Distributed Backstepping Control of Multiple Thrust-Propelled Vehicles on Balanced Graph

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Abstract: We propose novel distributed exponentially-converging control frameworks for flocking and centroid trajectory tracking of multiple thrust-propelled vehicles (TPVs), which consist of the under-actuated translation dynamics on E(3) with one-dimensional thrust-force input and the fully-actuated attitude kinematics on SO(3) with angular-rates inputs; and evolves on a strongly-connected, yet, balanced information graph \( G \). To address this issue of under-actuation, we utilize the backstepping technique; and, to decentralize the backstepping control over the balanced graph, we extend/generalize the (passive) decomposition of \([15]\).

Keywords: thrust-propelled vehicles, passive decomposition, backstepping, flocking, tracking, balanced graph

As pointed out in [2], many autonomous robotic vehicles share the following properties: 1) they are 6-degree-of-freedom (DOF) dynamic systems evolving on \( SE(3) = E(3) \times SO(3) \); 2) their attitude motion on \( SO(3) \) is fully-actuated; and 3) their translation motion on \( E(3) \) is under-actuated with only one-DOF actuation (i.e. thrust), whose direction is fixed w.r.t. the body-frame, thus, coupled with the vehicle’s attitude motion in \( SO(3) \). Some examples include quadrotor-type unmanned aerial vehicles (UAVs) \([5, 10, 4]\), autonomous underwater or surface-water vehicles \([6, 7]\) and wheeled mobile robots (WMRs) \([12, 13]\). Following \([10]\), we call such systems thrust-propelled vehicles (TPVs). See Fig. 1 for an example of such TPVs.

In this paper, we propose a novel distributed backstepping control framework for multiple TPVs, with the topology of communication (or sensing) among them being constrained to be a (time-invariant) balanced graph \( G \) (i.e. in-degree = out-degree), which is more general than an undirected graph. Similar to \([10]\), we particularly consider the TPVs, which consist of the translational dynamics on \( E(3) \) propelled by a one-DOF body-fixed thrust-force, and the attitude kinematics on \( SO(3) \) with the angular-rates as the control inputs. We consider these “mixed” TPVs, since: 1) such (desired) angular-rates can also be achieved for the TPVs with the angular-torque inputs, since their attitude dynamics in \( SO(3) \) is fully-actuated with some exponentially-converging angular-rate tracking controls likely attainable \([23]\); and 2) some commercially available TPVs only accept thrust-force and angular-rates, not angular-torques (e.g., Asptec Hummingbird®).

We then propose a (Cartesian) flocking control law for multiple of such (heterogeneous/nonlinear) TPVs on a balanced graph. The key challenge to achieve this is that the TPVs are under-actuated (i.e., translation dynamics on \( E(3) \) with one-DOF thrust control), preventing us from directly utilizing many consensus or flocking results derived for (simple) agents with full-actuation (e.g., \([24, 15, 25, 11]\)). To address this issue of under-actuation, we first extend/generalize the (passive) decomposition of \([15, \text{Lee and Li}]\) to explicitly incorporate the graph topology into it, with which we can not only analyze the TPVs’ flocking dynamics (similar to \([15]\)) and design the backstepping law \([22]\) for all the TPVs together; but also can show that this (combined) backstepping law is indeed decentralizable among the TPVs while respecting the topology of \( G \). We also show that: 1) under a similar condition as that in \([15, (20), \text{Th.1}]\), the TPVs exponentially converge to the desired formation shape and the common constant flocking velocity; yet, 2) unlike \([15]\), their centroid velocity is not invariant, which deviates from its initial value by a certain (exponentially-decaying) integrand term dependent on the initial condition.

Augmenting this distributed flocking control law for each TPV with a certain trajectory tracking action (scaled by its mass), we also achieve exponentially-converging backstepping centroid trajectory tracking control for the multiple TPVs, which is still distributed (i.e. respects the topology of balanced \( G \)), although it requires the (common) centroid’s desired trajectory be available to all the TPVs. A particularly interesting aspect of this distributed

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1 That is, all the TPVs converge to a certain desired formation shape, while their velocities reach a common constant value \([24]\).
backstepping centroid trajectory tracking control is that the augmented centroid tracking action does not at all perturb the original flocking control’s attempt to achieve the desired formation shape (i.e. formation-centroid decoupling); it rather further enforces (or improves the robustness of) this formation control’s exponential convergence.

There are many results available for controlling a single TPV on SE(3) [19, 2, 5, 10]. Yet, there are very rare results for controlling multiple of TPVs, particularly when the communication/sensing topology among them is not complete and directed. An exception for this is [1], which, however, utilizes absolute damping (in contrast to relative damping), more suitable for (strict) stabilization and not so for the (marginally-stable) flocking problem of this paper. The results of [1] are also limited to an undirected graph, and their stability conditions are more complicated and conservative than ours (e.g., TPVs’ acceleration ≤ g for all the three E(3) directions, making aggressive ascending or swaying impossible).

