Optimal State-Feedback Design for MIMO systems subject to multiple SNR constraints

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Abstract: This paper studies optimal control problems associated to networked systems. In particular, we focus on a case where a multiple-input multiple-output (MIMO) plant is controlled over a channel subject to multiple signal-to-noise ratio (SNR) constraints. For this setup, we establish necessary and sufficient conditions for the existence of static state-feedback controllers that stabilize the plant in a mean square sense. Such conditions are given in terms of a convex optimization problem involving linear matrix inequalities (LMIs). An optimal design procedure is also presented. As an application of our results, we study a class of networked systems closed over erasure channels.

Keywords: Networked control systems, signal-to-noise ratio, optimal control, data dropouts.

1. INTRODUCTION

Networked control systems (NCSs) are control systems closed over constrained communication channels. The study of such systems has received much attention in the recent literature (see, e.g., Antsaklis and Baillieul (2007)). Most of the work has focused on data-rate constraints (Nair et al. (2007)), data dropouts (Schenato et al. (2007)), random delays (Zhang et al. (2005)), and signal-to-noise ratio (SNR) constraints (Braslavsky et al. (2007); Rojas et al. (2008); Silva et al. (2010)). In this paper, we focus on MIMO plants controlled over an arbitrary number of (possible MIMO) additive noise channels subject to SNR constraints.

As a first step in the study of SNR constrained systems, Braslavsky et al. (2007) considered the stabilization of a single-input single-output plant closed over a scalar power constrained additive noise channel. The main results in Braslavsky et al. (2007) is a closed form characterization of the minimal channel SNR compatible with mean square stability. However, those results do not provide performance guarantees. Rojas et al. (2008) and Freudenberg et al. (2010), considered performance and robustness related issues, as well as the problem of disturbance attenuation over SNR constrained additive noise channels. Further work was carried out by Silva et al. (2010), where a general LTI control architecture involving one scalar SNR constrained channel was studied. The works referred to above provide several insights, but consider situations where only one scalar channel is present. Thus the problem of control system design for MIMO plants controlled over multiple SNR constrained channels is still open.

The recent paper by Pulgar et al. (2010) addresses the problem of optimal static state-feedback control design for MIMO LTI plants controlled over two SNR constrained channels. To that end, Pulgar et al. (2010) first showed that the corresponding design problem is equivalent to the optimal design of mode-independent static state-feedback controllers for (a class of) Markov jump linear systems (MJLSs; Costa et al. (2005)). Unfortunately, the results available in MJLS theory do not provide a solution to that problem. Indeed, most results related to the optimal control of MJLSs focus on mode-dependent controllers, where it is assumed that the state of the underlying Markov chain is perfectly known at any time instant, and without delay (Costa et al. (2005); Geromel et al. (2009); Xiong and Lam (2007)). Results for the mode-independent case are presented in Shu et al. (2010) and do Val et al. (2002) present sufficient conditions for the existence of stabilizing state-feedback controllers, and provide an upper bound on the best achievable performance, when only partial information on the Markov chain state is available. By using the results in do Val et al. (2002), Pulgar et al. (2010) established an upper bound on the best achievable performance in the considered two channel SNR constrained control architecture, and also sufficient conditions for the existence of stabilizing controllers.

As a first contribution of this paper, we provide a methodology to optimally design static state-feedback controllers for MIMO systems subject to multiple SNR constraints. The approach is based upon LMIs (see, e.g. Boyd et al. (1994)), and provides both necessary and sufficient conditions for the existence of optimal controllers of the considered class. By exploiting these results, and using the equivalence unveiled by Pulgar et al. (2010), we also
Fig. 1. Networked control system closed over an SNR constrained additive white noise channel.

solve an optimal static state-feedback control problem for NCSs closed over two analog erasure channels.

The remainder of this paper is organized as follows: Section 2 describes the problem setup. Section 3 presents the first contribution of this paper. Section 4 shows an application of our results to the solution to a problem left open by Pulgar et al. (2010) which involved NCSs closed over erasure channels. Section 5 presents a numerical example, and conclusions are drawn in Section 6.

