On Contraction of Piecewise Smooth Dynamical Systems

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Abstract: In this paper we extend to a class of piecewise-smooth dynamical systems, a fundamental property of dynamical systems which has been used in a number of different applications in the case of smooth dynamical systems: contraction theory. We give analytical conditions under which trajectories of discontinuous vector fields, satisfying Carathéodory conditions for the existence and unicity of a solution, converge towards each other. In particular, we prove that if each mode of the vector field is contracting then the dynamical systems of interest is contracting. We apply our results to the problem of synchronizing a network of piecewise linear dynamical systems.

1. INTRODUCTION

Piecewise-smooth dynamical systems are commonly used in Control Theory to model devices of interest and/or synthesize discontinuous control actions e.g., Cortes [2008], di Bernardo et al. [2008]. Despite the large number of available results on their well-posedness and stability, there are very few papers in the literature where the problem of assessing their contraction (or convergence) properties is discussed (Pavlov et al. [2005], Pavlov et al. [2007], Pavlov and van de Wouw [2008], El Rifi and Slotine [2006]).

Contraction theory is a classical tool of smooth dynamical systems theory, which has been used in a wide range of applications. For example, it has been shown that contraction is an extremely useful property to analyze coordination problems in networked control systems such as the emergence of synchronization or consensus (Lohmiller and Slotine [1998], Pham and Slotine [2002], Russo et al. [2010a], Wang and Slotine [2005], Russo and di Bernardo [2009a,b], Pavlov et al. [2010b]). Indeed, all trajectories of a contracting system can be shown to exponentially converge towards each other asymptotically. Therefore as shown in Wang and Slotine [2005], this property can be effectively exploited to give conditions for the synchronization of a network of dynamical systems of interest.

Historically, ideas closely related to contraction can be traced back to Hartman [1961] and even to Lewis [1949] (see also Pavlov et al. [2004], Angeli [2002], and e.g. Lohmiller and Slotine [2005], Jouffroy [2005] for a more exhaustive list of related references). For autonomous systems and with constant metrics, the basic nonlinear contraction result reduces to Krasovskii’s theorem (Slotine and Li [1990]) in the continuous-time case, and to the contraction mapping theorem in the discrete-time case (Lohmiller and Slotine [1998], Bertsekas and Tsitsiklis [1989]).

The aim of this paper is to discuss contraction properties of piecewise-smooth dynamical systems. In particular, we focus on systems whose vector fields satisfy the Carathéodory conditions for the existence and uniqueness of an absolutely continuous solution (see Filippov [1988] and Cortes [2008] for further details). We prove that for this class of piecewise-smooth systems, contraction of each individual mode is sufficient to guarantee convergence of all the system trajectories towards each other, i.e. contraction of the overall system of interest. We then present an application to the problem of proving convergence of a PWA system and that of synchronizing a network of piecewise linear systems with time-dependent switchings.

2. MATHEMATICAL PRELIMINARIES

Let $x$ be an $n$-dimensional vector. We denote with $|x|$ the norm of the vector. Let $A$ be a (real) matrix. Then, $\|A\|$ denotes the norm of $A$. We recall (see for instance Michel et al. [2007]) that, given a vector norm on Euclidean space $(\|\cdot\|)$, with its induced matrix norm $\|A\|$, the associated matrix measure $\mu$ is defined as the directional derivative of the matrix norm, that is,

$$
\mu(A) := \lim_{\delta \to 0^+} \frac{1}{\delta} \left( \|I + \delta A\| - 1 \right).
$$

For example, if $\|\cdot\|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of $A$. As we shall see, however, different norms will be useful for our applications. Matrix measures are also known as “logarithmic norms”, a concept independently introduced by Germund Dahlquist and Sergei Lozinskii in 1959, Dahlquist [1959], Lozinskii [1959]. The limit is known to exist, and the convergence is monotonic, see Dahlquist [1959]. In what follows we report the analytic expression of some matrix measure. We have:

- $\mu_1(A) = \max_j \left( a_{jj} + \sum_{i \neq j} |a_{ij}| \right)$;
- $\mu_2(A) = \frac{1}{2} \left( A + A^T \right)$;
Definition 1. A function \( g(t) : [t_0, +\infty) \to \mathbb{R}^n \) is said to be measurable if for any real number \( a \) the set \( \{ t \in [t_0, +\infty) : g(t) > a \} \) is measurable in the sense of Lebesgue.