On the other hand, there are numerous results for multia-gent consensus or flocking control on a directed/balanced graph, yet, most of them are for simple agents on vector spaces (e.g. first/second-order point mass) and/or with full-action [16, 24, 15, 25, 11, 21]. Some recent results, perhaps most related to our paper, are [18, 17], where some backstepping control laws are designed for the similar problem considered in this paper. Yet, due to their vector-space embedding, the results of [18, 17] are not directly applicable to the TPVs, whose configuration manifold is SE(3), not identifiable by such a vector space.

To our knowledge, our flocking and centroid trajectory tracking results in this paper are the very first results on the distributed control of multiple under-actuated TPVs in SE(3) on a balanced graph.

The rest of the paper is organized as follows. We provide some preliminary materials in Sec. 1. Distributed backstepping flocking control for multiple under-actuated TPVs on a balanced graph is presented in Sec. 2, and distributed backstepping centroid trajectory tracking control in Sec. 3. Concluding remarks and comments on future research are given in Sec. 4.

1. PRELIMINARY

Let us consider a team of N “mixed” thrust-propelled vehicles (TPVs), each with the following translation dynamics in E(3) and the attitude kinematics in SO(3) [10]: for the i-th agent,

\[ m_i \ddot{x}_i = -\lambda_i R_i e_3 + m_i g e_3 + \delta_i \]

\[ \dot{R}_i = R_i S(w_i) \]

where \( m_i > 0 \) is the mass, \( x_i \in \mathbb{R}^3 \) is the Cartesian position w.r.t. the inertial frame with \( e_3 \) representing the down-direction and \( e_1, e_2 \) being the two other canonical directions (respecting the right-hand rule), \( \lambda_i \in \mathbb{R} \) is the thrust along the body frame, \( R_i \in SO(3) \) describes the rotation of the body frame w.r.t. the inertial frame, \( w_i \in \mathbb{R}^3 \) is the body frame’s angular rate relative to the inertial frame represented in the body frame, \( g \) is the gravitational constant, \( \delta_i \) is the disturbance (e.g. aero-dynamics effect [8]), and \( S(\cdot) : \mathbb{R}^3 \rightarrow so(3) \) is the skew-symmetric operator defined s.t. for \( a, b \in \mathbb{R}^3 \),

\[ S(a)b = a \times b. \] Here, without loss of generality, we assume \( \lambda_i \) is along the direction of the body-fixed \( e_3 \). We also assume \( \delta_i \) to be negligible. Some robustness analysis against bounded/unknown \( \delta_i \) would likely be possible though, due to the exponential convergence of our control laws (e.g., ultimate boundedness) – see the paragraph before Th. 1.

We also consider the case where the communication (or sensing) among the N TPVs is constrained as specified by a partially-connected and directed graph \( G := (V,E) \), where the node set \( V := \{v_1, v_2, \ldots, v_N\} \) and the edge set \( E \subseteq V \times V \) respectively specify the N TPVs and the (directed) information flow among them, with the self-joining edges excluded from \( E \) (i.e., \( (v_i,v_i) \notin E \)). If we assign a weight \( w_{ij} \) > 0 on each \( (v_i,v_j) \in E \), we can then define the graph Laplacian matrix \( L \in \mathbb{R}^{N \times N} \) of G s.t.

\[ L_{ij} = \begin{cases} -w_{ij} & \text{if } i \neq j \ and \ (v_i,v_j) \in E \\ 0 & \text{if } i \neq j \ and \ (v_i,v_j) \notin E \\ \sum_{k \in N_i} w_{ik} & \text{if } i = j \end{cases} \] (3)

where \( L_{ij} \) is the \( ij \)-th component of \( L \), and \( N_i := \{j | (v_i,v_j) \in E \} \) is the set of neighbors of the \( i \)-th TPV. It is then well-known that: 1) all the eigenvalues of \( L \) have non-negative real-part; 2) \( 1_N := [1,1,\ldots,1]^T \in \mathbb{R}^N \) is an eigenvector with zero eigenvalue, i.e. \( 1_LL_N = 0 \); and 3) if \( G \) is strongly connected, this zero eigenvalue is simple.

Similar to [15], in this paper, we particularly consider the case where the graph \( G \) is balanced, that is, for each node, its in-degree \( m_{in}(G) := \sum_{j \in V} L_{ij} \) is the same as its out-degree \( m_{out}(G) := -\sum_{j \in V} L_{ij}, \forall i \in 1,2,\ldots,N \). For this balanced graph \( G \), we then have the followings: 1) the column sums of \( L \) are also zero with \( \sum_{i=1}^N L = 0 \) [15]; and 2) if \( G \) is strongly-connected, \( L + L^T \) becomes a valid Laplacian (of the mirror graph of \( G \)) and has a simple eigenvalue at zero with the associated eigenvector \( 1_N \).

