On the Absolute Stability Approach to Quantized $\mathcal{H}_\infty$ Control

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Abstract: The $\mathcal{H}_\infty$ performance analysis of quantized feedback control systems is revisited in this paper. By applying some intrinsic properties of the logarithmic quantizer and the sector bound conditions, we present a new approach to $\mathcal{H}_\infty$ performance analysis of control systems with input quantization. The results are expressed in linear matrix inequalities (LMIs) and are valid for both single-input and multiple-input discrete-time linear systems with logarithmic quantizers. The key point of our method is the utilization of Tsypkin-type Lyapunov functions in the absolute stability analysis problem, rendering generally less conservative results than those in the quadratic framework, which is shown both through theoretical and numerical analysis.

Keywords: $\mathcal{H}_\infty$ Control, Quantization, Absolute Stability, Lyapunov function

1. INTRODUCTION

There is an increasingly keen research interest in quantized feedback control systems (see Delchamps (1990), Fagnani and Zampieri (2004), Liberzon (2003) and the references given there), which is mainly the result of the wide application of digital computers in control systems and the development of network based control theory (see for example, Goodwin et al. (2004), Goodwin et al. (2008), Goodwin et al. (2010), Baillieul and Antsaklis (2007), Gao et al. (2008), Liu (2010), Xiong and Lam (2007) and Zhou et al. (2011)). In such systems, the measurement and control signals cannot be directly transmitted before being quantized such that the data are only available with finite precision (see for example, Liberzon (2006), Tatikonda and Mitter (2004), Zheng and Fu (2010), You and Xie (2010) and the references therein).

Many approaches have been developed to handle problems in quantized feedback control systems (see for example, Ishii and Francis (2003), Haimovich and Seron (2008), Nair et al. (2007) and You et al. (2011)). A novel result was proposed in Elia and Mitter (2001) that the coarsest quantizer which quadratically stabilizes discrete-time single-input single-output linear systems is logarithmic. However, these results are difficult to extend to the multiple-input case (Fu and Xie (2005)). By noting that the quantization error can be treated as a sector bound uncertainty, a sector bound approach for control systems with input quantization is proposed in Fu and Xie (2005). By using the classical sector bound approach, many quantized feedback design problems can be transformed into well-known robust control problems with sector bound uncertainties. Moreover, there is no difficulty in extending the results to the multiple-input case. By recognizing that only a quadratic Lyapunov function is used in Fu and Xie (2005), a quantization-dependent Lyapunov function approach which can lead to less conservative results was proposed in Gao and Chen (2008). The main difficulty coming from this approach is that it leads to a large number of LMIs. For example, for stability analysis, $4^m$ LMIs with $2^{n+1}$ variables should be satisfied simultaneously with $m$ being the number of input channels. The quantized $\mathcal{H}_\infty$ control problem was also considered in Fu and Xie (2005) and Gao and Chen (2008). In Coutinho et al. (2010), the case of feedback control systems subject to both input and output quantization was investigated with the sector bound approach in Fu and Xie (2005). In Fu and Xie (2010), several results for different controller-quantizer configurations were also proposed for linear uncertain systems by applying the sector bound approach.

Very recently, in Zhou et al. (2010), the stabilization problem of quantized control systems was considered by using the tools in the absolute stability analysis of nonlinear systems. Based on some intrinsic properties of the logarithmic quantizer explored there, a more general Popov type Lyapunov function (see, e.g., Kapila and Haddad (1996), Park and Kim (1998) and Larsen and Kokotovic (2001)) was constructed for the quantized feedback systems. Theoretical analysis and numerical results in Zhou et al. (2010) both proved to be generally less conservative than those...
in the quadratic framework (Fu and Xie (2005), Gao and Chen (2008)). What is more, different from Gao and Chen (2008), it is known that a memoryless and infinite-level logarithmic quantizer \( f(\cdot) \) can be equivalently written as:

\[
\begin{align*}
\eta_f & = \limsup_{\delta \to 0} \frac{\#g[\varepsilon]}{-\ln \varepsilon}, \\
\eta_f & = -\frac{2}{\ln \rho}.
\end{align*}
\]

Each quantization level \( u_i \) corresponds to a certain segment of \( R \) such that the quantizer maps different segments of \( R \) to corresponding quantization levels. Following the works in Elia and Mitter (2001) and Fu and Xie (2005), it is known that a memoryless and infinite-level logarithmic quantizer \( f(\cdot) \) can be equivalently written as:

\[
\begin{align*}
\eta_f & = \limsup_{\delta \to 0} \frac{\#g[\varepsilon]}{-\ln \varepsilon}, \\
\eta_f & = -\frac{2}{\ln \rho}.
\end{align*}
\]

where \( \delta = \frac{1 - \rho}{1 + \rho} \).

