\textbf{H}_\infty \text{ Control with Multi-Saddles}∗

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Abstract: This paper investigates the \textit{H}_\infty control problems with multi-saddles for linear autonomous systems and affine nonlinear autonomous systems. From the viewpoint of (sub)optimal control, one should find the saddles of the Hamilton function of the considered system in order to solve the \textit{H}_\infty control problem. Usually, solving the \textit{H}_\infty control problem is based on an assumption to assure the existence and uniqueness of the saddle, which is not generally true. This paper uses the theory of generalized inverses (esp. the group inverse) of matrices to solve the Hamilton-Jacobi-Isaacs (HJI) equations or inequalities; in particular, linear \textit{H}_\infty control problems can be solved by solving algebraic Riccati equations or inequalities, see Basar et al. [1990], Hong [2001], Isidori et al. [1995], van der Schaft [1991], Zhou et al. [1988, 2006], Doyle et al. [1989] and Zhang et al., [2010].

As mentioned in Isidori et al. [1995], in order to solve \textit{H}_\infty control problem from the viewpoint of (sub)optimal control, one should first ensure the existence and uniqueness of the saddle of the Hamilton function of the considered system. Most related literatures are working under such an assumption. But, in fact, a Hamilton function may either have no saddles or have more than one saddle. If a Hamilton function has no saddles, solving the \textit{H}_\infty control problem from the viewpoint of (sub)optimal control is possible. In this paper, with the help of the theory of generalized inverses of matrices, some solvability conditions and state feedback solutions for the \textit{H}_\infty control problems with multi-saddles are given via the generic saddle solution representations. In addition, the above results obtained are extended to the case without the existence of saddles.

1. INTRODUCTION

\textit{H}_\infty control problem (throughout this paper, \textit{H}_\infty control is only based on a state space approach and only autonomous systems are considered) is one of the important problems in systems theory and engineering. With the help of differential game theory, from the viewpoint of (sub)optimal control, \textit{H}_\infty control problems can be solved by solving Hamilton-Jacobi-Isaacs (HJI) equations or inequalities; in particular, linear \textit{H}_\infty control problems can be solved by solving algebraic Riccati equations or inequalities, see Basar et al. [1990], Hong [2001], Isidori et al. [1995], van der Schaft [1991], Zhou et al. [1988, 2006], Doyle et al. [1989] and Zhang et al., [2010].

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If a Hamilton function has multi-saddles, how to give a generic representation of the saddle solutions becomes the key, but it seems to be impossible, since one has to find an “inverse” of a singular matrix to do so. Luckily, the emergence of the theory of generalized inverse makes it possible. In this paper, with the help of the theory of generalized inverses of matrices and the frequency domain theory, Stoorvogel [1992] and Zhang et al., [2001] also deal with the similar case based on a linear space approach.

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and affine nonlinear systems respectively. Section 4 is a brief conclusion.

2. PRELIMINARIES

2.1 MS - H∞ Control Problem Description

Consider the following affine nonlinear system

\[
\begin{aligned}
\dot{x} &= f(x) + g_1(x)u + g_2(x)w, \\
z &= h(x) + k(x)u,
\end{aligned}
\]

where \( x = x(t) \in \mathbb{R}^n, u \in \mathbb{R}^m, w = w(t) \in \mathbb{R}^q \) and \( z \in \mathbb{R}^p \) denote the state, the control input, all exogenous disturbances and the control output respectively, \( f(0) = 0, h(0) = 0 \). \( f(x), g_1(x), g_2(x), h(x), k(x) \) are defined on \( D \), a neighborhood of the origin, and analytic.

Thus we have \( \ker(A) = \ker(A^2) \).

Proposition 2. For any given matrix \( A \) over \( F \), it follows that

\[
I - A(A^T A)^\#A^T = (A^T A)^\pi.
\]

Since \([1]-inverse is not generally unique, which may bring troubles to computation, it is needed to introduce another kind of general inverse which exists uniquely: Drazin inverse. The concept of Drazin inverse is first introduced in Drazin [1958].

Definition 1. (Drazin [1958]). For an element \( A \) in a ring (semigroup), the element \( A^D \) in the ring (semigroup) satisfying

\[
A^kA^D A = A^k \quad \text{(for some nonnegative integer } k),
\]

\[
A^D A A^D = A^D,
\]

\[
A A^D = A^D A
\]

is said to be a Drazin inverse of \( A \).