The main goal of this paper is to achieve the following two control objectives in E(3) for the N “mixed” TPVs (1)-(2) on the balanced graph \( G^2 \): 1) distributed flocking: \( ||x_i - x_j|| \rightarrow 0 \) with \( x_i \rightarrow \bar{v} \forall i,j \in 1,2,\ldots,N \), where \( \bar{v} \in \mathbb{R}^3 \) is a (unspecified) constant terminal flocking velocity for every TPV; and 2) distributed flocking with centroid trajectory tracking: \( ||x_i - \bar{x}|| \rightarrow 0 \) with \( (z_1(t),z_2(t)) \rightarrow (\bar{z}_1(t),\bar{z}_2(t)) \), where \( z_i \) is the centroid position as defined by \( z_i := \sum_{j=1}^N (m_j/m_i)x_j \in \mathbb{R}^3 \) with \( m_L = \sum_{i=1}^N m_i \), and \( \bar{z}_1, \bar{z}_2 \in \mathbb{R}^3 \) are the (smooth) desired centroid trajectory/velocity. We also want to achieve these two objectives in a distributed fashion, with the inter-TPV information flow constrained by the (partially-connected) balanced graph \( G \).

The key challenge to achieve these two objectives is that the TPVs (1)-(2) are under-actuated, that is, although the objectives require motion control in E(3), the TPV’s control for E(3) is only one-DOF thrust, \( \lambda_i \), whose direction is also coupled to the TPV’s attitude motion in SO(3). Due to this under-actuation, most of the consensus or flocking results (e.g. [16, 24, 15, 25, 11, 21]) become inapplicable

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2 Due to the symmetry of (1)-(2) along E(3), inter-TPV offset \( \alpha_{ij} \in \mathbb{R}^3 \) can be incorporated by replacing \( x_i \) and \( z_1^i \) with \( x_i - \alpha_i \) and \( z_1^i := z_1^i - \sum_{j=1}^N (m_j/m_i)\alpha_j \).
here. To address this challenge, in this paper, we will utilize the backstepping control [22], which is also implementable over a (given) balanced graph $G$.

2. DISTRIBUTED BACKSTEPPING FLOCKING CONTROL OF MULTIPLE TPVS

Let us first start with the distributed flocking control problem. For this, following [15], we design the desired control $\nu_i$ for the $i$-th TPV (1) s.t.

$$\lambda_i R_i e_3 = b \sum_{j \in N_i} w_{ij} e_{ij} + k \sum_{j \in N_i} w_{ij} e_{ij} + m_i ge_3 + \nu \in \mathbb{R}^3$$

where $b, k > 0$ are the gain coefficients, $w_{ij} > 0$ is the weight for the edge $(e_i, e_j) \in \mathcal{E}$, $e_{ij} := x_i - x_j$, $N_i$ is the set of neighbors of the $i$-th TPV, and $\nu$ is the control-generation error, which can be measured by using $\lambda_i R_i e_3$ and $\nu$, both known.

With this $\nu_i$ and $\eta_i$, we can then write the $i$-th TPV’s closed-loop dynamics as

$$m_i \ddot{x}_i + b \sum_{j \in N_i} w_{ij} \dot{e}_{ij} + k \sum_{j \in N_i} w_{ij} e_{ij} = -\nu \in \mathbb{R}^3$$

and, stacking them up, we can write the closed-loop dynamics of the $N$ TPVs in the “product” form:

$$M \ddot{z} + bL\dot{z} + kLz = -\nu$$

where $M := \mathcal{M} \otimes I_3 \in \mathbb{R}^{3N \times 3N}$ with $\mathcal{M} := \text{diag}[m_1, m_2, ..., m_N] \in \mathbb{R}^{N \times N}$, $\otimes$ and $I_3$ are the Kronecker product [3] and the 3 x 3 identity matrix; $L := \mathcal{L} \otimes I_3 \in \mathbb{R}^{3N \times 3N}$ with $\mathcal{L} \in \mathbb{R}^{N \times N}$ being the graph Laplacian of $G$; $z := [x_1; x_2; ...; x_N] \in \mathbb{R}^{3N}$; and $\nu \in \{
u_1; \nu_2; ..., \nu_N\} \in \mathbb{R}^{3N}$. Since $G$ is balanced, yet, not necessarily undirected, $\mathcal{L}$ and $L$ are in general not symmetric.

