &Q^{-1} L Q \otimes \Gamma = \\ &\Lambda \otimes \Gamma.\end{aligned}$$

That is, network dynamics can be written as:

$$\dot{Z} = \left[I_N \otimes A(\sigma(t)) - \Lambda \otimes \Gamma\right] Z, \tag{13}$$

or equivalently:

$$\dot{z}_i = \left[A(\sigma(t)) - \lambda_i \Gamma\right] z_i, \quad i = 1, \ldots, N, \quad z_i \in \mathbb{R}^n.$$

Now, recall that the eigenvector associated to the smallest eigenvalue of the Laplacian matrix, i.e. $\lambda_1 = 0$, is 1_N and spans $\mathcal{M}_s$. Therefore, the dynamics along $\mathcal{M}_s$ is given by

$$\dot{z}_1 = \left[A(\sigma(t))\right] z_1,$$

i.e. it is a solution of the uncoupled nodes' dynamics. The dynamics transversal to the invariant subspace are given by:

$$\dot{z}_i = \left[A(\sigma(t)) - \lambda_i \Gamma\right] z_i, \quad i = 2, \ldots, N.$$

Obviously $\left[A(\sigma(t)) - \lambda_i \Gamma\right]$ is bounded and measurable. Thus, by virtue of Corollary 2, all node trajectories globally exponentially converge a.e. towards $\mathcal{M}_s$, if all of the above dynamics are contracting. Now, it is straightforward to check that such a condition is fulfilled if

$$\dot{z}_2 = \left[A(\sigma(t)) - \lambda_2 \Gamma\right] z_2$$

is contracting. As this is true from the hypotheses, the result is then proved.

6.1 A numerical example

As a representative example, in this Section we use the results presented above to synchronize a network of the form (9), where the dynamics of each uncoupled node is given by:

$$\dot{x}_i := \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} = \begin{bmatrix} 0 & |\sin(t)| \\ -1 & 0 \end{bmatrix} x_i, \tag{14}$$

The matrix Γ, also known as the internal coupling matrix in the literature, is of the form:

$$\Gamma = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with k being the coupling gain that will be determined using Theorem 3. The network considered here consists of an all to all topology of three nodes. That is,

$$L := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

and $\lambda_2 = 3$. Thus, from Theorem 3 it follows that the network synchronizes if there exists some matrix measure such that:

$$\begin{aligned}&\mu\left(\begin{bmatrix} 0 & \sin(t) \\ -1 & 0 \end{bmatrix} - 3k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \le -c_1^2, \quad \text{if } \sin(t) \ge 0; \\ &\mu\left(\begin{bmatrix} 0 & -\sin(t) \\ -1 & 0 \end{bmatrix} - 3k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \le -c_2^2, \text{ if } \sin(t) < 0;\end{aligned}$$

with $c_i \neq 0$. That is, synchronization is attained if

$$\begin{aligned}&\mu\left(\begin{bmatrix} -3k & \sin(t) \\ -1 & -3k \end{bmatrix}\right) \le -c_1^2, \quad \text{if } \sin(t) \ge 0; \\ &\mu\left(\begin{bmatrix} -3k & -\sin(t) \\ -1 & -3k \end{bmatrix}\right) \le -c_2^2, \text{ if } \sin(t) < 0.\end{aligned}$$

Now, using the matrix measure induced by the vector-1 norm, it is straightforward to check that the above conditions are fulfilled if the coupling gains is selected as

$$k > \frac{1}{3}.$$

Fig. 1 and 2 show the evolution of the state x_1 for the three time-switching nodes in the network for $k = 0$ (i.e. when they are uncoupled) and $k = 0.4$ confirming the theoretical predictions.

7. CONCLUSIONS

We have studied the contraction properties of piecewise-smooth dynamical systems satisfying Caratheodory conditions for existence and uniqueness of a solution. Specifically, we derived a set of conditions guaranteeing that all of their trajectories exponentially converge towards each other. The main result is that such systems are contracting if each of their modes is contracting. We then presented applications to the problems of studying stability of switched linear systems and the emergence of a synchronous evolution in a network of diffusively coupled identical systems with time-dependent switchings. We wish to emphasize that much work is still needed to properly account for other classes of discontinuous dynamical systems for example those exhibiting sliding motion.

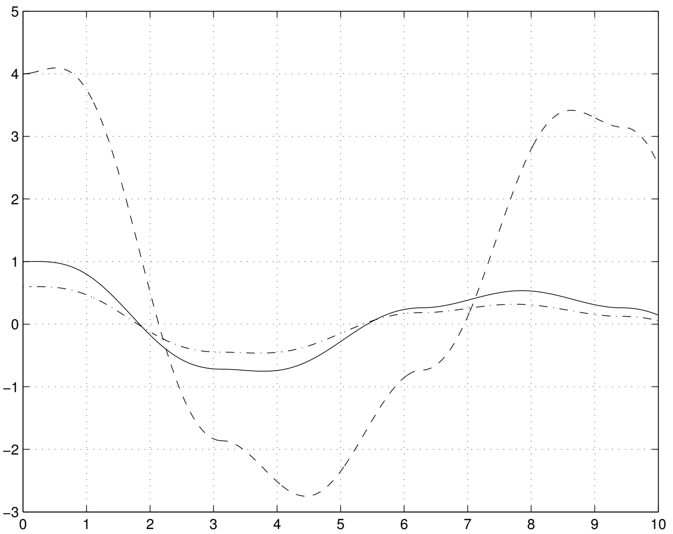

Fig. 1. State evolution of the three time-switching nodes in the network when $k = 0$