Definition 2. A function \( l(t) : [t_0, +\infty) \to \mathbb{R}^n \) is summable if for any real number \( a \) the set \( \{ t \in [t_0, +\infty) : l(t) > a \} \) is measurable in the sense of Lebesgue.

Definition 3. A function \( z(t) : [a, b] \to \mathbb{R}^n \) is absolutely continuous if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( [a_1, b_1], \ldots, [a_n, b_n] \) is a finite collection of disjoint sets in \([a, b]\) then
\[
\sum_k (b_k - a_k) < \varepsilon \Rightarrow \sum_k |z(b_k) - z(a_k)| < \varepsilon.
\]

Let \( C \subseteq \mathbb{R}^n \) be a convex set and consider a dynamical system of the form:
\[
\dot{x} = \varphi(x, t),
\]
where \( x \in C \) and \( \varphi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \). We say that the vector field \( \varphi(x, t) \) fulfills the Caratheodory conditions if:

1. For almost all \( t \in [0, \infty) \), the function \( \varphi(x, t) \) is continuous for all \( x \in C \);
2. For each \( x \in C \), the function \( \varphi(x, t) \) is measurable in \( t \);
3. For all \( (x, t) \in C \times [0, +\infty) \), there exist a summable function \( m(t) \) such that \( |\varphi(x, t)| \leq m(t) \).

Above, we denote with \( |\varphi| \) the supremum norm of the function \( \varphi(x, t) \):
\[
|\varphi| := \sup_{\theta \in \mathbb{R}^n} |\varphi(\theta)|.
\]

Notice that we use the same symbol to denote the norm of a vector and the norm of a function. However, the meaning of the symbol "\( |\cdot| \)" will be clear from the context.

It is well known (see Filippov [1988]) that, under the above assumptions, an absolutely continuous function \( x(t) \) exists that solves (1) so that:
\[
x(t) = x(t_0) + \int_{t_0}^{t} \varphi(x(\tau), \tau) d\tau.
\]

Moreover, such a solution is also unique if:
\[
(x - y)^T(\varphi(x, t) - \varphi(y, t)) \leq l(t)(x - y)^T(x - y),
\]
where \( l(t) \) is a summable function.

Notice that, as discussed in Filippov [1988], p. 10 and proved in Andreev and Bogdanov [1958], equations that satisfy Caratheodory conditions and those required for the uniqueness of a solution show continuous dependence on initial conditions.

### 3. CONTRACTION OF PIECEWISE-SMOOTH SYSTEMS

We consider piecewise-smooth dynamical systems of the form
\[
\dot{x} = f(x, t, \sigma) := \varphi(x, t),
\]
where \( \sigma(t) : [0, +\infty) \to \Sigma \) is defined as the switching signal with \( \Sigma \) being a finite index set. We assume that for any \( \sigma \), \( f(x, t, \sigma) \) is defined on the convex set \( C \subseteq \mathbb{R}^n \).

In what follows, we will denote as \( f_x(x, t, \sigma) \) the Jacobian of the vector field \( f(x, t, \sigma) \) and assume that Caratheodory conditions are satisfied by \( f \) so that its Jacobian \( f_x \) exists almost everywhere in \( t \).

### Definition 4.
We say that a piecewise smooth system of the form (3) is contracting (in the Caratheodory sense) in a convex set \( C \) (termed as contraction region) if there exist a unique matrix measure, \( \mu \), such that:
\[
\mu(f_x(x, t, \sigma)) \leq -c^2
\]
for all \( \sigma \in \Sigma, x \in C \) and for almost all \( t \), with \( c_\sigma \) being a set of real non-zero scalars. (In what follows, we will define \( c^2 := \max_{\sigma \in \Sigma} c_\sigma^2 \).

We say that the piecewise-smooth system is contracting if \( C \equiv \mathbb{R}^n \).

We will show next that, under certain conditions, trajectories of a contracting piecewise smooth system such as (3) rooted in \( C \) globally exponentially converge towards each other, almost everywhere and for almost all \( t \in \mathbb{R}^+ \).

Our main result can now be stated as follows.

**Theorem 1.** The trajectories of a time-switching system of the form (3) that:

1. fulfills the Caratheodory conditions;
2. satisfies the condition
\[
(x - y)^T(f(x, t, \sigma) - f(y, t, \sigma)) \leq l(t)(x - y)^T(x - y);
\]
3. is contracting according to Definition 4,

globally exponentially converge towards each other almost everywhere in \( C \) and for almost all \( t \in \mathbb{R}^+ \).

