Abstract: A novel methodology is proposed for coordination of dynamical systems. The scheme is based on the sliding mode reference conditioning technique in a sort of supervisory level. The approach addresses the problem of coordinating dynamical systems with possible different dynamics (e.g., linear and nonlinear, different orders, constraints, etc.). To achieve this, the dynamics of each subsystem are hidden from the coordination mechanism. The main idea is to shape the systems local references in order to keep them coordinated. This implies considering the global goals, the systems constraints and the achievable performances as well. Sliding Mode Reference Conditioning (SMRC) is used for this purpose by means of a hierarchical supervisory structure. To show the applicability of the approach, the problem of coordinating a number of different dynamical systems with control saturations is addressed as a particular case. Coordination will be understood as actuating on the systems references to achieve some collective behavior considering the individual restrictions of each system.

Keywords: Nonlinear control; process control; sliding modes; coordination; consensus.

1. INTRODUCTION

Coordination of dynamical systems is a very up to-date topic. This issue has been treated in the framework of consensus. Thereby in this context, consensus refers to the idea of reaching agreement on some information state among a set of individual systems in order to accomplish a common goal, generally depending on the initial conditions.

Moreover, in the literature is usually assumed that all the systems involved are identical, therefore they have the same dynamics, which are generally single integrators. Recently, the consensus problem has been addressed (see Olfati-Saber et al., 2007; Ren et al., 2007, and references therein) using algebraic graph theory and properties of the Laplacian Matrix of a graph, for single integrators. And extended for a chain of integrators in He and Cao (2011).

Synchronization can also be seen as a special case of coordination, where some output signal should take the same values for all the systems. This problem has been studied from many different approaches (Pikovsky et al., 2001). Again, it is assumed that all the systems have the same dynamics. The use of sliding mode (SM) techniques has been proposed in (Almeida and Alvarez, 2006) to address the robust synchronization of nonlinear systems. Sliding mode techniques are used in combination with a master-slave configuration, which has some constraints on the relative degree of the individual systems.

In this contribution we depart from the usual assumptions in the literature, and use SM techniques to induce coordination. In particular, we do not assume that the systems which are going to be coordinated have the same dynamics. On the contrary, the approach addresses the problem of coordinating systems with possible different dynamics (e.g., linear and nonlinear, different orders, constraints, etc.). To achieve this, each subsystem dynamics is hidden from the coordination mechanism.

The idea behind our approach to the coordination problem is that in order to coordinate the systems, we can shape their local references as function of the global goals and the achievable performance of each system. Sliding mode reference conditioning (SMRC) is used for this purpose by means of a hierarchical SMRC structure with two levels. The upper level manages the relationship between the global goal and constraints, and produces a feasible global set-point that is sent to the local subsystems as the desired one. The lower level isolates the local subsystems dynamics from the group. At this level, the global feasible set-point is managed and transformed into a feasible local desired set-point for each subsystem, i.e., compatible with the local dynamics.
The technique is inspired by recent proposals from the co-authors, where they have combined reference conditioning techniques and sliding mode ideas to bound cross-coupling interactions in multi-variable linear systems (Garelli et al., 2006a,b), and for set-point seeking in nonlinear systems with state dependent constraints (Picó et al., 2009).

1.1 Background on Sliding Mode Reference Conditioning

The Sliding Mode Reference Conditioning technique is an approach that clearly differs from conventional variable structure control, in which sliding modes are established within the main control loop by fast switching the input to the plant. Here, sliding regimes are established, as a transient mode of operation, in an auxiliary loop rather than in the main loop. The purpose is to shape the reference signal \( r \) given to a controlled loop (see figure 2) in such a way that a given region in the state space, where control specifications are fulfilled, is kept invariant.

An interesting property of this technique is that the main control loop and the auxiliary reference conditioning dynamics do not interact, thereby they can be designed independently. Additionally, the main drawbacks of variable structure control (i.e. chattering and open-loop surface reaching) do not occur in this application.