To decompose the product dynamics (5) into that of the centroid and that of the formation shape among the $N$ TPVs, we first extend/generalize the passive decomposition of [15] using the results of [Lee and Li, 13] s.t.:

$$z := Sx, \quad \eta := S^{-T}\nu$$

with $z := [z_1; z_2; ...; z_N]$, $\eta := [\eta_1; \eta_2; ...; \eta_N]$, and

$$S := \begin{bmatrix} I_T^N \mathcal{M} / m_L \end{bmatrix} \otimes I_3, \quad S^{-1} = [I_N \Delta_\perp] \otimes I_3 \quad (7)$$

with $1) \Omega_{\perp} \in \mathbb{R}^{(N-1) \times (N-1)}$ is defined s.t. $z_e = (\Omega_{\perp} \otimes I_3)x \in \mathbb{R}^{3(N-1)}$ can specify the formation shape among the $N$ TPVs (i.e. $z_e \to 0$ implies $x_1 - x_j \to 0$), while also satisfying $\Omega_{\perp} 1_N = 0$ and having full row rank (i.e. $N-1$); 2) with $m_L := \sum_{i=1}^N m_i$ and $I_{\mathcal{M}} / m_L = [m_1, m_2, ..., m_N] / m_L$, $z_1 = \sum_{i=1}^N (m_i / m_L) e_i \in \mathbb{R}^3$ (i.e. centroid position); 3) $\Delta_\perp \in \mathbb{R}^{\times (N-1)}$ is given by [Lee and Li, 13]

$$\Delta_\perp = M^{-1} \Omega_{\perp} (\Omega_{\perp} M^{-1} \Omega_{\perp}^T)^{-1}$$

and: 4) $\eta_1 = (I_T^N \otimes I_3) \nu = \sum_{i=1}^N \nu_i \in \mathbb{R}^3$, with $\eta_1$ and $\eta_e$ representing the portion of $\nu$ perturbing the $z_1$ and $z_e$ dynamics, respectively.

Here, in fact, $\Delta_\perp$ identifies the space orthogonal to $1_N$ w.r.t. the metric $\mathcal{M}$, as can be seen by $I^T_T \mathcal{M} \Delta_\perp = 0$ (from $\Omega_{\perp} 1_N = 0$) - see [Lee and Li, 13]. We can also show that, with $\Omega_{\perp}$ as defined above, $S$ is always full-rank (thus, invertible), since $I^T_T \mathcal{M} / m_L$ cannot be spanned by the rows of $\Omega_{\perp}$. This is because, if so, there should exist a non-zero $y \in \mathbb{R}^N$ s.t.

$$y^T \Omega_{\perp} = I^T_T \mathcal{M} / m_L$$

which, yet, is impossible, since, post-multiplying it by $1_N$, we have

$$0 = y^T \Omega_{\perp} 1_N = 1_N^T \mathcal{M} / m_L = 1.$$ 

We can also directly check $SS^{-1} = I_{3N}$ for (7) using properties of Kronecker product (e.g., $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ for compatible $A, B, C, D$ [3]).

The matrix $\Omega_{\perp}$, as specified above, can be chosen s.t., for instance,

$$\Omega_{\perp} := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & -1 \end{bmatrix}$$

similar to [15] (or any equivalent class of such [14]). If the graph $G$ is balanced and strongly-connected, we can also choose

$$\Omega_{\perp} := [L_1 l_2]$$

with $L_1 \in \mathbb{R}^{(N-1) \times (N-1)}$, $l_2 \in \mathbb{R}^{N-1}$ are the partitions of the Laplacian $L$ s.t.

$$L := \begin{bmatrix} L_1 & l_2 \\ l_1^T & l_3 \end{bmatrix}$$

with $l_3 \in \mathbb{R}^{N-1}$ and $l_4 \in \mathbb{R}$. We can then show that this $\Omega_{\perp}$ has indeed full row rank $N-1$, since, with $1^T T \bar{L} = 0$ (from $G$ being balanced), we have

$$\begin{bmatrix} I_{N-1} \ 0_{N-1} \\ I_{N-1} \ 1 \end{bmatrix} \begin{bmatrix} L_1 & l_2 \\ l_1^T & l_3 \end{bmatrix} = \begin{bmatrix} L_1 & l_2 \\ 0 & l_3 \end{bmatrix} \begin{bmatrix} 0_{N-1} \ 0 \end{bmatrix}$$

where the left most matrix is full-rank, while $\bar{L}$ has rank $N-1$ (from $G$ being strongly-connected), thus, the most right matrix (i.e., $\Lambda_{\perp}$) should have the rank $N-1$.

