Incremental-Dissipativity-Based Synchronization of Interconnected Systems

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Abstract: In this paper, the incremental-dissipativity is used to investigate synchronization of a class of dynamical networks from an input-output point of view. The network consists of subsystems described by input-output operators. Firstly, the concept of incremental-passivity is extended to incremental-dissipativity, and it shows that the incremental-dissipativity of the subsystems leads to the dissipativity of a group of virtual systems each of which is related to an individual subsystem. Then the synchronization problem is transformed into the stability problem of a system interconnected by the virtual systems. By utilizing the stability theory of large-scale systems, a sufficient condition is obtained which ensures the input-output synchronization of the network. The counterparts of the main result and the discussion on conditions of the incremental-dissipativity for the subsystems with state space representations are also presented. Finally, an example is given to show the effectiveness of the proposed result.

Keywords: Synchronization, interconnected systems, incremental-dissipativity.

1. INTRODUCTION

Synchronization of dynamical networks or interconnected systems and its related problems such as consensus of multi-agent systems have attracted a great deal of focus due to their extensive applications in physics, biology and engineering. Numerical as well as analytical approaches to the problems have been reported in the literature (see recent papers and a monograph Arenas et al. (2008); Ren et al. (2007); Olfati-Saber and Murray (2004); Moreau (2005); Pham and Slotine (2007); Wu (2007) for details).

Dissipativity theory of nonlinear systems introduced by Willems (1972) and further extended by Hill and Moylan (1977, 1980) has proved to be an effective tool for studying complex systems. It provides a framework for the analysis and design of complex systems using an input-output description based on energy-related considerations (Brogliato et al. (2007)). Dissipativity as well as its special case passivity has been exploited to analyze the stability of interconnected large-scale systems in Moylan and Hill (1978); Vidyasagar (1981); Hill and Moylan (1983); Arcak and Sontag (2006) to name just a few contributions. In particular, a general sufficient condition for the input-output stability of a large-scale system which consists of linearly coupled dissipative subsystems was obtained in Moylan and Hill (1978) and Hill and Moylan (1983). Since synchronization can be seen as a kind of stability; it is expected that properties of dissipativity can be also applicable to the analysis of synchronization. Actually, passivity has already been used to investigate state synchronization as well as output synchronization for networks with identical and non-identical nodes, respectively (Steur et al. (2009); Chopra and Spong (2006); Zhao et al. (2010)).

Recently, the concept of dissipativity was extended to incremental-dissipativity in Stan and Sepulchre (2007) from a Lyapunov approach point of view, and asymptotic synchronization of a network with identical incrementally passive oscillators was considered subsequently. In Stan et al. (2007), output synchronization of networks with cyclic biochemical oscillators was discussed by using the properties of incremental output-feedback passivity. Motivated by cellular networks, Scardovi et al. (2010) introduced a network of nonlinear systems, and investigated its synchronization from an input-output point of view by extending the incremental output-feedback passivity to the relaxed-cocoercivity.

Apparently, the incremental-dissipativity includes the incremental-passivity or relaxed-cocoercivity as special cases, and can describe a more extensive class of physical systems. Thus, it is important to study how this more general property of systems can benefit the analysis of network synchronization. In this paper, we introduce the concept of incremental-dissipativity with a quadratic form by input-
output standpoint, and extend the result of Scardovi et al. (2010) to the case where each subsystem satisfies such an incremental-dissipativity condition. The synchronization problem of the network is shown to be equivalent to an input-output stability problem of a large-scale system which is made up of coupled dissipative subsystems, and a synchronization criterion is developed by applying the method established by Moylan and Hill (1978).

The rest of the paper is organized as follows. In Section 2, the network model and some preliminaries are introduced. The concept of input-output (Q, S, R)-dissipativity is reviewed, and then the definition of incremental-dissipativity is given. Section 3 contains the main results of the paper including the input-output part and its extension to the state space counterpart. Discussion on systems which satisfy the incremental-dissipativity property is carried out in Section 4. Section 5 gives a network which consists of the chaotic Chua circuits to illustrate the main result. Finally conclusions are presented in Section 5.