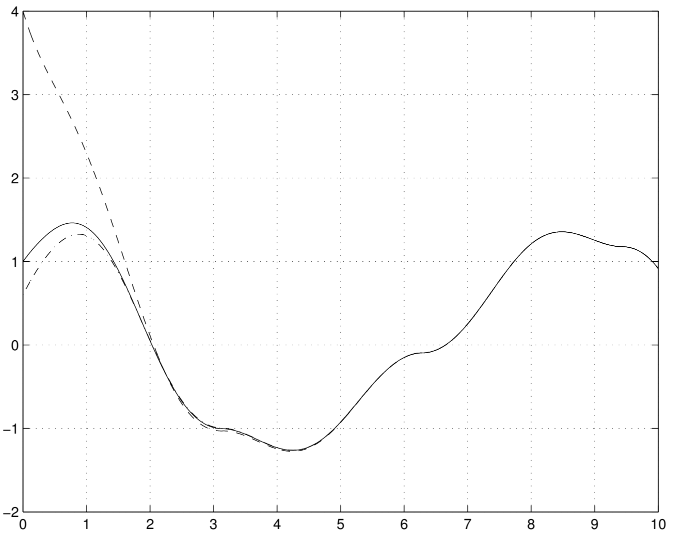

Fig. 2. State evolution of the three time-switching nodes in the network when $k = 0.4$

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REFERENCES

A.F. Andreev and Yu.S. Bogdanov. On continuous dependence of solution of the cauchy problem on initial data. *Upsekhi Mat. Nauk.*, 13:165–166, 1958.

D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47:410–321, 2002.

M. Arcak. On spatially uniform behavior in reaction-diffusion pde and coupled ode systems. Available at: http://arxiv.org/abs/0908.2614.

D. Bertsekas and J. Tsitsiklis. *Parallel and distributed computation: numerical methods.* Prentice-Hall (Upper Saddle River, NJ, USA), 1989.

J. Cortes. Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28:36–73, 2008.

G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations.* Transanctions of the Royal Institute Technology (Stockholm), 1959.

M. di Bernardo, C.J. Budd, A.R. Champneys, and P. Kowalczyk. *Piecewise Smooth Dynamical Systems: Theory and Applications.* Springer-Verlag (London), 2008.

K. El Rifai and J. J.E. Slotine. Compositional contraction analysis of hybrid nonlinear systems. *IEEE Transactions on Automatic Control*, 41:1536–1541, 2006.

A.F. Filippov. *Differential equations with discontinuous righthand sides.* Kluwer, 1988.

P. Hartman. On stability in the large for systems of ordinary differential equations. *Canadian Journal of Mathematics*, 13:480–492, 1961.

J. Jouffroy. Some ancestors of contraction analysis. In *Proceedings of the International Conference on Decision and Control*, pages 5450– 5455, 2005.

D. C. Lewis. Metric properties of differential equations. *American Journal of Mathematics*, 71:294–312, 1949.

W. Lohmiller and J. J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34:683–696, 1998.

W. Lohmiller and J. J.E. Slotine. Contraction analysis of non-linear distributed systems. *International Journal of Control*, 78:678–688, 2005.

S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izv. Vtssh. Uchebn. Zaved Matematika*, 5:222–222, 1959.

A. N. Michel, D. Liu, and L. Hou. *Stability of Dynamical Systems: Continuous, Discontinuous, and Discrete Systems.* Springer-Verlag (New-York), 2007.

A. Pavlov and N. van de Wouw. Convergent discrete-time nonlinear systems: the case of pwa systems. In *Proc American Control Conference*, pages 3452–3457, 2008.

A. Pavlov, A. Pogromvsky, N. van de Wouv, and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Systems and Control Letters*, 52: 257–261, 2004.

A. Pavlov, N. van de Wouw, and H. Nijmeier. Convergent piecewise affine systems: analysis and design - part i: continuous case. In *Proceedings of the 44th Conf. Decision and Control*, pages 5391–5396, 2005.

A. Pavlov, N. Pogromsky, N. van de Wouw, and H. Nijmeier. On convergence properties of piecewise affine systems. *International Journal of Control*, 80(8):1233–1247, 2007.

Q. C. Pham and J. J. E. Slotine. Stable concurrent synchronization in dynamic system networks. *Neural Networks*, 20:62–77, 2007.

G. Russo and M. di Bernardo. How to synchronize biological clocks. *Journal of Computationa Biology*, 16: 379–393, 2009a.

G. Russo and M. di Bernardo. Solving the rendezvous problem for multi-agent systems using contraction theory. In *Proceedings of the International Conference on Decision and Control*, 2009b.

G. Russo, M. di Bernardo, and J.J. Slotine. A graphical algorithm to prove contraction of nonlinear circuits and systems. *IEEE Transactions on Circuits And Systems I*, 58:336 – 348, 2010a. IEEE Transactions on Circuits and Systems I.

G. Russo, M. di Bernardo, and E. D. Sontag. Global entrainment of transcriptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):e1000739, 2010b.

J. J. Slotine and W. Li. *Applied nonlinear control.* Prentice Hall (Englewood Cliffs, NJ, USA), 1990.

W. Wang and J. J. E. Slotine. On partial contraction analysis for coupled nonlinear oscillators. *Biological Cybernetics*, 92:38–53, 2005.