## 4. PROOF OF THEOREM 1

Pick any two solutions \( x(t) \) and \( y(t) \) of (3), two points: \( \xi := x(t_0), \zeta := y(t_0) \) and a smooth curve, \( \gamma : [0, 1] \to C \) such that:
\[
\gamma(0) = \xi, \quad \gamma(1) = \zeta.
\]

Let \( \psi(t, r) := \psi(t, 0, r) \) be the solution of (3) rooted in \( \phi(t, r) = \gamma(r) \), with \( \gamma(t) \in [0, 1] \). Notice that \( \psi(t, r) \) is continuous with respect to \( r \) for all \( t \). Also, \( \gamma \) can be chosen so that \( \psi(t, r) \) is differentiable with respect to \( r \) for almost all the pairs \( (t, r) \) (i.e. almost everywhere). Let:
\[
w(t, r) := \frac{\partial \psi}{\partial r}, \quad a.e.
\]

Thus we have:
\[
\frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial r} \right) = \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial r} \right) = \frac{\partial}{\partial r} (f(\psi(t, r), t, \sigma)), \quad a.e.
\]

Now:
\[
\frac{\partial}{\partial r} (f(\psi(t, r), t, \sigma)) = \frac{\partial}{\partial \psi} f(\psi(t, r), t, \sigma) \frac{\partial \psi}{\partial r}, \quad a.e.
\]

The above expression can be written as:
\[
\frac{\partial}{\partial t} w(t, r) = A(\psi(t, r), t, \sigma)w(t, r), \quad a.e.
\]
with \( A \) being the Jacobian of \( f(x, t, \sigma) \), for almost all \( t \in \mathbb{R}^+, \quad r \in [0, 1] \). Then, for any \( t \in \mathbb{R}^+ \) and for any \( \tilde{r} \in [0, 1] \), solving the above equation for any given \( r \) we have (in what follows integration has to be understood in the sense of Lebesgue):

\[
w(t + \delta, \tilde{r}) = w(t, \tilde{r}) + \int_t^{t + \delta} A(\psi(\tau, \tilde{r}), \tau, \sigma)w(\tau, \tilde{r}) d\tau =
\]

\[
= w(t, \tilde{r}) + \int_t^{t + \delta} A(\psi(\tau, \tilde{r}), \tau, \sigma)w(\tau, \tilde{r} - \delta(\psi(\tau, \tilde{r}), t, \sigma)w(t, \tilde{r})) d\tau,
\]
almost everywhere. Thus, from the above expression we have:
\[
\frac{|w(t + \delta, \bar{r})| - |w(t, \bar{r})|}{\delta} \leq \|I + \delta A(\psi(t, \bar{r}), t, \sigma)\| - \cdots
\]
(Notice that such a vector spans Ms.) The network dynamics can then be written as:
\[
\dot{X} = (IN \otimes A(\sigma(t)))X - (L \otimes \Gamma)X.
\]

6. CONVERGENCE OF NETWORKS OF TIME-SWITCHING SYSTEMS

Now, we use the results presented above to derive conditions guaranteeing the synchronization/consensus of a network of diffusively coupled identical piecewise linear dynamical systems with time-dependent switchings of the form:
\[
\dot{x}_i = A(\sigma(t))x_i + \Gamma \sum_{j \in N_i} |x_j - x_i|, \quad (9)
\]
where \(x_i \in \mathbb{R}^n\) represents the state vector of node \(i\), \(N_i\) denotes the set of the neighbors of the \(i\)-th network node. The cardinality of \(N_i\), i.e. the degree of the \(i\)-th network node, is denoted with \(d_i\). In the above dynamics, \(\Gamma\) is some coupling matrix also termed as inner-coupling matrix. In what follows the eigenvalues of the network Laplacian matrix \((L)\) are denoted with \(\lambda_i\) with \(\lambda_2\) being the algebraic connectivity. We assume that \(A(\sigma(t))\) is bounded and measurable.