2. MODEL AND PRELIMINARIES

Let \( \mathbb{R} \) denote the field of real numbers, \( \mathbb{R}^n \) be the \( n \)-dimensional real vector space, and \( \mathbb{R}^{m \times n} \) be the set of \( n \times m \) real matrices. For a matrix \( P \in \mathbb{R}^{n \times m} \), \( P^T \) represents its transpose. If \( P \in \mathbb{R}^{n \times n} \) is a symmetric positive (negative) definite matrix, then denotes it as \( P > 0 \) (\( P < 0 \)). \( I_n \) is an \( n \times n \) identity matrix, and \( 1_n = (1,1,\ldots,1)^T \in \mathbb{R}^n \) is an \( n \)-order vector. \( \otimes \) represents the Kronecker product of matrices. We denote by \( \mathcal{U} \) an inner product space whose elements are functions \( u : \mathbb{R} \to \mathbb{R} \). Let \( \mathcal{U}_m \) be the space of \( m \)-tuples over \( \mathcal{U} \). Then for any \( u \in \mathcal{U} \) and any \( T \in \mathbb{R} \), a truncation \( u_T \) is defined as

\[
   u_T(t) = \begin{cases} u(t), & t < T \\ 0, & \text{otherwise} \end{cases}
\]

and the extended space of \( \mathcal{U} \) is defined by \( \mathcal{U}_e = \{ u \mid u_T \in \mathcal{U}, \forall T \in \mathbb{R} \} \). Given two functions \( u, v \in \mathcal{U}_e \) and any fixed \( T \in \mathbb{R} \), \( u_T \) represents the inner product of \( u_T \) and \( v_T \), and \( |u||v| = (u,v)_T \) is the norm of \( u_T \). Moreover, let \( \mathcal{L}_2 \) be the usual square integral space with elements \( u : [0,\infty) \to \mathbb{R} \), and \( \mathcal{L}_{2e} \) be the extended space of \( \mathcal{L}_2 \).

A system with \( m \) inputs and \( p \) outputs can be defined as a relation on \( \mathcal{U}_m^p \times Y^p \) with input \( u \in \mathcal{U}_m^p \) and output \( y \in Y^p \), and its input-output representation is an operator \( G : \mathcal{U}_m^p \to Y^p \).

**Definition 1.** (Hill and Moylan (1980)). The system \( G \) is dissipative with respect to the triple \((Q,S,R)\) if

\[
   (y,Qy)_T + 2(y,Su)_T + (u,Ru)_T \geq 0
\]

holds for all \( T \in \mathbb{R} \) and all \( u \in \mathcal{U}_m^p \), where \( Q = Q^T \in \mathbb{R}^{p \times p} \), \( S \in \mathbb{R}^{m \times n} \) and \( R = R^T \in \mathbb{R}^{m \times m} \) be constant matrices.

Here, we specify \( \mathcal{U}_e \) by \( \mathcal{L}_{2e} \), and consider the network model proposed in Scardovi et al. (2010). The network consists of \( n \) identical systems regarded as compartments, and each compartment is composed of \( N \) different subsystems described by operators. The network is given by:

\[
   y_{k,j} = G_{k} v_{k,j}, \quad k = 1, \ldots, N, \quad j = 1, \ldots, n \tag{1a}
\]

\[
   v_{k,j} = w_{k,j} + \sum_{i=1}^{N} h_{k,i} y_{i,j} + \sum_{z=1}^{n} a^k_{j,z} (y_{k,z} - y_{j,z}) \tag{1b}
\]

where \( y_{k,j} \in \mathcal{L}_{2e} \) and \( w_{k,j} \in \mathcal{L}_{2e} \) are the outputs and external inputs of the network, respectively. \( G_{k} : \mathcal{L}_{2e} \to \mathcal{L}_{2e} \) is a nonlinear operator describing the input-output behavior of the \( k \)-th subsystem of each compartment, and is to be further specified.