Drazin [1958] gives that \( A \) has at most one Drazin inverse. In particular, if \( k = 1 \), \( A^D \) is also called a group inverse of \( A \) when \( k = 1 \), and usually denoted by \( A^G \).

It is easy to get that if \( A \) is a square matrix over \( F \), \( A^D \) exists. We know there exist invertible matrices \( P \) and \( A_0 \) and a nilpotent matrix \( A_0 \) over \( F \) such that \( A = P \text{diag}(A_0, A_0^p)^{-1} \) (Fitting decomposition). Thus \( A^D = P \text{diag}(A_0^{-1}, 0)^{-1} \). Then \( A^G \) exists if and only if rank \((A) = \text{rank}(A_0^2) \), i.e., \( A_0 = 0 \).

Compared \([1]-inverse with group inverse, one sees immediately that group inverse is a special \([1]-inverse. Since group inverse is unique, it must have better properties than general \([1]-inverses. Due to the advantages of group inverse, if the matrix \( A \) in Eq. (3) has a group inverse, we might as well replace the matrix \( A^{(1)} \) in (4) by \( A^G \), i.e.,

\[
x = A^G b + (I - A^G A)u := A^G b + A^G u,
\]

which will be used to give the generic saddle solution representations of the Hamilton functions.

Hereafter, \( F \) denotes any field satisfying Assumption 1.

Assumption 1. For any given positive integer \( n \), any \( x_1, \ldots, x_n \in F, \sum_{i=1} x_i^2 = 0 \) implies \( x_1 = \cdots = x_n = 0 \).

Based on Assumption 1, we give the following three propositions which help to solve the \( MS - H_{\infty} \) control problem.

Proposition 1. If \( A \) is an \( n \times n \) matrix over \( F \) and \( A^T = A \), \( A^T \) exists.

Proof: We should prove rank \((A) = \text{rank}(A^2) \), i.e., ker \((A) = \text{ker}(A^2) \), ker \((A) \subset \text{ker}(A^2) \) holds obviously. Next we prove ker \((A^2) \subset \text{ker}(A) \). For any \( x \in \text{ker}(A^2), A^2 x = 0 \), \( 0 = x^T A^T A x = x^T (A^T A) x = (Ax)^T A x = (Ax)^T A x, \) then \( A x = 0 \), so \( x \in \text{ker}(A), i.e., \text{ker}(A^2) \subset \text{ker}(A) \). Thus we have ker \((A) = \text{ker}(A^2) \).

Proposition 2. For any given matrix \( A \) over \( F \), it follows that

\[
I - A(A^T A)^D A^T = (A A^T)^\pi.
\]
Proof. By Proposition 1, we have that $(A^T A)^\#$ and $(A^T A)^\#$ both exist. Denote by $\mathcal{L}(F^n, F^m)$, the set of all bounded linear operators from $F^n$ to $F^m$. Let $A$ be an $n \times n$ matrix, then $A \in \mathcal{L}(F^n, F^m)$. We should prove $A(A^T A)^\# A^T - (A A^T)^\# A^T \in \mathcal{L}(F^n, F^m)$ is a zero operator. It is obvious that $\text{Im}(A A^T) \oplus \ker(A A^T) = F^m$. Take $x = x_1 + x_2 \in F^m$, where $x_1 \in \text{Im}(A A^T) = \text{im}(A)$, $x_2 \in \ker(A A^T)$. Since $A^T A = \text{Im}(A^T A)$, there exists $y \in F^m$ such that $A^T x = A^T Ay$. Write $y = y_1 + y_2$, where $y_1 \in \text{im}(A A^T)$, $y_2 \in \ker(A A^T)$, then $A^T Ay = A^T A y_1$. Similarly, $A^T x = A^T x_1$. Hence $A^T (x_1 - Ay_1) = 0$ and $x_1 - Ay_1 \in \text{im}(A) \cap \ker(A^T A) = 0$. That is, $x_1 = Ay_1$.