The following definition is cited from Elia and Mitter (2001).

**Definition 1.**

The density of the quantizer \( f(\cdot) \) is defined as:

\[
\eta_f = \limsup_{\delta \to 0} \frac{\#g[\varepsilon]}{-\ln \varepsilon},
\]

where \( \#g[\varepsilon] \) denotes the number of quantization levels in the interval \([\varepsilon, \frac{1}{\varepsilon}]\).

**Remark 1.**

Clearly, a smaller \( \eta_f \) corresponds to a coarser quantizer. For example, a finite quantizer has \( \eta_f = 0 \) and a linear quantizer has \( \eta_f = \infty \). It is easy to show that (Elia and Mitter (2001) and Fu and Xie (2005))

\[
\eta_f = -\frac{2}{\ln \rho},
\]

which indicates that the smaller the \( \rho \), the smaller the \( \eta_f \) will be. Therefore, we will abuse the terminology by calling \( \rho \) (instead of \( \eta_f \)) the quantization density in this paper (see, for example, Fu and Xie (2005) and Gao and Chen (2008)).

In Zhou et al. (2010), we noted that \( f(\cdot) \) is bounded by a sector, namely,

\[
\frac{2 \rho}{1 + \rho} \sigma \leq f(\sigma) \leq \frac{2}{1 + \rho} \sigma.
\]

Also, we observed that \( f(\cdot) \) is nondecreasing, e.g., for two arbitrary numbers \( x \) and \( y \), we have

\[
\frac{f(x) - f(y)}{x - y} \geq 0,
\]

from which the following results can be obtained

\[
(y - x) f(x) \leq \int_x^y f(\sigma) d\sigma \leq (y - x) f(y).
\]

In Zhou et al. (2010), the following definition and lemma were introduced to present more intricate properties of the logarithmic quantizer. Less conservative results for the stabilization problem of quantized control systems were then obtained by constructing and applying a general Tsypkin-type Lyapunov function which makes use of these properties of the logarithmic quantizer.

**Definition 2.**

Let \( \varphi(x) : R \to R \) be a nonlinear function belonging to the sector \([k_1, k_2] \) and satisfy \( \varphi(x) = -\varphi(-x) \). For \( \forall x \in (0, \infty) \), let

\[
\begin{align*}
k^+(\varphi) & = \text{ess inf} \left\{ k : \int_0^x (k\sigma - \varphi(\sigma)) \, d\sigma \geq 0 \right\}, \\
k^-(\varphi) & = \text{ess sup} \left\{ k : \int_0^x (\varphi(\sigma) - k\sigma) \, d\sigma \geq 0 \right\},
\end{align*}
\]

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

We start this subsection by recalling the quantized state feedback stabilization problem for discrete-time linear system captured by the following equation:

\[
x(k + 1) = Ax(k) + Bu(k),
\]

where \( A \in R^{n \times n} \) and \( B \in R^{n \times m} \) are constant matrices, \( x(k) \) is the state vector, and \( u(k) \) is the input vector. Assume \((A, B)\) is stabilizable. The quantized state feedback problem is to design a controller in the form of

\[
\begin{align*}
u(k) &= \begin{bmatrix} f_1(v_1(k)) & \cdots & f_m(v_m(k)) \end{bmatrix} \vphantom{\begin{bmatrix} 1 \end{bmatrix}} \\
\end{align*}
\]

such that the origin of the closed-loop system is an asymptotically stable equilibrium, where \( K \in R^{m \times n} \) is feedback gain matrix and

\[
\begin{align*}
f(v(k)) &= \begin{bmatrix} f_1(v_1(k)) & \cdots & f_m(v_m(k)) \end{bmatrix} \\
v(k) &= \begin{bmatrix} v_1(k) & \cdots & v_m(k) \end{bmatrix}
\end{align*}
\]

It is assumed that the quantizer \( f(\cdot) \) is time-invariant and symmetric, i.e., \( f_i(-s) = -f_i(s), \forall s, i = 1, 2, \ldots, m \). For simplicity, throughout this subsection, we assume that \( f(\cdot) \) is a scalar logarithmic quantizer. From Elia and Mitter (2001), a quantizer is called logarithmic if the set of quantized levels is characterized by

\[
\mathcal{U} = \{ \pm u_i, u_i = \rho^i u_0, i = \pm 1, \pm 2, \ldots \} \cup \{ \pm u_0 \} \cup \{ 0 \},
\]

where \( 0 < \rho < 1 \) and \( u_0 > 0 \).
then $k^+(\varphi)$ and $k^- (\varphi)$ are, respectively, called the upper integral bound and lower integral bound of function $\varphi (x)$. In addition, $\varphi (x)$ is denoted by $\varphi (x) \in \text{Int} [k^- (\varphi), k^+ (\varphi)]$.