The network has two different kinds of interconnections, i.e., internal interconnections and external interconnections. \( H = (h_{k,i}) \in \mathbb{R}^{N \times N} \) is the subsystem coupling matrix which forms the compartment. It belongs to the internal interconnections, represents the interactions between different subsystem in a compartment, and is identical in every compartment. For any fixed \( k, \) \( 1 \leq k \leq N, \) \( \mathcal{A}_k = (a^k_{j,z}) \in \mathbb{R}^{n \times n} \) with \( a^k_{j,z} \geq 0 \) and \( a^k_{j,j} = 0, \) \( j, z = 1, \ldots, n, \) \( k \) is a compartmental coupling matrix. It represents the interplays of the \( k \)-th subsystem in different compartments and connects the compartments to constitute the whole network in a diffusive structure (Li et al. (2004)). Fig. 1 gives an example of such a network with 4 identical compartments which consists of 3 different subsystems described by input-output operator \( G_{k}, \) \( k = 1, \ldots, 3. \)

Let \( Y_k = (y_{k,1}, \ldots, y_{k,n})^T, \) \( V_k = (v_{k,1}, \ldots, v_{k,n})^T, \) \( W_k = (w_{k,1}, \ldots, w_{k,n})^T \) be the vectors of the outputs, inputs and external signals from the same subsystems \( k \) of all compartments in the network. Also, given a set of vectors \( Z_k, k = 1, \ldots, N, \) we denote the stacked vector by \( Z = \text{col}(Z_1, \ldots, Z_N) \). Then the feedback law (1b) can be rewritten as

\[
   V_k = W_k + \sum_{i=1}^{N} h_{k,i} Y_i - L_k Y_k, \quad k = 1, \ldots, N \tag{2}
\]

where \( L_k \), \( k = 1, \ldots, N \), are the Laplacian matrices associated to the compartmental coupling matrices \( \mathcal{A}_k \) with

\[
   L^k_{i,j} = \begin{cases} \sum_{z=1}^{n} a^k_{i,z}, & i = j \\ -a^k_{i,j}, & i \neq j \end{cases} \tag{3}
\]

Furthermore, let \( \bar{Y}_k = \frac{1}{N} Y_k, k = 1, \ldots, N \), denote the average of the outputs of the \( n \)-tuples of the subsystem \( k \), and define

\[
   Y_k^\Delta = \text{col}(y_{k,1} - \bar{Y}_k, \ldots, y_{k,n} - \bar{Y}_k), \quad k = 1, \ldots, N. \tag{4}
\]
Consequently, we define $\bar{V}_k$, $\bar{W}_k$, $V^\Delta_k$ and $W^\Delta_k$ corresponding to input signals $V_k$ and external signals $W_k$, respectively.

With the help of $\|Y^\Delta\|_T$ and $\|W^\Delta\|_T$, now we can give the definition of synchronization for the network (1). Similar to the input-output stability with finite gain of a nonlinear system, we define the finite gain input-output synchronization as follows.

**Definition 2.** The network (1) is said to achieve input-output synchronization with finite gain $\rho$ if

$$
\|Y^\Delta\|_T \leq \rho \|W^\Delta\|_T, \quad \forall T \geq 0
$$

(5)

for some $\rho > 0$ and all $W^\Delta \in \mathcal{L}^n_{2e}$.\]

**Remark 1.** Definition 2 can be regarded as the incremental stability of an input-output operator (Fromion et al. (1996)). It describes an input-output relationship of network (1) whose subsystems satisfy the incrementally-dissipative property.

**Definition 3.** Let the operator $G : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$. We say $G$ is $(\gamma_y, \gamma_{uy}, \gamma_n)$-incrementally-dissipative if

$$
\gamma_y \|G_{1u} - G_{2u}\|_T^2 + \gamma_u \|u_1 - u_2\|_T^2 + 2\gamma_{uy} \|G_{1u} - G_{2u}, u_1 - u_2\|_T \geq 0, \quad \forall T \geq 0
$$

(6)

for every pair of inputs $u_1, u_2 \in \mathcal{L}_{2e}$ with $\gamma_y \in \mathbb{R}$, $\gamma_{uy}, \gamma_n \in \mathbb{R}$, $\gamma_n \in \mathbb{R}$ being some constants.

**Remark 2.** Relaxed cocoercivity is $(\gamma_y, \gamma_{uy}, \gamma_n)$-incrementally-dissipativity with $\gamma_y = -\gamma_c$, $\gamma_{uy} = \frac{1}{T}$ and $\gamma_n = 0$.