By definition, we have $(A A^T)^\#(A^T A)$ is the projector of $F^n$ on $\text{Im}(A^T A)$ along $\ker(A A^T)$, and $(A^T A)^\#(A^T A)$ is the projector of $F^n$ on $\text{Im}(A^T A)$ along $\ker(A^T A)$. Then we have $(A A^T)^\#(A^T A)x = x$, and $A(A^T A)^\#(A^T x) = A(A A^T)^\# A^T A y = Ay = x$, which ends the proof. \(\square\)

Proposition 3. For any given matrix $A$ over $F$, we have $A(A^T A)^\# A = 0$.

Proof. The proof is similar to that of Proposition 2, we omit it. \(\square\)

Based on the above preliminaries, we revert to the main problem, $MS - H_\infty$ control.

3. MAIN RESULTS

3.1. $MS - H_\infty$ Control of Linear Systems

Consider the following linear system
\[
\begin{cases}
\dot{x} = A x + B_1 u + B_2 w, \\
\dot{z} = C x + K u,
\end{cases}
\]
where $x$, $u$, $w$ and $z$ are shown in (1), $A$, $B_1$, $B_2$, $C$, $K$ are constant matrices of appropriate dimensions. Here we give Assumption 2 to assure the existence of the saddle solutions instead of the nonsingularity of $K T^T$ to assure the existence of uniqueness of the saddle solution. Later, this assumption will be removed.

Assumption 2. $\text{Im}(B_1^T) \subset \text{Im}(K T^T)$. 

We solve the $MS - H_\infty$ control problem of system (6). The Hamilton function of system (6) is
\[
H(p, x, u, w) = p^T (A x + B_1 u + B_2 w) + z^T z - x^T w^T w. \tag{7}
\]
According to the idea of optimal control, we firstly find the weakest control $u^*$ and the strongest disturbance $w^*$. From
\[
\frac{\partial H}{\partial u} = p^T B_1 + 2 x^T C^T K + 2 u^T K T^T K = 0, \tag{8}
\]
$K T^T$ is symmetric, by Proposition 1, Assumption 2, (5) and $\text{Im}(K T^T) = \text{Im}(K T^T)$, we get the generic solutions of Eq. (8) can be represented as
\[
u^* = -(K T^T)^\# \left( \frac{1}{2} B_1^T p + K T^T C x \right) + (K T^T)^\# v, \tag{9}
\]
where $v \in F^m$ is arbitrary, since
\[
H(p, x, u, w) - H(p, x, u^*, w) = (u^* - u)^T K T^T K (u^* - u) \geq 0.
\]
Similarly, we get
\[
w^* = \frac{1}{2 \lambda^2} B_2^T p. \tag{10}
\]
then we have
\[
\begin{align*}
H(p, x, u^*, w) & \leq H(p, x, u^*, w^*) \\
& \leq H(p, x, u, w^*) \tag{11}
\end{align*}
\]
for any given $x, p \in F^n$, and
\[
\begin{align*}
H(p, x, u^*, w^*) & = p^T A x + x^T C^T C x + \frac{1}{4 \lambda^2} p^T B_2 B_2^T p \\
& - \left( \frac{1}{2} B_1^T p + K T^T C x \right)^T (K T^T K)^\# \left( \frac{1}{2} B_1^T p + K T^T C x \right), \tag{12}
\end{align*}
\]
and $\text{Im}(K T^T)$ in $\mathbb{R}^m$ for any given $x$ and $p$. In addition, from (12), we see the Hamilton function (7) is independent of $v$, thus it is constant with respect to its saddles.

Choose the state feedback control (9) and regard $p$ as a function of $x$, we have the following closed loop system
\[
\dot{x} = (A - B_1 (K T^T)^\# K T^T C) x - \frac{1}{2} B_1 (K T^T)^\# B_2^T p(x). \tag{13}
\]
We note, from (13), that $v$ neither affects the state, nor affects the control output, but it affects the input. Based on those, we can choose appropriate $v$ to regulate the input. Of course, we can set $v \equiv 0$.

From (11), (12) and (13), we have

Theorem 4. The $MS - H_\infty$ control problem of system (6) is solvable under Assumption 2, if there exists a continuous function $p : \mathbb{R}^n \to \mathbb{R}^n$, such that
\[
(1) \text{system (13) is asymptotically stable,}
\]
\[
(2) \text{(12) with } p \text{ the function } p(x) \text{ is nonpositive, for some positive constant } \lambda, \text{ for all } x \in \mathbb{R}^n,
\]
\[
(3) \int_0^T p(x(t)) (A x(t) + B_1 u^* + B_2 w(t)) dt \geq 0, \text{ for all } w \in L_2^2(0, T), \text{ where } T \in (0, +\infty), u^* \text{ is shown in (9),}
\]
and $u^*$ is a solution.