Using this decomposition matrix $S$, we can then decompose the product dynamics of the $N$ TPVs (5) s.t.

$$M \dot{z}_1 + bL \dot{z}_e + kLz_e = -\nu_e$$

where

$$S^{-T} M S^{-1} = \begin{bmatrix} m_L \otimes I_3 & 0 \\ 0 & M \end{bmatrix}$$

with $M := \Delta_\perp^T \mathcal{M} \Delta_{\perp} \otimes I_3 \in \mathbb{R}^{3(N-1) \times 3(N-1)}$ and $1^T_T \mathcal{M} \Delta_{\perp} = 0$ (from $\Omega_{\perp} 1_N = 0$) and

$$S^{-T} L S^{-1} = \begin{bmatrix} 0 & 0 \\ 0_{N-1} & \Delta_{\perp}^T L \Delta_{\perp} \end{bmatrix} \otimes I_3 = \text{diag}[0_3, \bar{L}]$$

with $\bar{L} := \Delta_{\perp}^T L \Delta_{\perp} \otimes I_3 \in \mathbb{R}^{3(N-1) \times 3(N-1)}$ and $1^T_L \mathcal{L} = 0$ (from $G$ being balanced).

Here, using a similar argument as in [15, Th.1], we can show that, if $G$ is strongly-connected and balanced, $L_{\text{sym}} := (L + L^T) / 2 \succ 0$ (i.e. positive-definite). This is because, from (11), we have

$$S^{-T} L_{\text{sym}} S^{-1} = \text{diag}[0_3, \bar{L}]$$

where $L_{\text{sym}} := (L + L^T) / 2 = L_{\text{sym}} \otimes I_3$, with $L_{\text{sym}} := (L + L^T) / 2$ being a Laplacian (of the strongly-connected mirror graph of $G$), which possesses a simple zero eigenvalue with all the other $N-1$ eigenvalues strictly-positive [20]; while the congruence transform in the left side preserves the signs of the eigenvalues. Therefore, all the eigenvalues of $L_{\text{sym}}$ are strictly-positive, implying $L_{\text{sym}} \succ 0$. 

8874
Then, for the \( z_e \)-dynamics (10), we can show that
\[
\frac{dV_1}{dt} = -\left( \frac{\dot{z}_e}{z_e} \right)^T \left[ bL_{sym} - \epsilon M \frac{k - eb}{2} L_{skew} \right] \left( \frac{\dot{z}_e}{z_e} \right)
\]
and
\[
V_1 := \frac{1}{2} \left( \frac{\dot{z}_e}{z_e} \right)^T \left[ \frac{M}{\epsilon M} (k + eb) L_{sym} \right] \left( \frac{\dot{z}_e}{z_e} \right)
\]
where
\[
\frac{dV_2}{dt} = -\left( \frac{\dot{z}_e}{z_e} \right)^T \left( \frac{\dot{z}_e}{z_e} \right) + \frac{1}{2} \eta e \dot{z}_e T \eta e - \frac{1}{2} \eta e \dot{z}_e + \frac{1}{2} \eta e \dot{z}_e + \frac{1}{2} \eta e \dot{z}_e + \frac{1}{2} \eta e \dot{z}_e
\]

and \( L_{skew} := (I - L^T)/2 \). Thus, if the last term of (12) is absent, and \( P > 0 \) and \( Q > 0 \), we would have \((\dot{z}_e, z_e) \rightarrow 0\), which is the basic requirement of the flocking (i.e., desired formation shape achieved).

Now, to address the last term of (12), we design a backstepping control. For this, we augment \( V_1 \) s.t.
\[
V_2 := V_1 + \frac{1}{2} \eta e T \eta e - V_1 + \frac{1}{2} \eta e T \eta e
\]
where \( \eta e := (z_e + \eta e) T \eta e \) and \( \dot{\eta e} := \dot{z}_e + \eta e \), according to the following target update law for \( \eta e \):
\[
\dot{\eta e} = (z_e + \eta e) T \eta e - \alpha \eta e
\]
which suggests the following target update law for \( \eta e \):
\[
\dot{\eta e} = (\Delta \otimes I_3) T \eta e = \gamma \dot{L} (x + ex) = -\alpha \eta e
\]
from (6). We can then transform the distributed update law (20) into \( \eta e \) s.t.
\[
\dot{\eta e} = (\Delta \otimes I_3) T \eta e = \gamma (\Delta \otimes I_3) T (x + ex) - \alpha \eta e
\]
which will match the target update law (15), if
\[
\Gamma (\dot{z}_e + \epsilon z_e) = (\Delta \otimes I_3) T (x + ex)
\]
and we use the fact that
\[
\dot{x} = (1 \otimes I_3) z_1 + (\Delta \otimes I_3) \dot{z}_1
\]
for \( \dot{x} \) and (7) with \( \dot{L} = L(\Delta \otimes I_3) \dot{z}_1 \). Since \( L(1 \otimes I_3) = (\Delta \otimes I_3)(1 \otimes I_3) = (I \otimes \Lambda_I) I_3 = 0 \) (also hold also for \( x_1 \) and \( x_2 \)). This shows that the distributed update law (20) will produce the target update law (15), if we choose \( \Gamma \) as
\[
\dot{x} = (1 \otimes I_3) z_1 + (\Delta \otimes I_3) \dot{z}_1
\]
which is uniformly bounded for a given \( \gamma > 0 \). Moreover, if the graph \( G \) is balanced and strongly-connected, following (11): 1) \( \Gamma \) is invertible, since so is \( L \); and 2) \( \Gamma^{-1} \) s.t. \( \gamma > 0 \), we have, for any \( y \in \mathbb{R}^{3(N-1)} \)
\[
y \Gamma^{-1} y = y \Gamma^{-1} y = y \Gamma^{-1} y \Gamma^{-1} y \Gamma^{-1} y \geq 0
\]
with equality hold only with \( y = 0 \), since \( \Gamma^{-1} y \). This \( \Gamma^{-1} y \) is used only for analysis, thus, not needed to be computed for our control implementation.