Let the system

\[
\begin{align*}
\frac{dx}{dt} &= f(x) + g(x)u, \\
y &= h_1(x) \\
v &= h_2(x)
\end{align*}
\]

(1)

where \( x \in \mathbb{X} \subset \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R} \) is a control input (possibly discontinuous), \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) a vector field, and \( h_1(x) \), \( h_2(x) : \mathbb{R}^n \rightarrow \mathbb{R} \), scalar fields; all of them defined in \( \mathbb{X} \).

Variables \( y \) and \( v \) are both real valued system outputs, \( y \) being the main controlled variable, while \( v \) is a variable (e.g. a measurable state or given function of the states and/or control signal) to be bounded so as to fulfill user-specified system constraints. The bounds on \( v \) define the set:

\[
\Phi = \{ x \in \mathbb{X} \mid \phi(v^*) = v - v^* \leq 0 \}
\]

(2)

From a geometrical point of view, the goal is to find a control input \( u \) such that the region \( \Phi \) becomes invariant (i.e. trajectories originating in \( \Phi \) remain in \( \Phi \) for all times \( t \)), while \( y \) is driven as close as possible to its desired value \( r \). To ensure the invariance of \( \Phi \), the control input \( u \) must guarantee that the right hand side of the first equation in (1) points to the interior of \( \Phi \) at all points on the border surface \( \partial \Phi = \{ x \mid \phi(v^*) = 0 \} \), as shown in figure 1.

Mathematically, this condition can be expressed as:

\[
\| \nabla \phi \| \| f + gu \| \cos \theta = \nabla \phi^\top \dot{x} = \dot{\phi}(x,u) \leq 0, \forall x \in \partial \Phi
\]

(3)

which constitute in standard form the implicit invariance condition (Amann, 1990; Mareczek et al., 2002):

\[
\sup_u \phi(x,u) \leq 0, \text{ when } x \in \partial \Phi
\]

(4)

Solving equation (4) for \( u \), the explicit invariance condition for system 1 is obtained:

\[
(u - u^\phi) L_y \phi \leq 0, \forall x \in \partial \Phi \implies (5)
\]

with \( u^\phi = - \frac{L_y \phi}{L_y^\phi} \) and \( L_y \phi \) assumed to be positive.

Note that the transversality condition \( (L_y \phi \neq 0, \text{ Sira-Ramírez, 1988}) \), must hold on \( \partial \Phi \) for \( u^\phi \) to exist. If this is not the case, \( \Phi \) should be redefined accordingly.

For this purpose, the reference conditioning loop depicted in figure 2 is implemented. It is conformed by two elements: a filter \( F \) which purpose is to smooth out the conditioned signal \( r_f \), and a discontinuous decision block driving the search so as to fulfill the constraints and force the system to remain in the invariance set.

Fig. 2. SM Reference conditioning general scheme.

Notice that the block \( \Sigma \) in the figure may represent a control loop, in this case \( r_f \) is the reference, and \( x \) in equation (1) is the extended state comprising the process, controller, and filter states.

The discontinuous decision block is implemented by means of the variable structure control law:

\[
\begin{align*}
u &= \begin{cases} 
  u^+ & \text{if } \phi_{SM}(v^*) > 0 \\
  0 & \text{if } \phi_{SM}(v^*) \leq 0
\end{cases}
\end{align*}
\]

(7)

where

\[
\phi_{SM}(v^*) = v - v^* + \sum_{i=1}^{l-1} \tau_i v^{(i)}
\]

(8)
with \( l \) being the relative degree between the input \( v \) and the output \( u \), \( v^{(i)} \) the ith derivative of \( v \), and \( \tau_i \) constant gains. The filter \( \mathcal{F} \) is implemented as the first-order filter

\[
\dot{r}_f = -\alpha (r_f + u - r),
\]

with \( \alpha \) a design parameter of the filter.

In this paper, the reference conditioning technique is used in the individual systems and it is also conveniently reformulated allowing it to fit on the proposed coordination scheme. The rest of the paper is organized as follows. Next section describes the problem statement and the elements conforming the coordination structure. Section 3 describes how the switching surfaces are designed inducing coordination among the dynamical systems. Section 4 shows the results obtained in the example used as approach demonstration. Finally, in section 5 some conclusions are presented and open issues for future study are considered.