Notation: \( \mathbb{R} \) and \( \mathbb{N} \) refer to the real and natural numbers, respectively. \( \mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\} \) and \( \mathbb{R}^+ \triangleq \{x \in \mathbb{R} : 0 < x < \infty\} \). \( P \{\ast\} \) stands for the probability of \( \{\ast\} \) and \( \mathcal{E} \{\ast\} \) denotes the expectation of \( \{\ast\} \). Given a matrix \( W, W^T \) and \( W^H \) denote its transpose and conjugate transpose, respectively. \( 0_{n \times m} \) denotes the \( n \times m \) zero matrix, \( I_n \) denotes the \( n \times n \) identity matrix, and \( 0_n \triangleq 0_{n \times n} \). The notation \( \text{diag}(x_1, \cdots, x_n) \) or simply \( \text{diag}(x_i) \) refers to a block diagonal matrix with diagonal blocks given by \( x_i \). If \( x \) is a wide sense stationary (wss) (resp. asymptotically wss) process, then \( P_x \) denotes its covariance matrix (resp. stationary covariance matrix) and \( \sigma_x^2 \triangleq \text{trace}\{P_x\} \) its variance (resp. stationary variance). We say that a random variable (process) is a second order one if and only if it has finite mean and finite second order moments for all time instants \( k \in \mathbb{N}_0 \) (and also when \( k \to \infty \)). We use \( \rho \) for the forward shift operator.

2. PROBLEM SETUP

In this paper we focus on the NCS of Fig. 1, where \( G \) is a MIMO LTI system whose state \( x \) is available for measurement, \( K \) is a static state-feedback controller, \( u \) is the controller output, \( d \) is a disturbance, and \( z \) is a signal that reflects closed loop performance. The NCS of Fig. 1 also comprises a possibly MIMO additive noise channel, with input \( v \), output \( w \), and noise \( q \).

We assume that \( G \) has the state space description

\[
\begin{bmatrix}
  x(k+1) \\
  v(k) \\
  z(k)
\end{bmatrix} = \begin{bmatrix}
  A & B_w & B_d & B_z \\
  C_x & D_{uw} & D_{dv} & D_{dz} \\
  C_z & D_{uz} & D_{dz} & D_{uw}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  u(k) \\
  d(k) \\
  w(k)
\end{bmatrix},
\]

with \( k \in \mathbb{N}_0 \), and where \( x(k) \in \mathbb{R}^n, x(0) = x_0, u(k) \in \mathbb{R}^m, d(k) \in \mathbb{R}^p, w(k) \in \mathbb{R}^q, z(k) \in \mathbb{R}^r, \) and \( v(k) \in \mathbb{R}^s \).

We will work under the following assumptions:

**Assumption 1.**

\( a) \) \( x_0 \) is a second order random variable.

\( b) \) The disturbance \( d \) is a zero mean second order white noise sequence, uncorrelated with \( x_0 \), and with covariance matrix \( P_d \triangleq \Omega_d \Omega_d^H > 0 \).

\( c) \) \( D_{wv} \) has a strictly lower (or upper) triangular structure.

We note that Assumption 1\( c) \) guarantees that the system of Fig. 1 is well posed \( ^{1} \) for any choice of the controller \( K \).

The channel of Fig. 1 is formally defined next:

**Definition 1.** The channel of Fig. 1, with input \( v \) and output \( w \), is an SNR constrained additive white noise channel if and only if, \( \forall k \in \mathbb{N}_0, \forall v(k) \in \mathbb{R}^q \),

\[
w(k) = q(k) + v(k), \quad q(k) \triangleq \left[q_1(k)^T \cdots q_c(k)^T\right]^T, \quad (2)
\]

where \( q_i(k) \in \mathbb{R}^{n_i}, i \in \{1, \ldots, c\}, \ell = n_1 + \cdots + n_c, q_i \) is a zero mean white noise sequence uncorrelated with \( (x_0,d) \), and \( q_i \) is not correlated with \( q_j \) \( \forall i \neq j \). In addition, the covariance matrix of \( q_i \), i.e., \( P_{q_i} \) is a design variable that can be chosen within the class of all \( n_i \times n_i \) positive semidefinite matrices subject to the stationary SNR constraint

\[
P_{q_i} \leq \Gamma_i P_{q_i}, \quad (3)
\]

where \( P_{q_i} \) is the stationary variance of \( v_i \) and \( \Gamma_i \in \mathbb{R}^+ \) is the maximum admissible SNR for channel \( i \).