One last thing that remains to show is the stability of the \( z_i \)-dynamics (9). For this, from (6) that \( \eta_i = (1 \otimes I_3)^T \nu e \). With the distributed update (20), we then have
\[
\eta_i = (1 \otimes I_3) T \eta e = (1 \otimes I_3) T \eta e - \alpha \eta_i
\]
and (15) also specifies
\[
\eta_i = (1 \otimes I_3)(1 \otimes I_3) = 0 \text{ (also hold also for } x_1 \text{ and } x_2 \).
\]

Integrating (9) with \( \eta_i(t) = \eta_i(0) e^{-\alpha t} \), we can also compute the (constant) terminal centroid velocity s.t.
\[
\dot{z}_1(t) = \dot{z}_1(0) + \eta_i(0) \frac{\eta_i(0)}{\alpha_m} e^{-\alpha t} - 1 \rightarrow \dot{z}_1(0) + \frac{\eta_i(0)}{\alpha_m}
\]
which shows that: 1) if \( \eta_i(0) = 0 \), \( \dot{z}_1(t) = \dot{z}_1(0) \) (i.e. invariant centroid velocity); and 2) the larger \( \alpha_i \) and \( \alpha \) are, the closer \( \dot{z}_1(t) \) is to \( \dot{z}_1(0) \) (i.e. almost invariant \( \dot{z}_1 \)). Note here that, due to the under-actuation of TPVs (i.e. \( \eta(0) = 0 \)), in contrast to [15], \( \dot{z}_1 \) is in general unbounded even with invariant \( G \) being balanced. Also, note that, with \( \dot{z}_e \), \( \dot{z}_e \rightarrow 0 \) (i.e. \( \dot{z}_i = \dot{z}_i \rightarrow 0 \), we have \( \dot{z}_1 = \sum_{i=1}^{N} (m_i/m_l) \dot{z}_i \rightarrow \dot{x} \). This means (22) also specifies the (constant) constant terminal flocking velocity for each TPV s.t.
\[
\dot{z}_i(t) = \sum_{j=1}^{N} (m_j/m_l) \dot{z}_j(0) + \eta(0)/\alpha_m \quad \forall i = 1, 2, ..., N.
\]
The distributed backstepping flocking control, designed above, then needs to be decoded for the (real) control variables \( \lambda_1, w_i \) of each TPV. For this, using (19) with \( \nu_{ei} = \lambda_1 R_i e_3 - \nu_i \) (from (4)) and \( \dot{\nu}_{ei} = (\lambda_1 R_i e_3 + \lambda_i R_i S(w_i) e_3 + \dot{\nu}_i) \) (using (2)), we can obtain
\[
[(\dot{\lambda}_i + \alpha \lambda_i) R_i + \lambda_i R_i S(w_i)] e_3 + \dot{\nu}_i = \nu_i + \alpha \nu_i + \gamma \sum_{j \in N_i} w_{ij}(\dot{e}_{ij} + \epsilon e_{ij}) =: \dot{\nu}_i
\]
from which we can extract the control law for each TPV:
\[
[\lambda_i w_i^2 - \lambda_i w_i 1 \lambda_i + \alpha \lambda_i]^T = R_i^T \dot{\nu}_i
\]  
(23)

where \( w_i =: [w_i^1, w_i^2, w_i^3] \). To compute \( \dot{\nu}_i \) in \( \dot{\nu}_i \), we may also use
\[
\dot{\nu}_i = -b \sum_{j \in N_i} w_{ij} \left( \frac{\nu_i}{m_i} R_i - \frac{\nu_j}{m_j} R_j \right) e_3 + k \sum_{j \in N_i} w_{ij} \dot{e}_{ij}
\]
to avoid the usage of (unavailable) \( \dot{x}_i \), which can be obtained by differentiating \( \nu_i \) in (4) (with (1)).