Set $p(x) = 2 P x$, where $P$ is a real symmetric matrix, then Theorem 4 can be simplified to Theorem 5.

Theorem 5. The $MS - H_\infty$ control problem of system (6) is solvable under Assumption 2, if there exists a real symmetric positive definite matrix $P$ and a positive constant $\lambda$, such that
\[
PA + A^T P + C^T C + \frac{1}{\lambda^2} P B_2 B_2^T P - \left( P B_1 + C T K \right)^T \left( K T^T K \right)^\# \left( \frac{1}{2} B_1^T P + K T^T C \right) < 0, \tag{14}
\]
and the control input $u^* = -(K T^T)^\# \left( \frac{1}{2} B_1^T p + K T^T C x \right) + (K T^T)^\# v$ is a solution, where $v \in \mathbb{R}^m$ is arbitrary.

Note that (14) is similar to algebraic Riccati inequality, we might as well call it singular algebraic Riccati inequality.

Now we give Assumption 3 to make an approximation of the $L_2$ gain, which is equivalent to the stabilizability of the linear system $(A, B_1 G^2 B_2^T)$ (see Assumption 3), which was first proposed in Andreini et al. [1988].
Assumption 3. There exists a real symmetric positive definite matrix $P$, such that
\[
\ker(GB \bar{P}) \subset \{ x \in \mathbb{R}^n | x^T \bar{P} \bar{A} x < 0 \} \cup \{ 0 \},
\]
where $G = (K^T K)^{1/2}$. 

Theorem 6. The $MS-H_\infty$ control problem of system (6) is solvable under Assumption 2 and Assumption 3 with a finite $L_2$ gain
\[
\lambda = \max \left\{ \max_{x \in \Lambda_1} \left\{ \frac{a(x)}{b(x)} \right\}, -\min_{x \in \Lambda_2} \left\{ \frac{c(x)}{2a(x)} \right\}, \sup_{x \in \Lambda_3} \left\{ \frac{a(x) + \sqrt{a(x)^2 + b(x)e(x)}}{b(x)} \right\} \right\} + \epsilon,
\]
where $\epsilon$ is an arbitrary positive number,
\[
a(x) = x^T \bar{P} \bar{A} x, \\
b(x) = \|GB_1^T \bar{P} x\|_2^2, \\
c(x) = \|(K \bar{K}^T + \bar{C} \bar{x})\|_2^2 + \|B_2^T \bar{P} x\|_2^2,
\]
\[
\Lambda_1 = \{ x \in \mathbb{R}^n \mid a(x) \geq 0 \}, \\
\Lambda_2 = \mathbb{R}^{n-1} \cap \ker\left(GB_1^T P\right), \\
\Lambda_3 = \mathbb{R}^{n-1} - \ker\left(GB_1^T P\right),
\]
and
\[
u(x) = -G^2(\lambda^2 + K^T K)^v
\]
is a solution, where $v \in \mathbb{R}^m$ is arbitrary.

Proof. Substitute (15) into system (6) and let $w = 0$, by Proposition 3, we get the closed loop system
\[
x = (\bar{A} - \lambda \bar{B}_1 G^2 \bar{K}^T C)x + (K^T K)^v v.
\]
Substitute (15) and (10) into the Hamilton function (7), by Proposition 2 and Proposition 3, we have
\[
H(2\lambda P x, u, v, u^*) = -b(x)^2 + 2a(x)\lambda + c(x).
\]

By Assumption 3, we have $b(x) = 0$ implies $x = 0$ or $a(x) < 0$. By notations we have $b(x)$ and $c(x)$ are both positive semidefinite.

Firstly we prove the Hamilton function (17) is nonpositive.

If $b(x) = 0$ and $x = 0$, (17) vanishes obviously.

If $b(x) = 0$ and $x \neq 0$, then $a(x) < 0$. By $\lambda > -\max_{x \in \Lambda_2} \left\{ \frac{c(x)}{2a(x)} \right\}$, (17) is negative.