**Remark 1.** Given Definition 1, \( P_{q_i} \) is block diagonal with diagonal blocks \( P_{q_i} \), i.e., \( P_{q_i} \triangleq \text{diag}\{P_{q_i}\} \). However, each \( P_{q_i} \) may be non-diagonal.

The partition of \( q \) in Definition 1 induces a corresponding partition on \( w \) and \( v \), namely \( w \triangleq \left[w_1^T \cdots w_c^T\right]^T \) and \( v \triangleq \left[v_1^T \cdots v_c^T\right]^T \), with \( w_i(k), v_i(k) \in \mathbb{R}^{n_i} \). Also define, \( \forall i \in \{1, \ldots, c\} \),

\[
\eta_i \triangleq \left[0_{n_i \times (n_1 + \cdots + n_{i-1})} I_{n_i} \ 0_{n_i \times (n_{i+1} + \cdots + n_c)}\right]. \quad (4)
\]

The situation described above arises when, e.g., several subsystems (within a distributed architecture) exchange information over power constrained additive white noise channels and pre- and post-scaling factors are used around them (see, e.g., Braslavsky et al. (2007); Cover and Thomas (2006)). By doing so, and as discussed by Silva et al. (2010), the resulting communication links become equivalent to SNR constrained additive white noise channels. In that setup, choosing the channel scaling factors amounts to choosing the variance of the equivalent noise \( q \), and the ratio between the maximum admissible channel input power and the underlying channel noise variance corresponds to the maximum admissible SNR of the equivalent SNR constrained channel.

As foreshadowed before, we are interested in static state-feedback control laws. Our aim is to design such control law so as to minimize the stationary variance of the controlled output \( z \), subject to the stationary SNR constraints imposed by the channel (see (3)). We can thus define the problem of interest as follows:

**Problem 1.** Consider the NCS of Fig. 1, where \( G \) has the realization in (1), Assumption 1 holds, and the link between \( v \) and \( w \) is an SNR constrained additive white noise channel. Given \( \Gamma \triangleq \{\Gamma_1, \ldots, \Gamma_c\} \), find

\[\text{...}\]

\(^{1}\) In the standard sense defined in, e.g., Zhou et al. (1996)
where \( i = 1, \ldots, c \), \( \sigma^2_z \) is the stationary variance of \( z \), and 
\[ S = \{ K \in \mathbb{R}^{m \times n} : \text{the closed loop of Fig. 1 is internally stable} \}. \]

In (5), the notation \([\sigma^2_z]\) \( K \) is used to emphasize the fact that the optimal value of \( \sigma^2_z \) depends on the maximum admissible channel SNRs \( \Gamma_1, \ldots, \Gamma_c \). We note that Problem 1 can be infeasible if \( \Gamma \) is not properly chosen (see Braslavsky et al. (2007)). In that case, we set \([\sigma^2_z]\) \( K \) = \( \infty \), as usual (see, e.g., Boyd and Vandenberghe (2004)).

3. OPTIMAL DESIGN SUBJECT TO SNR CONSTRAINTS

In this section we provide a solution to Problem 1. To do so, we use results from the literature on control system design subject to upper bounds on the state covariance (see Skelton et al. (1997)).

When there is no feedback from \( x \) to \( u \), a state space representation of the system of Fig. 1 is given by
\[
\begin{align*}
    x(k+1) &= A_p x(k) + B_p u(k) + D_{pd} d(k), \quad x(0) = x_0, \quad (6a)
    z(k) &= C_p x(k) + B_u u(k) + D_{dz} d(k), \quad (6b)
\end{align*}
\]

where \( A_p \triangleq A + B_w \Delta C_v \), \( B_p \triangleq B_u + B_w \Delta D_w, \)
\( B_z \triangleq D_{uw} + D_{dz} \Delta D_w, \)
\( B_{\Psi} \triangleq C_p + D_{uw} \Delta C_v \), \( D_{\Psi} \triangleq [D_{pd} D_{dz}], \)
\( D_z \triangleq [D_{dz} D_{zu}], \)
\( D_{\Delta} \triangleq D_{dw} + D_{dz} \Delta D_w, \)
\( D_{\Delta} \triangleq (I - D_{dw})^{-1}, \) and \( d(k) \) \( q(k) \).