### 3. MAIN RESULTS

In this section, we will investigate synchronization of the network (1) whose subsystems satisfy the incrementally-dissipative property from an input-output point of view, and give a sufficient condition under which synchronization can be achieved. Next, we will give a useful lemma which will be used to prove the main result.

**Lemma 1.** Consider the systems (1a). If the operators $G_k$ are $(\gamma_y^k, \gamma_{uy}^k, \gamma_n^k)$-incrementally-dissipative, then

$$
\gamma_y^k \|Y_k^\Delta\|_T^2 + \gamma_{uy}^k \|V_k\|_T^2 + 2\gamma_{uy}^k \|\bar{Y}_k, \bar{V}_k\|_T \geq 0, \quad \forall T \geq 0
$$

(7)

for every $k = 1, \ldots, N$, and $V_k \in \mathcal{L}^n_{2e}$, where $\bar{Y}_k = \Phi Y_k$, $\bar{V}_k = \Phi V_k$, and $\Phi \in \mathbb{R}^{(n-1) \times n}$ is given by

$$
\Phi = \begin{pmatrix}
-1 + (n-1)\nu & -\nu & \ldots & -\nu \\
-1 + (n-1)\nu & -\nu & \ldots & -\nu \\
\vdots & \vdots & \ddots & \vdots \\
-1 + (n-1)\nu & -\nu & -\nu & \ldots & -\nu
\end{pmatrix}
$$

(8)

with $\nu = \frac{n-\sqrt{n}}{n(n-1)}$.

**Proof.** Since $\Phi$ has the form of (8), we have $\Phi I_n = 0$, $\Phi \Phi^T = I_{n-1}$, and $\Phi^T \Phi = I_{n-1} - \frac{1}{n-1} I_{n-1}$. Thus, we have $\Phi^T \Phi Y_k = Y_k^\Delta$ from (4), which is to say that $\bar{Y}_k$ and $Y_k^\Delta$ are related by $Y_k^\Delta = \Phi^T \bar{Y}_k$, therefore

$$
\|Y_k^\Delta\|_T^2 = \sum_{k=0}^{T} \Phi^T \Phi Y_k dt = \|Y_k\|_T^2, \quad k = 1, \ldots, N
$$

(9)

Now we divide the proof into two different cases by distinguishing the value of $\gamma_{uy}^k$, i.e., the cases $\gamma_{uy}^k = 0$ and $\gamma_{uy}^k \neq 0$.

**Case a:** When $\gamma_{uy}^k = 0$, from Definition 3, we have

$$
\gamma_y^k \|y_{k,i} - y_{k,j}\|_T^2 + \gamma_{uy}^k \|v_{k,i} - v_{k,j}\|_T^2 \geq 0
$$

(10)

for $i, j = 1, \ldots, n$. By summing (10) over $i, j = 1, \ldots, n$, we can get that (7) is satisfied with

$$
\gamma_y^k \sum_{i,j=1}^{n} \|y_{k,i} - y_{k,j}\|_T^2 = \gamma_y^k \|Y_k\|_T^2
$$

(11)

and

$$
\sum_{i,j=1}^{n} \|v_{k,i} - v_{k,j}\|_T^2 \geq \frac{\gamma_y^k}{2\gamma_{uy}^k} \|V_k\|_T^2.
$$

(12)

**Case b:** When $\gamma_{uy}^k \neq 0$, define $z_{k,j} = v_{k,j} + \frac{\gamma_y^k}{2\gamma_{uy}^k} y_{k,j}$, which can be written in the vector form

$$
Z_k = V_k + \frac{\gamma_y^k}{2\gamma_{uy}^k} Y_k
$$

(13)

Thus we have

$$
2\gamma_y^k \langle V_k, Z_k \rangle_T = 2\gamma_y^k \langle \bar{V}_k, \bar{Z}_k \rangle_T - \gamma_y^k \|\bar{Y}_k\|_T^2
$$

(14)

where $\bar{Z}_k = \Phi Z_k$. Moreover, from (6) we have

$$
2\gamma_y^k \langle z_{k,i} - z_{k,j}, y_{k,i} - y_{k,j} \rangle_T = 2\gamma_y^k \langle \bar{V}_k, \bar{Z}_k \rangle_T - \gamma_y^k \|\bar{Y}_k\|_T^2
$$