In what follows, we will derive a sufficient condition ensuring that all the solutions of the network globally exponentially converge, almost everywhere, towards \(n\)-dimensional linear subspace \(M_{\alpha} := \{x_1 = \ldots = x_N\}\) (it is straightforward to check that the subspace is flow invariant for the network dynamics). In what follows, we will denote by \(s(t)\) the common asymptotic behavior of all nodes on \(M_{\alpha}\). Note that \(s(t)\) is obviously a solution of each isolated node of (9), i.e. \(s(t) = A(\sigma(t))s(t)\). We will also say that the network nodes are coordinated (or that the network is coordinated) if
\[
|x_i(t) - s(t)| \to 0, \quad \text{a.e.}
\]
In the special case where \(s(t)\) exhibits an oscillatory behavior, we will say that all network nodes are synchronized (or that the network is synchronized).

\textbf{Theorem 3.} The trajectories of all nodes in the network (9) exponentially converge towards each other almost everywhere (i.e., the network is coordinated a.e.) if (i) the topology of the network is connected and (ii) there exist some matrix measure, \(\mu\), such that:
\[
\mu(A(\sigma(t)) - \lambda_2 \Gamma) \leq -\gamma^2, \quad c \neq 0,
\]
for all \(\sigma \in \Sigma\) and for almost all \(t\).

Before starting with the proof of the Theorem, we report here two useful results, Arcak

\textbf{Lemma 4.} Let \(\otimes\) denote the Kronecker product. The following properties hold:

- \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\);
- if \(A\) and \(B\) are invertible, then \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\);

\textbf{Lemma 5.} For any \(n \times n\) real symmetric matrix, \(A\), there exist an orthogonal \(n \times n\) matrix, \(Q\), such that:
\[
Q^T A Q = U, \quad (10)
\]
where \(U\) is an \(n \times n\) diagonal matrix.

\textbf{Proof of Theorem 3} Define:
\[
X := [x_1^T, \ldots, x_N^T]^T, \quad S := 1_N \otimes s, \quad E := X - S,
\]
where \(1_N\) denotes the \(N\)-dimensional vector consisting of all ones. (Notice that such a vector spans \(M_{\alpha}\)). The network dynamics can then be written as:
\[
\dot{X} = (I_N \otimes A(\sigma(t)))X - (L \otimes \Gamma)X.
\]
Thus, we have:
\[ \dot{E} = (IN \otimes A(\sigma(t)))E - (L \otimes \Gamma)X \quad (11) \]
Notice that coordination is attained for the network if all the dynamics of (11) transversal to \( M_\alpha \) are contracting. Furthermore, notice that
\[
\begin{align*}
(L \otimes \Gamma)X &= (L \otimes \Gamma)(E + S) = (L \otimes \Gamma)E + (L \otimes \Gamma)S = (L \otimes \Gamma)E + (L \otimes \Gamma)(I_N \otimes s) = (L \otimes \Gamma)E,
\end{align*}
\]
where the last equality follows from Lemma 4 and from the fact that \( L \cdot 1_N = 0 \), since the network is connected by hypotheses. Thus:
\[ \dot{E} = (IN \otimes A(\sigma(t)))E - (L \otimes \Gamma)X \quad (12) \]
Since \( L \) is symmetric, by means of Lemma 5 we have that there exist an \( N \times N \) orthogonal matrix \( Q \) (\( Q^TQ = I_N \)) such that:
\[ \Lambda = Q^TLQ, \]
where \( \Lambda \) is the \( N \times N \) diagonal matrix, having on its main diagonal the eigenvalues of \( L \).