The closed loop system that arises when the static state-feedback control law
\[
u(k) = K x(k), \quad K \in \mathbb{R}^{m \times n}
\]

is used to control the system described by (6) can be represented as
\[
\begin{align*}
    x(k+1) &= A_d x(k) + B_d d(k), \quad x(0) = x_0, \quad (8a)
    z(k) &= C_d x(k) + D_d d(k), \quad (8b)
\end{align*}
\]

where \( A_d \triangleq A_p + B_p K, \quad B_d \triangleq D_p, \quad C_d \triangleq C_p + B_z K \) and \( D_d \triangleq D_z \).

Lemma 1. [See Lemma 6.1.2 in Skelton et al. (1997)] Consider the discrete time LTI system in (8) with \( x_0 \) and \( d \) satisfying Assumption 1a)-b), and \( q \) as in Definition 1. If a positive semidefinite matrix \( \Lambda \) is given, then the following statements are equivalent:
\[ a) \text{The system in (8) is asymptotically stable and the stationary covariance matrix of the output } z \text{ is upper bounded by } \Lambda, \text{ i.e., } \lim_{k \to \infty} E \{ z(k) z(k)^T \} < \Lambda. \]
\[ b) \text{There exists } X > 0 \text{ such that} \]
\[
\begin{align*}
    X &> A_d X A_d^T + B_d D_p B_d^T, \quad (9a)
    \Lambda &> C_d X C_d^T + D_d D_p D_d^T, \quad (9b)
\end{align*}
\]

where \( P_d \triangleq \text{diag} \{ P_d, P_q \} \). \( \square \)

Lemma 1 allows one to characterize all stabilizing LTI controllers that achieve a stationary output variance bounded from above by a given positive semidefinite matrix. With the help of this result, we are now in a position to state the main result of this section:

**Theorem 1.** Consider Problem 1. Define the following optimization problem in the matrix variables \( \Lambda, X, Z \) and \( P_q \) of appropriate dimensions:
\[
\begin{align*}
    \text{Find: } \gamma \triangleq \inf \{ \alpha : \gamma \in \mathbb{R} \}, \quad & \alpha > 0, \quad (10) \\
    \text{s.t.:} \quad & \Lambda > 0, \quad X > 0, \quad P_q > 0 \quad (11) \\
    & \left[ \begin{array}{cc}
        \Lambda - D_{dz} P_q D_{dz}^T & D_{\Psi} X + B_z Z \\
        D_{\Psi} X + B_z Z & X
    \end{array} \right] > 0, \quad (12) \\
    & X - D_{dz} P_d D_{dz}^T \quad \text{subject to} \quad P_q \geq 0, \quad (13) \\
    & \left[ \begin{array}{cc}
        \Gamma_i \gamma & \Psi \eta \Delta C_v X + \eta \gamma \Delta D_w Z \\
        \Psi \eta \Delta C_v X + \eta \gamma \Delta D_w Z & X
    \end{array} \right] > 0, \quad \forall \Gamma_i \kappa \quad \text{subject to} \quad \gamma > 0 \quad (14)
\end{align*}
\]

where \( \Psi \eta \Delta C_v X + \eta \gamma \Delta D_w Z \) is upper bounded by \( \Lambda \), \( \text{and all the matrices involved are defined in (4) and immediately after (6)}. \)

Theorem 1. There exists a static state-feedback gain \( K \in \mathcal{S} \) and state variables \( P_q, v_i \in \{ 1, \ldots, c \} \), satisfying \( 0 \leq P_q < \infty \) and \( v_i \leq \Gamma_i \kappa_q \), if and only if the LMIs in (11)-(14) are feasible.