2. PROBLEM STATEMENT

Let consider \( N \) individual systems, not necessarily with the same dynamics, which will be coordinated. To this regard, each one of the individual systems is going to be considered fully autonomous, in the sense of having its own controller, thus being able to follow a bounded reference. In addition, each system has a sliding mode reference conditioning (SMRC) auxiliary loop, to ensure meeting its restrictions. This SMRC loop is designed independently of the main loop controllers and the global coordination design, in order to hide its dynamics to the supervisory global system while providing useful information about its local constraints.

Coordination will be understood acting on the systems global \( r \) and local \( r_{fi} \) references, to achieve some collective behavior considering the individual restrictions of each system. These constraints may also be different for each one. Desired collective behaviors could be, for instance:

- Maintaining the global reference direction, i.e. keeping the global reference in a given neighborhood or permitted band of a function \( \chi \) of the individual references.
- Maintaining a distance (or possible interval of distances) between its local feasible references, one to one, or between flock averages, etc.
- Achieving a generalized synchronization, as a combination of the previous options.

Note that the function \( \chi \) could be any local references combination, e.g. the average, the root mean square, maximum or minimum, etc., making this definition to be very general.

The coordination scheme is shown in figure 3. There are local inner loops in each system with some restriction \( \phi_i \) in the form of a switching function, a switching signal \( \omega_i \) and a filter \( F_i \). This one shapes the feasible conditioned reference \( r_{fi} \), obtained from the global external reference \( r \). On top of these, there is a global conditioning loop. It comprises several switching functions \( \sigma \) defining a regular sliding manifold, the global goal constraints, a switching law \( v \), and a global coordination filter. The last one synthesizes and determines the global behavior combining in some way the discontinuous actions, and hence obtaining, together with the global target \( c_g \), a smooth global reference \( r \).

2.1 Individual Systems

Each system \( i \) has a SMRC to overcome some kind of limitation. For the sake of demonstrating the proposed approach, it will be assumed that the individual systems have saturations as actuator limitations. But in the framework of SMRC, the restriction can be of any type and comprises the controller, systems states and output. In this case, the SMRC inner loop provides a feasible reference that avoids saturation of the given actuator.

![Coordination scheme](image)

Fig. 3. Coordination general scheme.

Each subsystem SMRC has a first order filter \( F_i \) (10) and a switching function (auxiliary output) \( \phi_i \) (11).

\[
F_i: \dot{r}_{fi} = -\alpha_i (r_{fi} - r + w_i) \quad (10)
\]

\[
\phi_i^\pm = u_i - u_{pi}^\pm \quad (11)
\]

Here, \( u_i \in \mathcal{U} \in \mathbb{R} \) and \( u_{pi}^\pm \) are the control actions and the plant input constraints of each subsystem (defined in Appendix A), respectively. All the controllers are biproper for exposition simplicity. Therefore they meet the SM requirement concerning relative degree (Utkin et al., 1999).

The discontinuous signal (12) is used as an input to the local filter.

\[
w_i = \begin{cases} 
    w_i^+ & \text{if } \phi_i^+ > 0 \\
    w_i^- & \text{if } \phi_i^- < 0 \\
    0 & \text{otherwise} 
\end{cases} \quad (12)
\]

This switching law generates in fact a linear band \( \Phi_i \) where the auxiliary sliding mode is off, and two limit surfaces \( \partial \Phi_i^\pm \) that will have to be active, in order to ensure that the restrictions are never violated.

\[
\partial \Phi_i^+ = \{ u \in \mathcal{U} : \phi_i^+(u) = 0 \} \\
\Phi_i = \{ u \in \mathcal{U} : \phi_i^+(u) < 0 \land \phi_i^-(u) > 0 \} \\
\partial \Phi_i^- = \{ u \in \mathcal{U} : \phi_i^-(u) = 0 \} 
\]

The reader is referred to (Garelli et al., 2006a) for a more comprehensive analysis of this reference conditioning loop.
2.2 Coordination Filter

The interaction among systems will be synthesized by the coordination filter \( F_g \), which determines, together with the restriction functions, the coordination policy and hence the global dynamics.