(15)

Similar to (11), we have

$$
\sum_{i,j=1}^{n} \langle z_{k,i} - z_{k,j}, y_{k,i} - y_{k,j} \rangle_T = 2\gamma_y^k \langle \bar{Z}_k, \bar{V}_k \rangle_T
$$

(16)

Summing (15) over $i, j = 1, \ldots, n$, and combining (12) and (16), we have

$$
2\gamma_y^k \langle \bar{Z}_k, \bar{V}_k \rangle_T \geq -\gamma_y^k \langle \bar{V}_k, \bar{V}_k \rangle_T
$$

(17)

(14) and (17) give (7). This completes the proof.

**Remark 3.** Lemma 1 shows that if the subsystem $G_k$ with input $u_{k,j}$ and output $y_{k,j}$ is $(\gamma_y^k, \gamma_{uy}^k, \gamma_n^k)$-incrementally-dissipative, then a corresponding virtual system with input-output pair $(\bar{V}_k, \bar{Y}_k)$ is $(Q, S, R)$-dissipative with $Q = \gamma_y^k I_{n-1}$, $S = \gamma_{uy}^k I_{n-1}$, and $R = \gamma_n^k I_{n-1}$.

Now we are ready to give the main result of the paper which is addressed in the following theorem.
Theorem 1. Consider the network (1). Suppose that each operator $G_k$ is $(\gamma_k, \gamma_y^k, \gamma_u^k)$-incrementally-dissipative. If
\[
\dot{Q} = -Q - SH - H^T S^T - H^T RH > 0,
\]
then we have
\[
\|Y^\Delta\|_T \leq \rho\|W^\Delta\|_T, \quad \forall T \geq 0
\]
for some $\rho > 0$, and all $W \in L_2^{n,n}$, where $Q = \text{diag}(Q_1, \ldots, Q_N)$, $S = \text{diag}(S_1, \ldots, S_N)$ and $R = \text{diag}(R_1, \ldots, R_N)$.\(^\text{(18)}\)

Proof. Let $V_t = U_k - L_k Y_k$, and from Lemma 1, we have
\[
\frac{d}{dt} \|V_k\|^2_T = -2\gamma_y^k \langle \dot{Y}_k, V_k \rangle_T - 2\gamma_y^k \langle \dot{V}_k, Y_k \rangle_T + \gamma_y^k \langle \dot{V}_k, Y_k \rangle_T - 2\gamma_y^k \langle \dot{U}_k, Y_k \rangle_T + \gamma_y^k \langle \dot{U}_k, Y_k \rangle_T - 2\gamma_y^k \langle \dot{S}_k, Y_k \rangle_T + \gamma_y^k \langle \dot{S}_k, Y_k \rangle_T - 2\gamma_y^k \langle \dot{R}_k, Y_k \rangle_T + \gamma_y^k \langle \dot{R}_k, Y_k \rangle_T.
\]

From (19) to the third one, we use the property that
\[
\Phi L_k Y_k = \Phi L_k \Phi^T Y_k = \Phi L_k \Phi^T \dot{Y}_k = \dot{L}_k Y_k,
\]
which comes from $L_k 1_n = 0$ and $L_k (I_n - \Phi^T \Phi) Y_k = 0$.

Thus we have that each subsystem with input $\dot{U}_k$ and output $\dot{Y}_k$ is $(Q_k, S_k, R_k)$-dissipative.

By summing (19) over all $k$, we have
\[
\langle \dot{Y}, Q \dot{Y} \rangle_T + 2\langle \dot{Y}, S \dot{U} \rangle_T + \langle \dot{U}, R \dot{U} \rangle_T \geq 0,
\]
which is to say that the overall system with $\dot{U}$ as input and $\dot{Y}$ as output, is also $(Q, S, R)$-dissipative. Furthermore, from (2), we have $U = W + (H \otimes I_n) Y$ which leads to
\[
\dot{U} = \dot{W} + (H \otimes I_n - 1) \dot{Y} = \dot{W} + H \dot{Y}.
\]