If $b(x) > 0$, by $\lambda > \sup_{x \in \Lambda_3} \left\{ \frac{a(x) + \sqrt{a(x)^2 + b(x)e(x)}}{b(x)} \right\}$, (17) is negative.

Secondly we prove system (16) is asymptotically stable.

Take $V(x) = x^T \bar{P} x$ as a Lyapunov function candidate, then $\dot{V}(x) = 2a(x) - 2\lambda b(x)$, which should be proved negative definite.

$x = 0$ implies $\dot{V}(x) = 0$.

If $x \neq 0$ and $b(x) = 0$, $\dot{V}(x) = a(x) < 0$.

If $b(x) > 0$, by $\lambda > \max_{x \in \Lambda_2} \left\{ \frac{a(x)}{b(x)} \right\}$ we have $\dot{V}(x) < 0$.

At last,
\[
\int_0^T (2x(t)^T P (Ax(t) + B_1 u(t) + B_2 w(t)) dt = V(x(T)) - V(x(0)) = V(x(T)) \geq 0,
\]
where $u(t)$ is shown in (15), $w(t) \in L_2^2[0,T]$ is arbitrary, $T \in (0, +\infty)$.

Remark 2. From the proof of Theorem 6, we see that Theorem 6 still holds without Assumption 2 under the condition $v \equiv 0$, and so are Theorem 4 and 5. That is to say, although the three theorems are obtained under an assumption to assure the existence of saddles, they are still right when saddles do not exist if $v$ vanishes.

For the sake of comparison research, let us recall the corresponding result given in Zhou et al. [1988]. Zhou et al. [1988] deals with this problem via rewriting $K$ as $K = U \Sigma$, where $U$ is of full column rank and $\Sigma$ is of full row rank. The result is the following theorem.

Theorem 7. (Zhou et al. [1988]). 1 For system (6), if $(A, B_1)$ is stabilizable, then the $H_\infty$ control problem of system (6) is solvable only if there exist positive constants $\gamma$ and $\epsilon$ and a real symmetric positive definite matrix $P$ such that
\[
(A - B_1 H_F K^T C)^T P + P (A - B_1 H_F K^T C) + \frac{1}{\gamma^2} P B_2 B_2^T P - P B_1 H_F B_1^T P - \frac{1}{\epsilon} P B_1 \Phi_F \Phi_F B_2^T P + (18)
\]
and a solution is
\[
u(x) = -\left( \frac{1}{2\epsilon} \Phi_F \Phi_F + H_F \right) B_2^T P + H_F K^T C \right) x, (19)
\]
where $H_F = \Sigma^T (\Sigma \Sigma^T)^{-1} (U^T U)^{-1} (\Sigma \Sigma^T)^{-1} \Sigma$. $\Phi_F$ is a real matrix such that $\Phi_F \Sigma^T = 0$ and $\Phi_F \Sigma F = I$.

Theorem 7 is obtained by proving the $H_\infty$ norm of the closed loop transfer function matrix is less than $\gamma$ based on the frequency domain theory, thereby it does not involve the concept of saddles. Besides, the proof is complex and cannot be extended to nonlinear systems. But it gives a necessary and sufficient condition. Comparatively, this paper gives solutions to the $MS-H_\infty$ control problem by finding the multi-saddles from the viewpoint of (sub)optimal control, which is direct and computable. The proof is simple and can be easily extended to nonlinear $H_\infty$ control. But the conclusions given in this paper are sufficient.

By comparing (14) with (18), we see that $H_F$ precisely equals $(K^T K)^p$ and $\Phi_F B_2^T = 0$ under Assumption 2. That is to say, (18) coincides with (14) and (19) coincides with (15) with $v$ vanishing under Assumption 2.

Since the theory of generalized inverses of matrices over a general field is also ripe (see Subsection 2.2) and affine nonlinear systems are indeed “linear” with respect to their control inputs, the theory of generalized inverse can be sufficient.

1 In Zhou et al. [1988], the output equation is $z = Cx + Ku + Dw$.

Zhou et al. [1988] gives that if $(A, B_1)$ is stabilizable, the $H_\infty$ control problem is solvable only if $D^T D < \gamma^2 I$. We set $D = 0$ here since $\gamma$ is usually small and that the exogenous disturbances affects the output only via affecting the state directly is reasonable.
conveniently used to handle the $MS-H_\infty$ control problem of affine nonlinear systems.