Notice that the control law (23) is indeed distributed, since \( \dot{\nu}_i \) can be computed only by using the information from the neighboring TPVs in \( N_i \). The conditions (17)-(18) for the control gains \( b, k, \epsilon, \alpha \) can also always be achieved by choosing \( b, k \) large enough. These conditions (17)-(18) can be further relaxed, when the graph \( G \) is undirected, since, in this case, \( L \) and \( \Gamma = \gamma L \) become both symmetric, thus, all the off-block-diagonal terms in (16) vanish, and, consequently, 1) instead of the second item of the condition (17), we can choose any \( \epsilon > 0 \) s.t. \( bL - \epsilon M > 0 \); and 2) the condition (18) becomes unnecessary with any \( \alpha > 0 \) usable. Note that we also have \( \nu_{ei} \rightarrow 0 \) exponentially, since \( \eta_i \rightarrow 0 \) (with \( P, Q \geq 0 \) for (16)) and \( \eta_i \rightarrow 0 \) (see the equation before (22)), both exponentially. Since the \( z_1 \) and \( z_2 \) dynamics are both exponentially-converging, we may also obtain some robustness measures (e.g. ultimate boundedness) against bounded disturbance \( \delta_i \) and/or uncertain \( m_i \) for computing the above \( \dot{\nu}_i \). We now summarize our results in the following Th. 1.

**Theorem 1.** Consider the \( N \) TPVs (1)-(2) on a balanced and strongly-connected graph \( G \) under the distributed backstepping flocking control (23), with its gains \( b, k, \epsilon, \alpha \) chosen according to (17)-(18) (or the first item of (17) with any \( \epsilon > 0 \) and \( \alpha > 0 \), when \( G \) is undirected). Then, the flocking is achieved with:

1. \( ||\dot{x}_i - \dot{x}_j|| \rightarrow 0 \) and \( ||x_i - x_j|| \rightarrow 0 \) exponentially;
2. \( \dot{x}_i \rightarrow \sum_{k=1}^N \frac{m_k}{m_i} \dot{x}_k(0) + \frac{n_i(0)}{m_i} \) exponentially;
3. \( \nu_{ei} \rightarrow 0 \) exponentially; and
4. \( \dot{z}_1(t) = \dot{z}_1(0) + \frac{n_i(0)}{m_i} [1 - e^{-\alpha t}], \forall i, j = 1, 2, ..., N \).

**3. DISTRIBUTED FLOCKING AND CENTROID TRACKING OF MULTIPLE TPVS**

Now, suppose that we not only want to achieve the desired formation shape among the \( N \) TPVs, but also to control their centroid position \( z_1 \) to follow a certain timed trajectory \( z_1(t) \) in \( \mathbb{R}^3 \) in a distributed way as specified by the communication/sensing graph \( G \). For this, we modify the flocking control law (4) s.t.
\[
\lambda_i R_i e_3 = \nu_i + m_i(\lambda_i (\dot{x}_i - \dot{z}_1(t)) + \lambda_k (z_i - z_1(t)) - \dot{z}_1|t|) + \nu_{ei}
\]  
(24)

where \( \lambda_k, \lambda_k > 0 \) are the control gains, \( \nu_i \in \mathbb{R}^3 \) is defined in (4), \( \nu_{ei} \in \mathbb{R}^3 \) is the newly added control to achieve the centroid trajectory tracking, and \( \nu_{ei} \in \mathbb{R}^3 \) is again the control-generation error. Here, we also assume that \( z_1(t), z_2(t), z_3(t) \) are available for all the TPVs, although inter-TPV communication/sensing is constrained by the balanced and strongly-connected graph \( G \).

Given that the control \( \nu_{ei} \) in (24) is seemingly designed with no explicit consideration on how this \( \nu_{ei} \) will affect the \( z_2 \)-dynamics through the balanced graph \( G \), a rather surprising fact here is that this control \( \nu_{ei} \) does not at all perturb the convergence of \( z_2 \rightarrow 0 \); it rather “helps” the exponential convergence of \( z_2 \rightarrow 0 \) (i.e., improves its robustness by relaxing the conditions (17)-(18)).