From (21) and $\dot{Q} > 0$, (20) can be written as
\[
\langle \dot{Y}, Q \dot{Y} \rangle_T - 2\langle \dot{Y}, Q \dot{S} \dot{U} \rangle_T \leq (W, R\dot{W})_T
\]
with $S = \dot{Q}^{-\frac{1}{2}} (S + H^T R)$. Since there always exists a finite scalar $\alpha > 0$ such that $R + S^T S \leq \alpha^2 I_{N(n-1)}$, then it follows that
\[
\|Q^{-\frac{1}{2}} \dot{Y} - \dot{S} \|_T \leq \alpha\|\dot{W}\|_T,
\]
which gives
\[
\|\dot{Y}\|_T \leq \rho\|\dot{W}\|_T)
\]
with $\rho = \|Q^{-\frac{1}{2}} (\alpha + S)\|$. Since $\|\dot{W}k\| = \|W^\Delta k\|_T$ and $\|\dot{Y}k\| = \|Y^\Delta k\|_T$ for every $T \geq 0$ and $k = 1, \ldots, N$, we have (18) hold. Consequently, if $W \in L_2^{n,n}$, then $\|\dot{Y}\|_T \leq \rho\|\dot{W}\|_T$.

Remark 4. In Theorem 1, we extend the condition of relaxed cocoercivity to a more general case, namely incremental-dissipativity. Thanks to the special property of incremental-dissipativity shown in Lemma 1, the network can be seen as a large-scale system interconnected by the virtual systems with input $\dot{U}_k$ and output $\dot{Y}_k$ which are $(Q_k, S_k, R_k)$-dissipative, and the input-output synchronization problem can be handled as an input-output stability problem of such a large-scale system. Then we could apply Moylan and Hill (1978)’s method to derive a single matrix condition $\dot{Q} > 0$ for the network synchronization. Moreover, let $\gamma_y^k = -\gamma_k$, $\gamma_y^k = \frac{1}{2}$ and $\gamma_u^k = 0$, then Theorem 1 in Scardovi et al. (2010) follows.

Also, we can extend Theorem 1 to a network with subsystems described by state space equations. Consider a group of systems

\[
\dot{x}_{k,j} = f_k(x_{k,j}, y_{k,j}), \quad k = 1, \ldots, N, \quad j = 1, \ldots, n
\]
where $x_{k,j} \in \mathbb{R}^p$, $y_{k,j} \in \mathbb{R}^q$, and $v_{k,j} \in \mathbb{R}$ are states, inputs and outputs of the systems, respectively. Moreover, functions $f_k : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^p$ are assumed to be locally Lipschitz in the first argument, and $h_j : \mathbb{R}^p \to \mathbb{R}$ are continuous. We suppose that for each fixed $k$, the system is $L_2$-well-posed, and has a nonlinear input-output operator $G_k : L_2 \to L_2$, with well-defined zero initial conditions, and the closed-loop system (25) and (1b) is zero-state reachable (van der Schaft (2000)). Then we have the following corollary.

Corollary 1. Consider the closed-loop system (25) and (1b) with no external signals (i.e., $w_{k,j} = 0$). Suppose the closed-loop system is zero-state reachable. If the conditions in Theorem 1 are satisfied, then the asymptotic output synchronization of the network is achieved, i.e.,
\[
\lim_{t \to \infty} (y_{i,k}(t) - y_{j,k}(t)) = 0,
\]
for all $k = 1, \ldots, N, i, j = 1, \ldots, n$.

Proof. The proof is similar to that of Corollary 1 in Scardovi et al. (2010), and thus it is omitted here.

4. DISCUSSION ON INCREMENTAL-DISSIPATIVITY

Generally, systems which are incremental-dissipative must be checked case by case. Even though, as a special case of incremental-dissipativity, the relaxed-cocoercive systems such as Goodwin oscillator satisfy the incremental-dissipativity condition automatically, it is still important and of interest to classify systems which are incrementally-dissipative rather than relaxed cocoercive, and to give some sufficient conditions which guarantee the incremental-dissipativity of the input-output operator related to special classes of systems described by state state space equations. In this section, we will distinguish a class of systems which is incrementally-dissipative, but not relaxed-cocoercive.