The coordination filter dynamics is:

\[
\dot{r} = -\alpha (r - c_g - k^T \cdot v) \tag{14}
\]

where \( v \) is the vector of global discontinuous actions, and \( k \) is a weighting vector defining the coordination policy in case of having more than one global restriction. From now on, the vector \( v \) will become a scalar \( v \), because without losing generality only one switching function will be defined.

3. SWITCHING SURFACES DESIGN

To meet the coordination specification, for example to maintain the global reference near the individual references function \( \chi \), the desired invariant set, and the corresponding switching surface must be defined.

It is interesting to remark that the reference conditioning (SMRC) will force the trajectories to remain in the invariant set bounded by the sliding manifold. Then the invariant set design, and of course, the manifold design, will carry out the coordination goals.

Let us define the invariant set

\[
S_{inv} = \{ x \in \mathbb{X} \mid \| r - \chi(r_f) \| \leq \Delta \} \tag{15}
\]

that will make the system trajectories evolve in such way that the distance between the global reference \( r \) and the function \( \chi \) of the individual references \( r_f \) is always less or equal to a predefined quantity \( \Delta \). Notice that the “natural” tendency of \( r \) is to converge toward the global target \( c_g \) (14). Thus, \( r \) is shaped away from \( c_g \) only to comply with coordination constraints.

3.1 Switching surface \( \sigma \chi \)

Consider now the following switching functions

\[
\sigma^+_\chi = r - \chi(r_f) - \Delta \tag{16}
\]

and its corresponding discontinuous action

\[
v = \begin{cases} 
  v^+ & \text{if } \sigma^+_\chi > 0 \\
  v^- & \text{if } \sigma^-_\chi < 0 \\
  0 & \text{otherwise} 
\end{cases} \tag{17}
\]

The two switching functions defined in (16), will result in a permitted region,

\[
S = \{ x \in \mathbb{X} : \sigma^+_\chi(x) < 0 \land \sigma^-_\chi(x) > 0 \} \tag{18}
\]

and two sliding surfaces as their upper an lower bounds

\[
\partial S^+_\chi = \{ x \in \mathbb{X} : \sigma^+_\chi(x) = 0 \} \tag{19}
\]

\[
\partial S^-_\chi = \{ x \in \mathbb{X} : \sigma^-_\chi(x) = 0 \}
\]

where \( x \in \mathbb{X} \in \mathbb{R}^n \) is the state vector. Then the definition of the invariant set \( S_{inv} \) will match the equation (15), being the union of the open set \( S \) and its closure, i.e. the switching manifolds \( \partial S^+_\chi \) and \( \partial S^-_\chi \) as follows,

\[
S_{inv} = S \cup \partial S^+_\chi \cup \partial S^-_\chi \tag{20}
\]

3.2 Sliding regime

Consider for the sake of notation clarity the case of having two local subsystems (\( N = 2 \)). The closed-loop system state vector representation (when the global reference conditioning is taking place) is:

\[
\dot{x} = f(x) + g(x)v + h(x) + p \tag{21}
\]

with

\[
x = \begin{bmatrix} r_{f1} \\ r_{f2} \end{bmatrix}, \tag{22}
\]

\[
f(x) = \begin{bmatrix} -\alpha_1(r_{f1} - r) \\ -\alpha_2(r_{f2} - r) \end{bmatrix} - \lambda r \tag{23}
\]

and

\[
g(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad h(x) = \begin{bmatrix} -\alpha_1 v_1 \\ -\alpha_2 v_2 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \tag{24}
\]

where \( f \) is the drift vector field, \( g \) is the control field, \( p \) acts as a perturbation and \( h \) contains the inner loop discontinuous actions, which are not considered as perturbations because these two signals are responsible for the difference between \( r \) and \( r_f \).

Now the gradient of \( \sigma \chi \) is

\[
\frac{\partial \sigma \chi}{\partial x} = \begin{bmatrix} -\frac{\partial \chi}{\partial r_{f1}} & -\frac{\partial \chi}{\partial r_{f2}} \end{bmatrix} \tag{25}
\]

Note that the only difference between \( \sigma^+_\chi \) and \( \sigma^-_\chi \) will not be reflected in their gradients, so \( \frac{\partial \sigma}{\partial x} \) can be either \( \sigma^+_\chi \) or \( \sigma^-_\chi \)'s gradient.