Moreover, if \( (Z_w, X_o, P_q) \) are the corresponding optimal values of \( (Z, X, P_q) \), then the choice \( K = K_o \triangleq Z_w X_o^{-1} \) and \( \kappa_q = P_q \) guarantees that the closed loop system of Fig. 1 is internally stable, that the SNR constraints \( v_i \leq \Gamma_i \kappa_q \) are satisfied \( \forall i \in \{ 1, \ldots, c \} \), and that \( \sigma^2_z \). \( \gamma = \gamma \).
Fig. 2. NCS closed over a two-block erasure channel.

\[
\begin{align*}
\Gamma_i \eta_i P_{q_i} \eta_i^T & \geq \\
& \eta_i \Delta (C_v + D_{w,v} K) X (C_v + D_{w,v} K)^T \Delta^T \eta_i^T \\
& + \eta_i \Delta D_{d,v} P_d D_{d,v}^T \Delta^T \eta_i^T + \eta_i \Delta D_{w,v} P_w D_{w,v}^T \Delta^T \eta_i^T, \\
\end{align*}
\]

(19)

where \( \eta_i \) and \( \Delta \) have been defined in (4) and (6), respectively. Using the procedure employed to obtain (11) and (13), one can rewrite (19) as in (14). We thus complete the proof.

Theorem 1 provides a solution to Problem 1 in terms of the solution to a convex optimization problem subject to LMI constraints that, as such, can be solved by using standard numerical algorithms (Boyd et al. (1994); Grant and Boyd (2010)). A key feature of the problem at hand is that the decision variables include not only the static state-feedback controller \( K \), but also a number of unknown covariance matrices \( P_{q_i} \). Thus, the solution to Problem 1 presented in Theorem 1 required a (slight) modification of the standard procedures for writing optimal control problems in terms of LMIs.

It is worth noting that the feasibility of (11)-(14) is not only sufficient, but also necessary for the existence of a stabilizing controller \( K \) and of noise covariances \( P_{q_i} \) satisfying the channel SNR constraints. Hence, one can use (11)-(14) to numerically characterize the set of all channel SNRs \( \Gamma_i \) that allow one to stabilize, by means of a static state-feedback controller, a given MIMO plant using multiple SNR constrained channels. The above stands in contrast to the results in Pulgar et al. (2010), where only sufficient conditions for the existence of optimal controllers were presented and, when such conditions are satisfied, only bounds on the optimal performance are provided.

4. AN APPLICATION: OPTIMAL CONTROL OVER TWO UNRELIABLE CHANNELS

In this section, we use the results of Section 3 to solve an optimal control problem for NCSs closed over two unreliable channels. In particular, we focus on the setup considered by Pulgar et al. (2010), which has been reproduced in Fig. 2. In that figure, all symbols with no bars are as defined before, \( \bar{G} \) has the state space description

\[
\begin{bmatrix}
\bar{x}(k + 1) \\
v_g(k) \\
\bar{z}(k)
\end{bmatrix} =
\begin{bmatrix}
A & \tilde{B}_o & \tilde{B}_d & \tilde{B}_p \\
C_v & D_{w,v} & 0 & 0 \\
C_z & D_{w,z} & 0 & D_{w,z}
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
w(k) \\
d(k) \\
v_g(k)
\end{bmatrix},
\]

(20)

where \( \bar{x} \) is the state, \( \bar{x}(0) = \bar{x}_o \), \( \bar{z} \) is an output, and the link between \( v_g \) and \( w_g \) is given by a two-block erasure channel:

\begin{align*}
\theta(k) & \triangleq \text{diag} \left[ \theta_1(k) I_{n_1}, \theta_2(k) I_{n_2} \right], \\
\theta_1(k) & \in \{0, 1\}, \quad \ell = n_1 + n_2, \quad \theta_i \text{ is a sequence of i.i.d. Bernoulli random variables such that } \mathbb{P} \{ \theta_i(k) = 1 \} = p_i, \\
& \text{with } 0 < p_i < 1, \quad \theta_1 \text{ is independent of } \theta_2, \quad \theta_i \text{ is independent of } (\bar{x}_o, d).
\end{align*}

Remark 2. In order to directly use the results in Pulgar et al. (2010), we assume that \( \bar{G} \) is such that there exists no feedthrough between \( (d, w_g) \) and \( v_g \), and between \( d \) and \( \bar{z} \). However, it is straightforward to extend the following results to more general cases.