### 3.2 MS–H∞ Control of Affine Nonlinear Systems

First we introduce some preliminaries. Let $s$ be an indeterminate and $\mathbb{R}[s] = \{ \sum_{n=0}^{\infty} a_n s^n | a_n \in \mathbb{R} \}$ be the set of all formal power series with indeterminate $s$, and coefficients elements of the real number field $\mathbb{R}$. It is well known that $\mathbb{R}[s]$ forms an integral domain with addition and multiplication defined by

$$
\sum_{n=0}^{\infty} a_n s^n + \sum_{n=0}^{\infty} b_n s^n = \sum_{n=0}^{\infty} (a_n + b_n) s^n
$$

and

$$
\left( \sum_{n=0}^{\infty} a_n s^n \right) \left( \sum_{n=0}^{\infty} b_n s^n \right) = \sum_{n=0}^{\infty} \left( \sum_{i+j=n} a_i b_j \right) s^n.
$$

It is also well known that each integral domain can be embedded in the field of fractions of itself uniquely up to isomorphism. For example, the integer domain $\mathbb{Z}$ can be embedded in the rational number field $\mathbb{Q}$. The field of fractions of $\mathbb{R}[s]$ is $\mathbb{R}(s) = \{ s^n \sum_{n=0}^{\infty} a_n s^n | m \in \mathbb{Z}, a_n \in \mathbb{R}, a_0 \neq 0 \}$. Details are referred to Sahai et al. [2008]. It is obvious that the field $\mathbb{R}(s)$ satisfies Assumption 1.

Recall that $C^\infty(D)$ denotes the set of all real valued analytic functions defined on $D \subset \mathbb{R}^n$. Then, if the variable $x \in D$ is regarded as an indeterminate, $C^\infty(D)$ can also form an integral domain with addition and multiplication similar to (20) and (21) respectively. Denote by $F_Q$, the field of fractions of $C^\infty(D)$. Similarly, $F_Q$ also satisfies Assumption 1.

Let $k(x)$ be an $m \times n$ matrix over $C^\infty(D)$, then $(k(x)^T k(x))$ is symmetric. Note that $(k(x)^T k(x))$ may have no group inverses (e.g. $\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$). But if we regard $(k(x)^T k(x))$ as a matrix over $F_Q$, by Proposition 1 we have $(k(x)^T k(x))$ has a group inverse. That is to say, the function $(k(x)^T k(x))^{\#}$ may have a discontinuous point. If $x$ is a discontinuous point of $(k(x)^T k(x))^{\#}$, we can first substitute $\bar{x}$ into $(k(x)^T k(x))^{\#}$, then compute $(k(\bar{x})^T k(\bar{x}))^{\#}$. Thus, we can say the function $(k(x)^T k(x))$ has a group inverse pointwise, but may not uniformly. Of course, if we compute $(k(x)^T k(x))^2$ pointwise, there needs not to assume $k(x)$ is analytic. Based on these, the functions $f(x), g_1(x), g_2(x), h(x), k(x)$ in system (1) can be regarded as matrices over $F_Q$.

Now, we revert to the $MS-H_\infty$ control problem of affine nonlinear systems. Similar to the discussion on the $MS-H_\infty$ control problem of the linear system (6), here we give Assumption 4. $\text{Im}(g_1^*(x)) \subset \text{Im}(k(x)^T k(x))$

to guarantee the existence of the saddles of the Hamilton function of system (1). Similar to Assumption 2, this assumption will be also removed later.

Note that Assumption 2 is just the linear case of Assumption 4. By Proposition 1, we obtain $(k^T(x) k(x))^2$ exists.

Based on the previous preliminaries, we can intuitively see that solving the affine nonlinear $MS-H_\infty$ control problem is similar to solving a linear one.

The Hamilton function of system (1) is

$$H(p, x, u, w) = p^T f(x) + g_1(x) u + g_2(x) w + z^T z - \lambda^2 w^T w.$$  

By a similar computation to the linear case, we get a generic representation of the saddle solutions of the Hamilton function (22) as

$$u^* = -(k(x)^T k(x))^2 \frac{1}{2} g_1(x)^T p + k(x)^T h(x)$$

$$+ (k(x)^T k(x))^2 v(x),$$

and

$$w^* = \frac{1}{2\lambda^2} g_2(x)^T p,$$

where $v(x) \in F_Q^m$ is arbitrary.