Then the Lie derivative of \( \sigma \chi \) along the direction of \( f \) is

\[
L_f \sigma \chi = \frac{\partial \chi}{\partial r_{f1}} \alpha_1 (r_{f1} - r) + \frac{\partial \chi}{\partial r_{f2}} \alpha_2 (r_{f2} - r) - \lambda r \tag{26}
\]

and along the direction of \( g \) is

\[
L_g \sigma \chi = -\lambda \tag{27}
\]

Now, to ensure that the set \( S \) will be invariant, the transversality condition (\( L_g \sigma \chi \neq 0 \), Sira-Ramírez, 1988) and the explicit invariance condition (6) must hold.

If the filter \( F_g \) is stable, then \( \lambda > 0 \), and then

\[
L_g \sigma \chi = -\lambda < 0 \tag{28}
\]

which makes the transversality condition hold. To ensure (6), we calculate the equivalent control and check under which conditions this will be bounded

\[
v^* = \alpha_1 \frac{\partial \chi}{\lambda \partial r_{f1}} r_{f1} + \alpha_2 \frac{\partial \chi}{\lambda \partial r_{f2}} r_{f2} - (\alpha_1 \frac{\partial \chi}{\lambda \partial r_{f1}} + \alpha_2 \frac{\partial \chi}{\lambda \partial r_{f2}} + 1) r \tag{29}
\]

Suppose fixed \( |v^+|, |v^-| < \infty \), and \( v^* > 0 \). Then if \( F_g \) is a BIBO stable filter with bounded inputs \( |v| \leq \max \{ |v^+|, |v^-| \} = v^* \) and \( c_g \) (we suppose the global target will be bounded), notice that \( |v| < K_r \). Then since \( F_i \) filters have bounded inputs \( r \) and \( v_i \), hence \( |r_{f1}| < K_{r_{f1}} \).

Regarding the bound \( \| \frac{\partial \chi}{\partial r_{f1}} \| < K_{\chi} \), which is also necessary, it depends only on the selection of the \( \chi \) function.
So, \( \exists v^* \) such
\[
|v^*| \leq \frac{1}{\lambda} \left[ \sum_{i=1}^{N} (K_{ri} K_{\chi_i}) + K_r \right] \leq K \leq v^* \tag{30}
\]
and it is possible to choose some \( v^- \leq -v^* \) and \( v^+ \geq v^* \) from the previous inequality, to ensure the invariance condition (6).

### 3.3 Example of coordination policy selection: Average as \( \chi \) function

As an example, consider the conditioned references average as function \( \chi \),
\[
\chi(r_{fi}) = \frac{1}{N} \sum_{i=1}^{N} r_{fi} \tag{31}
\]
and then
\[
\frac{\partial \chi}{\partial r_{fi}} = \frac{1}{N} \tag{32}
\]
which is bounded and ensures the invariant set \( S_{inv} \) existence, according to (30). The coordination goal resulting from this selection of \( \chi \) function will be within the following idea:

*The global reference will be shaped in such way to permit all the individual systems references to evolve sufficiently near the global reference to make the average not be apart for more than a preestablished amount \( \Delta \).*

### 4. SIMULATION EXAMPLE

The proposed sliding surface and the coordination goal are going to be demonstrated through an example. Let us consider five dynamical systems with different dynamics, controllers and constraints (see Appendix A). Each one has its own local reference conditioning scheme.

![Fig. 4. Simulation example: (a) Global target, global reference, avg. cond. reference, permitted band. (b) Discontinuous action.](image)

The global target \( c_g \) was set to a positive step at \( t = 0.5 \text{ sec} \), and a negative one at \( t = 3 \text{ sec} \). Figures 4 to 6 show the simulation results. In the upper part of Figure 4 is possible to see the global target in blue, the global reference in green, the \( \chi \) function of the local conditioned references (in this case being the average) in red, and the permitted band around the \( \chi \) function in dashed lines. As a consequence of the coordination strategy, the distance between the \( \chi \) function of the local references and the global reference, is always less than the preestablished amount \( \Delta \).