For the situation described above, we are interested in finding the state-feedback gain \( K \) that minimizes the stationary variance of the controlled output \( \bar{z} \): 

\begin{align*}
\text{Problem 2.} \quad & \text{Consider the NCS in Fig. 2, where } \bar{G} \text{ has the realization in (20), } \bar{x}_o \text{ is a second order random variable, the disturbance } d \text{ is as in Assumption 1b) and is uncorrelated with } \bar{x}_o, \\
& \text{and the link between } v_g \text{ and } w_g \text{ is a two-block erasure channel. Find}
\end{align*}

\[
\left[ \sigma^2 \right]_{p} \triangleq \inf_{K \in S_M} \left[ \sigma^2 \right]_{p},
\]

(23)

where \( \sigma^2 \) is the stationary variance of \( \bar{z} \), \( p \triangleq \{p_1, p_2\} \), and

\[
S_M \triangleq \{ K \in \mathbb{R}^{m \times n} : \text{the closed loop of Fig. 2 is stable in the mean square sense} \}.
\]

By exploiting MJLS theory (see do Val et al. (2002) and Costa et al. (2005)), Pulgar et al. (2010) provided a sufficient condition for the existence of \( K \in S_M \), and an upper bound on the best achievable performance \( \left[ \sigma^2 \right]_{p} \). If a solution to the problem of optimally designing mode-independent state-feedback controllers for MJLSs were available, then a characterization of \( \left[ \sigma^2 \right]_{p} \) could be readily obtained by proceeding as in Pulgar et al. (2010). However, to the best of our knowledge, that problem is still open in the MJLS literature.

We will now show how to use the results of Section 3 to solve Problem 2. To that end, we start by considering the

\[\text{2 See, e.g., Costa et al. (2005).}\]
auxiliary NCS of Figure 3, where the two-block erasure channel has been replaced by a matrix gain
\[ F \triangleq \text{diag} \{ p_1I_{n_1}, p_2I_{n_2} \}, \] (24)
and an SNR constrained additive white noise channel with \( c = 2 \) where
\[ \Gamma_1 = p_1(1-p_1)^{-1} \quad \text{and} \quad \Gamma_2 = p_2(1-p_2)^{-1}. \] (25)
Note that the LTI system inside the uppermost dashed box in Figure 3 plays the role of \( G \) in Figure 1.

For the sake of clarity, we will henceforth use \( \tilde{z}^H \) and \( \tilde{z}^L \) to refer to the signal \( \tilde{z} \) in the switched system of Figure 2, and in the LTI system of Figure 3, respectively.

**Theorem 2.** Consider Problem 2. Also consider, under the same assumptions as those of Problem 2, the NCS of Figure 3, where \( F \) is as in (24) and the link between \( v \) and \( w \) is an SNR constrained additive noise channel with \( c = 2 \) and SNRs given by (25). Then,
\[
\left[ \sigma_{\tilde{z}^M}^2 \right]_p = \inf_{\mathcal{P} \in \mathcal{S}_M, \mathcal{P} \prec \infty} \sigma_{\tilde{z}^L}^2. \] (26)

where \( \mathcal{S} \) is the set of all static gains \( K \) that make the LTI system of Fig. 3 internally stable. Moreover, the static state-feedback controller \( K \in \mathcal{S} \) that solves the right-hand side optimization problem in (26) is also the controller \( K \in \mathcal{S}_M \) that solves Problem 2.

**Proof.** Immediate from Theorem 2 and Corollaries 2 and 3 in Pulgar et al. (2010).

Theorem 2 states that solving Problem 2 is, essentially, equivalent to solving Problem 1 for a specific choice for the LTI system \( G \). Provided that the SNR constraints of Problem 1 are active at the optimum. By exploiting Theorem 2, the following characterization of the solution to Problem 2 becomes immediate:

**Corollary 1.** Consider Problem 2. Define the following optimization problem in the matrix variables \( \Lambda, X, Z \) and \( P_q \) (of appropriate dimensions):