**Lemma 1.** Let $f(\cdot)$ be the logarithmic quantizer (4) with quantization density $\rho$, then

$$
\int_0^y f(\sigma) \, d\sigma = y f(y) - \frac{1}{2} y^2, \quad \forall y \in \mathbb{R},
$$

$$
f(\sigma) \in \text{Int} \left[ \frac{4\rho}{(1 + \rho)^2}, 1 \right].
$$

In Zhou et al. (2010), by noting that the logarithmic quantizer $f(\cdot)$ satisfies sector condition (5) and nondecreasing relation (6), we found the closed-loop quantized system (1) and (2), namely,

$$
x(k + 1) = Ax(k) + B f(Kx(k))
$$

is in the standard form of the absolute stability problem considered in Kapila and Haddad (1996).

The original system (8) was firstly augmented to construct a modified Tsypkin type criterion. Let

$$
\xi(k) = \begin{bmatrix} x(k + 1) \\ v(k) \end{bmatrix},
$$

then the closed-loop quantized system (8) can be written as

$$
\begin{cases}
\xi(k + 1) = \mathcal{A} \xi(k) + \mathcal{B} f(v(k + 1)), \\
v(k + 1) = \mathcal{K} \xi(k)
\end{cases}
$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{K}$ are, respectively, given by (Kapila and Haddad (1996))

$$
\mathcal{A} = \begin{bmatrix} A & 0 \\ K & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \mathcal{K} = [K \ 0].
$$

Then based on Lemma 1, a Tsypkin-type Lyapunov function was constructed to consider the problem of stability analysis of system (10).

### 2.2 Problem Formulation

In this paper, we will revisit the problem of $\mathcal{H}_\infty$ performance analysis of control systems with input quantization. Consider the following system:

$$
\begin{cases}
x(k + 1) = Ax(k) + Bu(k) + E \omega(k), \\
z(k) = Cx(k) + Du(k) + E \omega(k)
\end{cases}
$$

where, $x(k) \in \mathbb{R}^n$ and $z(k) \in \mathbb{R}^q$ are the state vector and controlled output, respectively; and $\omega(k) \in \mathbb{R}^q$ is the disturbance input satisfying $\omega = \{\omega(k)\} \in l_2 [0, \infty)$; $u(k) = f(Kx(k))$ is the input vector.

We are interested to know under what conditions the quantized closed-loop system (12) is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level $\gamma$, that is, $\|z\|_2 < \gamma \|\omega\|_2$ for all $\omega \in l_2 [0, \infty)$ under zero initial condition. If we set

$$
\phi(k) = \omega(k + 1), \quad \eta(k) = \begin{bmatrix} z(k + 1) \\ 0 \end{bmatrix},
$$

the above the following augmented system can be obtained:

$$
\begin{cases}
\xi(k + 1) = \mathcal{A} \xi(k) + \mathcal{B} f(v(k + 1)) + \mathcal{G} \phi(k), \\
\eta(k) = \mathcal{G} \xi(k) + \mathcal{D} f(v(k + 1)) + \mathcal{F} \phi(k)
\end{cases}
$$

where, $\xi(k)$, $\mathcal{A}$ and $\mathcal{B}$ are the same as those in (9), (11), respectively; $\mathcal{G}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ are as following:

$$
\mathcal{G} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}.
$$

If we denote the following two matrices:

$$
\mathcal{K}_a = [K \ I], \quad \mathcal{K}_b = [0 \ I],
$$

then by using (9), it is easy to show that

$$
\begin{align*}
\nu(k + 1) - \nu(k) &= \mathcal{K}_a \xi(k), \\
\nu(k) &= \mathcal{K}_b \xi(k).
\end{align*}
$$

In the sequel, we will consider the $\mathcal{H}_\infty$ control of the augmented system (13).

### 3. MAIN RESULTS

The main results of this paper are presented as follows.