In the lower part of Figure 4 the discontinuous action \( v \) is shown. Here it is possible to see that when \( v \neq 0 \) (i.e. the global conditioning loop is active) the global reference \( r \) is on the boundary of the permitted region. When it goes back inside the permitted area, driven by the collective dynamics, the discontinuous action vanishes. The result of this is the shaping of \( r \) to meet the restrictions.

![Fig. 5. Global target and local conditioned references.](image)

In figure 5 the local references are shown. Here there are a few interesting things to keep in mind. The global dynamics is obviously slowed down, to let all the systems follow the global reference. As a natural consequence, a flocking phenomenon occurs dividing the systems set in two subsets. The local systems within each of those sets share some kind of restriction or dynamic property. The flocking is an emergent collective behavior, that arises when the coordination is mediating the systems.

![Fig. 6. Outputs of the local systems.](image)

Finally in figure 6 the systems outputs are shown. Here the transient behavior results from each individual controller and respective restriction. It is worth noticing that the slower responses are those corresponding to the systems 2 and 3, which also have slower changing references (see figure 5).
5. CONCLUSION

A novel strategy using ideas of sliding mode reference conditioning is developed to deal with the coordination of dynamical systems. The proposed methodology has an interesting potential to be expanded in order to overcome more general coordination problems. This is inherent to its definition, i.e., the coordination goals are reflected in the design of the sliding manifolds and coordination filter.

The fact that the individual systems dynamics are hidden to the coordination system, and only the necessary information about the subsystems constraints is communicated to it, makes the proposed methodology transparent and allows dealing with a broad kind of systems to be coordinated, as soon as they can be reference conditioned.

Additionally, the features of the SMRC and the SM itself are inherited by the proposal, such as robustness properties of SM control, but not the usual problems of SM like chattering and other, because the technique will be implemented as a part of a numeric algorithm in a digital environment. Moreover, stability of the overall coordinated systems is accomplished under mild assumptions on the individual systems internal loops.

Finally, the proposed methodology can be extended to a non-hierarchical architecture where the individual systems interact among them generating a network of information exchange. In this case the resultant structure shares some features with recently developed consensus algorithms with nonlinear terms.

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REFERENCES


Appendix A. INDIVIDUAL SYSTEMS DYNAMICS

The individual systems dynamics, as already said, are different from each other, with different input restrictions and controllers. Also every system has a Sliding Mode Reference Conditioning scheme as explained in 2.1, to deal with their restriction.

A.1 System 1

\[ \dot{x} = \begin{bmatrix} -10 & 0 \\ 1 & 2 \end{bmatrix} u, \quad y(x) = [1 \ 0] x \quad (A.1) \]

With an input restriction: \( u_{1,2}^\pm = \pm 10 \)

and a PI Controller with \( K_p = 15 \) and \( T_i = 0.5 \text{seg} \)

\[ PI : K_p \left( 1 + \frac{1}{sT_i} \right) \quad (A.2) \]

A.2 System 2

\[ \dot{x} = \begin{bmatrix} -15 & 0 \\ 1 & 0 \end{bmatrix} u, \quad y(x) = [1 \ 0] x \quad (A.3) \]

With an input restriction: \( u_{1,2}^\pm = \pm 6 \)

and a PI Controller with \( K_p = 10 \) and \( T_i = 0.5 \text{seg} \)

A.3 System 3

\[ \dot{x} = \begin{bmatrix} -18 & 0 \\ 1 & 0 \end{bmatrix} u, \quad y(x) = [1 \ 0] x \quad (A.4) \]

With an input restriction: \( u_{1,2}^\pm = \pm 15 \)

and a PI Controller with \( K_p = 8 \) and \( T_i = 0.5 \text{seg} \)

A.4 System 4

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} u, \quad y(x) = [5 \ 2] x \quad (A.5) \]

With an input restriction: \( u_{1,2}^\pm = \pm 8 \)

and a PI Controller with \( K_p = 5 \) and \( T_i = 0.5 \text{seg} \)

A.5 System 5

\[ \dot{x} = -4x + 10u, \quad y(x) = x \quad (A.6) \]

With an input restriction: \( u_{1,2}^\pm = \pm 6 \)

and a PI Controller with \( K_p = 5 \) and \( T_i = 0.5 \text{seg} \)