Define the following coordinate transformation:
\[ Z = (Q \otimes I_n)^{-1}E. \]
In the new coordinates (12) becomes
\[ \dot{Z} = (Q \otimes I_n)^{-1}((IN \otimes A(\sigma(t))) - (L \otimes \Gamma))(Q \otimes I_n)Z. \]
Then, using Lemma 4, we have:
\[
\begin{align*}
(Q \otimes I_n)^{-1}(IN \otimes A(\sigma(t)))Q \otimes I_n) &= (Q^{-1} \otimes I_n)(IN \otimes A(\sigma(t)))Q \otimes I_n) = (Q^{-1} \otimes A(\sigma(t)))Q \otimes I_n) = (IN \otimes A(\sigma(t))).
\end{align*}
\]
Analogously:
\[
\begin{align*}
(Q \otimes I_n)^{-1}(L \otimes \Gamma)Q \otimes I_n) &= (Q^{-1} \otimes I_n) (L \otimes \Gamma)Q \otimes I_n) = (Q^{-1}L \otimes \Gamma)Q \otimes I_n) = Q^{-1}LQ \otimes \Gamma = \Lambda \otimes \Gamma.
\end{align*}
\]
That is, network dynamics can be written as:
\[ \dot{Z} = [I_N \otimes A(\sigma(t)) - \Lambda \otimes \Gamma]Z, \quad (13) \]
or equivalently:
\[ \dot{z}_i = [A(\sigma(t)) - \lambda_i \Gamma]z_i, \quad i = 1, \ldots, N, \quad z_i \in \mathbb{R}^n. \]
Now, recall that the eigenvector associated to the smallest eigenvalue of the Laplacian matrix, i.e. \( \lambda_1 = 0 \), is \( 1_N \) and spans \( M_\alpha \). Therefore, the dynamics along \( M_\alpha \) is given by
\[ \dot{z}_1 = [A(\sigma(t))]z_1, \]
\[ \dot{z}_i = [A(\sigma(t)) - \lambda_i \Gamma]z_i, \quad i = 2, \ldots, N. \]
Obviously \( [A(\sigma(t)) - \lambda_i \Gamma] \) is bounded and measurable. Thus, by virtue of Corollary 2, all node trajectories globally exponentially converge a.e. towards \( M_\alpha \), if all of the above dynamics are contracting. Now, it is straightforward to check that such a condition is fulfilled if
\[ \dot{z}_2 = [A(\sigma(t)) - \lambda_2 \Gamma]z_2 \]
is contracting. As this is true from the hypotheses, the result is then proved.

6.1 A numerical example
As a representative example, in this Section we use the results presented above to synchronize a network of the form (9), where the dynamics of each uncoupled node is given by:
\[ \dot{x}_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \begin{bmatrix} 0 & \sin(t) \\ -1 & 0 \end{bmatrix} x_i, \quad (14) \]
The matrix \( \Gamma \), also known as the internal coupling matrix in the literature, is of the form:
\[ \Gamma = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]
with \( k \) being the coupling gain that will be determined using Theorem 3. The network considered here consists of an all to all topology of three nodes. That is,
\[ L := \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \]
and \( \lambda_2 = 3 \). Thus, from Theorem 3 it follows that the network synchronizes if there exists some matrix measure such that:
\[ \mu \left( \begin{bmatrix} 0 & \sin(t) \\ -1 & 0 \end{bmatrix} - 3k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \leq -c_1^2, \quad \text{if} \; \sin(t) \geq 0; \]
\[ \mu \left( \begin{bmatrix} 0 & -\sin(t) \\ -1 & 0 \end{bmatrix} - 3k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \leq -c_2^2, \quad \text{if} \; \sin(t) < 0; \]
with \( c_1 \neq 0 \). That is, synchronization is attained if
\[ \mu \left( \begin{bmatrix} -3k \sin(t) \\ -1 \end{bmatrix} \right) \leq -c_1^2, \quad \text{if} \; \sin(t) \geq 0; \]
\[ \mu \left( \begin{bmatrix} -3k \sin(t) \\ -1 \end{bmatrix} \right) \leq -c_2^2, \quad \text{if} \; \sin(t) < 0. \]
Now, using the matrix measure induced by the vector-1 norm, it is straightforward to check that the above conditions are fulfilled if the coupling gains is selected as
\[ k \geq \frac{1}{3}. \]
Fig. 1 and 2 show the evolution of the state \( x_1 \) for the three time-switching nodes in the network for \( k = 0 \) (i.e. when they are uncoupled) and \( k = 0.4 \) confirming the theoretical predictions.

7. CONCLUSIONS
We have studied the contraction properties of piecewise-smooth dynamical systems satisfying Carathéodory conditions for existence and uniqueness of a solution. Specifically, we derived a set of conditions guaranteeing that all of their trajectories exponentially converge towards each other. The main result is that such systems are contracting if each of their modes is contracting. We then presented applications to the problems of studying stability of switched linear systems and the emergence of a synchronous evolution in a network of diffusively coupled identical systems with time-dependent switchings. We wish to emphasize that much work is still needed to properly account for other classes of discontinuous dynamical systems for example those exhibiting sliding motion.
Fig. 1. State evolution of the three time-switching nodes in the network when $k = 0$

Fig. 2. State evolution of the three time-switching nodes in the network when $k = 0.4$

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