As pointed out in Scardovi et al. (2010), for a family of one dimensional systems
\[
\dot{x}_i = -f_i(x_i) + u_i, \quad y_i = x_i, \quad i = 1, \ldots, m
\]
where $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ are states, inputs and outputs of the systems, if for each $i = 1, \ldots, m$, the system in (26) is $L_2$-well-posed with a nonlinear input-output operator $G_i : L_2 \to L_2$. Furthermore, if $f_i$ satisfies the QUAD condition (27), i.e., for every $x_{i,1}, x_{i,2} \in \mathbb{R}$ the following inequality hold (Li et al. (2004); De Lellis et al. (2008))
\[
(x_{i,1} - x_{i,2})(f_i(x_{i,1}) - f_i(x_{i,2})) \geq \gamma_i (x_{i,1} - x_{i,2})^2,
\]
Now, we claim that the cascade interconnected system of $G_i$ – see Fig. 2 – is incrementally-dissipative provided some additional conditions hold. The details are given in Corollary 2.

**Corollary 2.** Consider the system $G$ which is a cascade interconnection of relaxed-cocoercive systems $G_i$, $i = 1, \ldots, m$ in the family of (26). If $\gamma_i > 0$, then for any

$$\kappa > \frac{1}{\gamma_1} \cdots \frac{1}{\gamma_m} \cos \left( \frac{\pi}{m+1} \right)^{m+1},$$

(28)

the interconnected system $G$ with input-output pair $(u, y)$ and zero initial conditions is $(-\varepsilon, \frac{1}{2}, \kappa)$-incremental-dissipative for some $\varepsilon > 0$.

**Proof.** Let $\delta x_i = x_{i,1} - x_{i,2}$, $\delta y_i = y_{i,1} - y_{i,2}$, $\delta u_i = u_{i,1} - u_{i,2}$, for $x_{i,j}$, $y_{i,j}$, $u_{i,j} \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, 2$. Define

$$V_i = \frac{1}{2\gamma_i} (x_{i,1} - x_{i,2})^2 = \delta x_i^2, \quad i = 1, \ldots, m,$$

(29)

then we have

$$\dot{V}_i = \frac{1}{\gamma_i} (-f_i(x_{i,1}) + f_i(x_{i,2}) + \delta u_i) \delta x_i$$

$$\leq -\varepsilon \|\delta Y\|_2^2 + \frac{1}{\gamma_i} \delta u_i \delta x_i$$

$$\leq -\varepsilon \delta y_i^2 + \frac{1}{\gamma_i} \delta u_i \delta y_i,$$

(30)

Let

$$V = \sum_{i=1}^{m} d_i V_i,$$

(31)

where $d_i > 0$. Applying Corollary 5 in Arcak and Sontag (2006) with condition (28), gives

$$\dot{V} \leq -\varepsilon \|\delta Y\|_2^2 + \kappa \|\delta u\|_2^2 + \|\delta y\|_2^2 + \|\delta u\|_2 \|\delta y\|_2$$

(32)

Integrating both side of (32) form 0 to $T$ and with the zero initial conditions, we have

$$0 \leq \sum_{i=1}^{m} d_i \|\delta x(T)\|_2^2 \leq -\varepsilon \|\delta y\|_2^2 + \kappa \|\delta u\|_2^2 + \|\delta y\|_2^2 + \|\delta u\|_2 \|\delta y\|_2$$

(33)

with the norm and inner product defined in $\mathcal{L}_{2e}$ space. This completes the proof.

**Remark 5.** Corollary 2 only gives a special case for systems which are incrementally-dissipative. Except for this particular case and the cocoercivity, many systems in practice satisfy the property of incremental-dissipativity. An important class of these systems are incrementally stable systems, i.e., $(-1, 0, \gamma^2)$-incremental-dissipativity, which have been studied since 1960s (see Fromion et al. (1996) and references therein). An example for such a system is

![Fig. 2. The cascade interconnection.](image)

![Fig. 3. The compartment of the network: Chua circuit saturated linear feedback system (Romanchuk and Smith (1999)).](image)
Fig. 4. The double-scroll Chua’s attractor.

Fig. 5. The synchronization errors of the network.

for synchronization of the network has been obtained. Conditions for a class of systems represented by state space equations which are incrementally-dissipative, but not relaxed cocoercivity have been identified.

REFERENCES