**Theorem 1.** Consider system (13) where $f(\cdot)$ is given by (3) in which $f_i(\cdot), i = 1, 2, \ldots, m$ are the logarithmic quantizers defined in (4) with quantization densities $\rho_i, i = 1, 2, \ldots, m$. Then the closed-loop system (13) is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level $\gamma$ if there exists a matrix $P > 0 \in \mathbb{R}^{(n+m) \times (n+m)}$ and three diagonal matrices $G^- \geq 0 \in \mathbb{R}^{m \times m}$, $G^+ \geq 0 \in \mathbb{R}^{m \times m}$ and $S \geq 0 \in \mathbb{R}^{m \times m}$ such that

$$
Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix} < 0,
$$

is satisfied, where

$$
\begin{align*}
Q_{11} &= \mathcal{A}^T \mathcal{P} \mathcal{A} - \mathcal{K}_a^T (D_2 S D_1 + D_1 S D_2) \mathcal{K}_a + \mathcal{G}^T \mathcal{C} \\
&\quad - (\mathcal{P} + \mathcal{K}_b^T G^- \mathcal{K}_b - \mathcal{K}_a^T G^+ \mathcal{K}_a), \\
Q_{12} &= \mathcal{A}^T (\mathcal{P} \mathcal{B} - \mathcal{K}_a^T G^-) + \mathcal{K}_b^T G^+ + 2 \mathcal{K}_a^T S + \mathcal{G}^T \mathcal{F} \\
Q_{13} &= \mathcal{A}^T \mathcal{P} \mathcal{C} - \mathcal{K}_b^T \mathcal{F} \\
Q_{22} &= \mathcal{B}^T \mathcal{P} \mathcal{B} - G^- \mathcal{K}_b \mathcal{B} - \mathcal{B}^T \mathcal{K}_a^T G^- - 2 S + \mathcal{F}^T \mathcal{F} \\
Q_{23} &= \mathcal{B}^T \mathcal{P} \mathcal{C} - G^- \mathcal{K}_b \mathcal{C} + \mathcal{F}^T \mathcal{F} \\
Q_{33} &= \mathcal{B}^T \mathcal{P} \mathcal{C} - \mathcal{G}^T \mathcal{F} - \gamma^2 I
\end{align*}
$$

in which

$$
\mathcal{P} = P + \mathcal{K}_a^T G^- D^+ \mathcal{K}_a,
$$

and

$$
D^+ = I,
$$

$$
D^- = \text{diag} \left\{ \frac{4 \rho_1}{(1 + \rho_1)^2}, \ldots, \frac{4 \rho_m}{(1 + \rho_m)^2} \right\},
$$

$$
D_1 = \text{diag} \left\{ \frac{2 \rho_1}{1 + \rho_1}, \ldots, \frac{2 \rho_m}{1 + \rho_m} \right\},
$$

$$
D_2 = \text{diag} \left\{ \frac{2}{1 + \rho_1}, \ldots, \frac{2}{1 + \rho_m} \right\}.
$$

**Proof.** Firstly, we recall the Tsypkin-type Lyapunov function defined in Zhou et al. (2010). Denote the $i$-th element of the vector $v$ as $v_i$. Consider the following functions
\[
\begin{aligned}
V_1 (k, P ) &= \xi^T (k) P \xi (k), \quad P > 0 \\
V_2 (k) &= 2 \sum_{i=1}^{m} \int_{v_i (k)}^{v_i (k+1)} \lambda_i^+ (f_i (\sigma) - k_i^- \sigma) \, d\sigma \\
V_3 (k) &= 2 \sum_{i=1}^{m} \int_{v_i (k+1)}^{v_i (k)} \lambda_i^- (k_i^+ \sigma - f_i (\sigma)) \, d\sigma, \\
V_4 (k) &= 2 \sum_{i=1}^{m} (D_2 v (i) - f (v (i)))^T \\
&\times S (f (v (i)) - D_1 v (i))
\end{aligned}
\]

where \( \lambda_i^+ \geq 0, k_i^- = \frac{4\rho_i}{(1+\rho_i)^2}, \lambda_i^- \geq 0, k_i^+ = 1, S > 0 \) is a diagonal matrix, and \( D_1 \) and \( D_2 \) are defined in (20)–(21). Then according to Lemma 1, we know that

\[
V_2 (k) \geq 0, \quad V_3 (k) \geq 0, \quad V_4 (k) \geq 0.
\]

Consequently, the following function

\[
V (k, P ) = V_1 (k, P ) + V_2 (k) + V_3 (k) + V_4 (k)
\]

is a suitable Lyapunov function candidate for the nonlinear system (13).