Find: \( \gamma \triangleq \inf \text{trace} \{ A \} \)
s.t.: \( \Lambda \geq 0, X > 0, P_q \geq 0 \)
\[
\begin{bmatrix}
\Lambda - D_{uu} P_q D_u^T & C_x X + B_z Z \\
* & X
\end{bmatrix} > 0, \] (27)
\[
\begin{bmatrix}
X - B_d P_d B_d^T & - B_w P_q B_w^T & A_p X + B_p Z \\
* & * & X
\end{bmatrix} > 0, \] (28)
\[
\begin{bmatrix}
\Gamma_1 \eta_1 P_q \eta_1^T & \eta_H F C_x X + \eta_H F D_u Z \\
* & * & X
\end{bmatrix} \geq 0, \] (29)
where \( i = 1, 2, * \) corresponds to entries that can be inferred by symmetry, \( \eta_q \) is as in (4), and the remaining matrices are defined in terms of the state space description of \( G \) (see (20)) as \( A_p \triangleq A + B_w F C_x, B_p \triangleq B_w + B_w F D_u, C_p \triangleq C + D_u F C_u, \) and \( B_z \triangleq D_{uu} + D_{uu} F D_u. \) Then, if the LMI s in (27)-(31) are feasible, and the optimal values of \( (X, P_q, \eta) \), say \( (X_o, P_{q_o}, \eta_{q_o}) \), are such that, \( \forall i \in \{ 1, 2 \}, \)
\[
\Gamma_i \eta_{q_o} \eta_{q_o}^T = \eta_H F(C_x X_o + D_{uu} Z_o) X_o^{-1}(C_x X_o + D_{uu} Z_o) F^T \eta_{q_o}^T \] (30)
holds, then Problem 2 is feasible, \( [\sigma_{\tilde{z}^M}^2]_p = \gamma \), and the choice \( K = K_o \triangleq Z_o X_o^{-1} \) guarantees that the NCS of Fig. 2 is mean square stable and that \( \sigma_{\tilde{z}^M}^2 = \gamma \).

5. SIMULATION STUDY

In this section, we present an example to illustrate the results of this paper. Consider the control system of Fig. 4, where the plant \( H \) has the state space description
\[
\begin{bmatrix}
x_H (k+1) \\
z_H (k)
\end{bmatrix} =
\begin{bmatrix}
A_H & B_H \\
C_H & 0
\end{bmatrix}
\begin{bmatrix}
x_H (k) \\
w (k)
\end{bmatrix} +
\begin{bmatrix}
d (k)
\end{bmatrix},
\]
with
\[
A_H = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1.5 & 0 & 0 \\
0 & 0 & 0.8 & 0 \\
0 & 0 & 0 & 0.2
\end{bmatrix},
B_H = 
\begin{bmatrix}
1 \\
0 \\
1 \\
\sqrt{2}
\end{bmatrix},
\]
\[
C_H = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix},
\]
and where \( x_H \) is the state, \( z_H \) the output, \( d \) is a zero mean white noise disturbance with covariance matrix \( P_d = I_2, \) and \( w \) is the control input.

In Fig. 4, the controller output \( u \triangleq [u_1 \ u_2]^T \) has to be transmitted over an SNR constrained additive noise channel with \( n_1 = n_2 = 1, \) and maximum admissible SNRs \( \Gamma_1 = 1.5 \) and \( \Gamma_2 = 100. \) The control aim is to minimize the stationary variance of \( z_H. \)

The results in Pulgar et al. (2010) yield the upper bound \( [\sigma_1^2]_{\Gamma_1, \Gamma_2} = [\sigma_2^2]_{1.5,100} \leq 115.244. \) In turn, the approach proposed in this paper allows one to see that \( [\sigma_2^2]_{1.5,100} = \)

\footnote{We used CVX for Matlab (Grant and Boyd (2010)).}
The system of Fig. 4 was simulated using this optimal parameters, obtaining a measured stationary variance for $z_H$ equal to 10.9445, and the measured channel SNRs $P_{v_1}/P_{R_1} = 1.4855$ and $P_{v_2}/P_{R_2} = 99.8364$. As expected, the simulation results match our theoretical predictions.

It is worth noting that, although the results by Pulgar et al. (2010) allowed one to obtain an upper bound on the best achievable performance in the second case, this upper bound is over-conservative (about ten times the actual optimal performance).

We end this section by showing, in Fig. 5, a plot of the best achievable performance as a function of the channel SNRs $\Gamma_1$ and $\Gamma_2$.

6. CONCLUSIONS

This paper has studied the problem of optimal control system design for MIMO LTI systems closed over multiple SNR constrained channels. For this type of systems, and by focusing on the state-feedback case, we provided an LMI based convex optimization problem to fully characterize the best achievable performance. Our methodology was also applied to optimal control system design for MIMO LTI systems closed over unreliable channels.

Future work should focus on dynamic output-feedback control laws, and on an extension of Theorem 2 to the $n$-channel case.

REFERENCES