We also get

$$H(p, x, u^*, w) \leq H(p, x, u^*, w^*) \leq H(p, x, u, w^*)$$

for all $x, p \in \mathbb{R}^n$, and

$$H(p, x, u^*, w^*) = p^T f(x) + h(x)^T h(x) + \frac{1}{4\lambda^2} p^T g_2(x) g_2(x)^T p$$

$$- \frac{1}{2} g_1(x)^T p + k(x)^T (k(x)^T k(x))^2 \times$$

$$\left( \frac{1}{2} g_1(x)^T p + k(x)^T h(x) \right)$$

by Proposition 3 and Assumption 4. Now choose the state feedback control as the control input $u^*$ in (23), and replace $p$ by a function $p(x)$, also by Proposition 3 and 4, we get the following closed loop system

$$\dot{x} = f(x) - g_1(x) (k(x)^T k(x))^2 k(x)^T h(x)$$

$$- \frac{1}{2} g_1(x) (k(x)^T k(x))^2 g_1(x)^T p(x).$$

**Remark 3.** Note that both (26) and (27) are independent of $v(x)$, thus the Hamilton function (22) is constant with respect to its saddles, and the state and the control output are both independent of $v(x)$ by Proposition 3. Again $v(x)$ affects the control input (23). By (23) and (24), the saddle set

$$\{ (u^*, w^*) | v(x) \in F_Q^m \}$$

is a linear manifold in $F_Q^m \times F_Q^m$ for any given $p$.

We obtain the first result about the $MS-H_\infty$ control problem of the affine nonlinear system (1).

**Theorem 8.** The $MS-H_\infty$ control of system (1) is solvable under Assumption 4, if there exists a continuous function $p : \mathbb{R}^n \to \mathbb{R}^n$, such that

(1) system (27) is asymptotically stable,

(2) (26) with $p$ the function $p(x)$ is nonpositive, for some positive constant $\lambda$, for all $x \in \mathbb{R}^n$,

(3) $\int_{-T}^{T} p(x(t))^T (f(x(t)) + g_1(x(t)) u^* + g_2(x(t)) w(t)) \, dt \geq 0$, for all $w \in L_2^2[0, T)$, where $T \in (0, + \infty)$, $u^*$ is shown in (23), and $w^*$ is a solution.

Now we give an assumption to make system (27) stabilizable with $p(x)$ the control input, the linear case of which is Assumption 3. Under this Assumption, a finite $L_2$ gain can be estimated.
First we introduce some notations. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function vanishing at the origin. Substitute $p(x) = \lambda \frac{\partial V(x)}{\partial x}$ into (26), by Proposition 2, we get
\[
H \left( \lambda \frac{\partial V(x)}{\partial x}, x, u^*, w^* \right) = -\frac{1}{4} b(x) \lambda^2 + a(x) \lambda + c(x),
\]
where
\[
a(x) = \frac{\partial V(x)}{\partial x} \left( f(x) - g_1(x) (k(x)^T k(x))^\top k(x)^T h(x) \right),
\]
\[
b(x) = \frac{\partial V(x)}{\partial x} g_1(x) (k(x)^T k(x))^\top g_1(x)^T \frac{\partial V(x)}{\partial x},
\]
\[
c(x) = \frac{1}{4} \left\| g_2(x)^T \frac{\partial^2 V(x)}{\partial x^2} \right\|^2_2 + \left\| (k(x)^T k(x))^\top h(x) \right\|^2_2.
\]

**Remark 4.** Similar to the linear case, Theorem 8 and 9 still hold without Assumption 4 if $v(x)$ vanishes.

4. CONCLUSIONS

This paper investigates the $H_\infty$ control problem with multi-saddles (short for $MS - H_\infty$ control) for linear autonomous systems and affine nonlinear autonomous systems. This paper firstly uses the theory of generalized inverses of matrices over a general field to give generic multi-saddle solution representations of the Hamilton functions of the considered systems; secondly, uses the given generic multi-saddle solution representations to give some solvable conditions for the $MS - H_\infty$ control problems and construct solutions to the $MS - H_\infty$ control problems.

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