To show this, similar to Sec. 2, stacking up the individual TPV dynamics, we first obtain the product dynamics of the \( N \) TPVs s.t.
\[
M \ddot{x} + bL \dot{x} + kL x + \nu_L = -\nu_c
\]
where \( \nu_L =: [\nu_{L1}; \nu_{L2}; \nu_{LN}] \in \mathbb{R}^{3N} \). Then, applying the passive decomposition (6) and some properties of its associated matrices, we can decompose and rewrite this product dynamics into
\[
m_L [(z_1 - \dot{z}_1) + \lambda_b (z_1 - \dot{z}_1) + \lambda_k (z_1 - \dot{z}_1)] = -\eta_1
\]  
(25)

\[
M \ddot{z}_2 + (bL + \lambda_L A) \dot{z}_2 + (kL + \lambda_k A) z_2 = -\eta_c
\]  
(26)

where \( A := (\Omega_L M^{-1} \Omega_L^T)^{-1} \otimes I_N > 0 \). Detailed derivation of (25)-(26) is omitted here due to the page limitation and will be reported in a future publication.

This then clearly shows that: 1) if \( \eta_1 \rightarrow 0 \), we will have the centroid trajectory tracking \( (z_2 - \dot{z}_2, z_1 - \dot{z}_1, \ldots) \rightarrow 0 \) with any \( \lambda_c, \lambda_k > 0 \); and 2) the centroid tracking control \( \nu_{ei} \) in (24) indeed favorably helps the exponential convergence of \( \dot{z}_2, z_2 \rightarrow 0 \) with \( A > 0 \). From A. Thus, we can also obtain the same relation (12) for (25)-(26), with only \( P, Q \) in (12) replaced by
\[
P' := P + \begin{bmatrix} 0 & 0 \\ 0 & -\lambda_k A \end{bmatrix}, \quad Q' := Q + \begin{bmatrix} \lambda_k A \\ 0 \end{bmatrix}
\]
with \( A > 0 \). This shows that the conditions (17)-(18) can still be used to choose the control gains \( b, k, \epsilon, \alpha \) (or only the first item of (17) for undirected graph \( G \), with any \( \alpha, \epsilon > 0 \), although they can be further relaxed by explicitly taking into account the presence of \( A > 0 \) in \( P' \) and \( Q' \). Moreover, since we still have (12) for (25)-(26) (with \( Q \) replaced by \( Q' \)), we can again utilize the same distributed backstepping law (19) of Sec. 2 to achieve the exponential convergence of \( z_2, \dot{z}_2, \eta_c \) and \( \eta_1 \), which will then, in turn, imply the exponential convergence of \( z_1, \dot{z}_1 \) as well with any \( \lambda_k > 0 \) and \( \lambda_k > 0 \).

Combining (19) and (24), we can also obtain a relation similar to (23) with \( \dot{\nu}_i \) modified to be
\[
\dot{\nu}_i = \dot{\nu}_i + \dot{\nu}_L + \alpha(\nu_i + \nu_{Ld}) + \gamma \sum_{j \in N_i} w_{ij}(\dot{e}_{ij} + \epsilon e_{ij})
\]  
(27)

and decode the backstepping centroid tracking control (with \( \dot{\nu}_i \) given by (27)) into the (real) control variables \( \lambda_1, w_i \) for each TPV. For this, similar to Sec. 2, we may use the following relation
\[
\dot{\nu}_{Ld} = -\lambda_b (\lambda_1 R_i e_3 - m_i e_3 + m_i \dot{z}_1(t) + m_i \lambda_k (\dot{x}_i - \dot{z}_1))
\]
to bypass the usage of \( \dot{x}_i \). We now present Th. 2, which summarizes our results so far on the distributed and
simultaneous flocking and centroid trajectory tracking for the N TPVs on the balanced graph $G$.

**Theorem 2.** Consider the N TPVs (1)-(2) on a balanced graph $G$. Under the distributed control law (23) with $\rho_i$ given by (27). Also, assume that $b, k, \epsilon, \alpha$ satisfy the conditions (17)-(18) (or only the first item of (17) with any $\epsilon, \alpha > 0$ for undirected $G$); and $\lambda_0 > 0$ and $\lambda_2 > 0$. Then, we have:

1. $(z_e, \dot{z}_e) \to 0$ exponentially;
2. $\nu_{\epsilon i} \to 0$ exponentially $\forall i = 1, 2, \ldots, N$; and
3. $(z_1 - z^i_1, \dot{z}_1 - \dot{z}^i_1) \to 0$ exponentially.

4. **SUMMARY AND FUTURE RESEARCH**

In this paper, we propose novel distributed control frameworks for TPVs on a strongly-connected and balanced graph $G$. To address the issue of under-actuation, we adopt the backstepping technique; while, to decentralize the controller over $G$, we extend/generalize the (passive) decomposition of [15]. Two control objectives are achieved: flocking and flocking plus centroid tracking, both exponentially converging. Some future research directions include: non-balanced and switching graphs; TPVs with attitude dynamics; and full robustness analysis against bounded $\delta_1$ and uncertain $m_i$.

**REFERENCES**