LMIs (16) imply the stability analysis results concerning system (10) in Zhou et al. (2010), which renders system (13) to be asymptotically stable. To prove the \( \mathcal{H}_\infty \) performance, assume zero initial condition, and consider the following index:

\[
\mathcal{J} = \sum_{k=0}^{\infty} (\xi^T (k) \xi (k) - 2 \phi^T (k) \phi (k))
\]

By using (7), \( \Delta V (k, P ) \) can be continued as follows:

\[
\Delta V (k, P ) \leq \xi^T (k+1) P \xi (k+1) - \xi^T (k) P \xi (k)
\]

\[
+ 2 \sum_{i=1}^{m} \lambda_i^+ (f_i (v (k+1)) - f_i (v (k+1)))
\]

\[
- 2 \sum_{i=1}^{m} \lambda_i^- (f_i (v (k+1)) - f_i (v (k))
\]

\[
+ \sum_{i=1}^{m} \lambda_i^+ k_i^- (v_i^2 (k+1) - v_i^2 (k+1))
\]

\[
+ 2 \lambda_i^- k_i^+ (v_i^2 (k) - v_i^2 (k+1))
\]

\[
\times S (f (v (k+1)) - D_1 v (k+1)).
\]

Then by using (15), along the solution of the closed-loop system (13), we have

\[
\mathcal{J} \leq \sum_{k=0}^{\infty} (\xi^T (k) \xi (k) - 2 \phi^T (k) \phi (k) + \Delta V (k, P )]
\]

\[
\leq \sum_{k=0}^{\infty} \left[ f (v (k+1)) \right]^T Q \left[ f (v (k+1)) \right],
\]

where \( Q \) and \( \mathcal{D} \) are in the form of (16) and (17) respectively. The details are omitted here for simplicity. Therefore, if (16) is satisfied, we can conclude that \( \mathcal{J} < 0 \) for all nonzero \( \phi \in l_2 [0, \infty) \). This completes the proof. \( \blacksquare \)

If the functions \( V_2 (k) \) and \( V_3 (k) \) in (22) are, respectively, replaced by

\[
\hat{V}_2 (k) = 2 \sum_{i=1}^{m} \int_{v_i (k)}^{v_i (k+1)} \lambda_i^+ (f_i (\sigma) - \hat{k}_i^- \sigma) \, d\sigma
\]

\[
\hat{V}_3 (k) = 2 \sum_{i=1}^{m} \int_{v_i (k+1)}^{v_i (k)} \lambda_i^- (\hat{k}_i^+ \sigma - f_i (\sigma)) \, d\sigma
\]

where

\[
\lambda_i^+ \geq 0, \quad \hat{k}_i^- = \frac{2\rho_i}{1+\rho_i}, \lambda_i^- \geq 0, \quad \hat{k}_i^+ = \frac{2}{1+\rho_i},
\]

then we can obtain the following Theorem which generalizes Proposition 8 in Zhou et al. (2010). For the sake of
simplicity, the proof of this Theorem is omitted. One can refer to Proposition 8 in Zhou et al. (2010) for a similar proof procedure.

Theorem 2. Consider system (13) where \( f(\cdot) \) is given by (3) in which \( f_i(\cdot), i = 1, 2, \ldots, m \) are the logarithmic quantizers defined in (4) with quantization densities \( \rho_i, i = 1, 2, \ldots, m \). Then the closed-loop system (13) is asymptotically stable with an \( \mathcal{H}_\infty \) disturbance attenuation level \( \gamma \) if there exists a matrix \( P > 0 \in \mathbb{R}^{(n+m) \times (n+m)} \) and three diagonal matrices \( G^- \geq 0 \in \mathbb{R}^{m \times m}, G^+ \geq 0 \in \mathbb{R}^{m \times m} \) and \( S > 0 \in \mathbb{R}^{m \times m} \) such that (16) is satisfied, in which

\[
\mathcal{P} = P + \mathcal{X}^T G^- D^+ \mathcal{X} + \mathcal{X}^T R G^+ \mathcal{X}_b + \mathcal{X}^T L G^- \mathcal{X},
\]

(29)

where

\[
R = \text{diag} \left\{ \frac{2\rho_1 (1 - \rho_1)}{(1 + \rho_1)^2}, \ldots, \frac{2\rho_m (1 - \rho_m)}{(1 + \rho_m)^2} \right\} > 0,
\]

\[
L = \text{diag} \left\{ \frac{1 - \rho_1}{1 + \rho_1}, \ldots, \frac{1 - \rho_m}{1 + \rho_m} \right\} > 0.
\]

Remark 2. In view of the sector condition (5), we know

\[
\sigma(k) = V_4(k+1) - V_4(k) \geq 0.
\]

Note that if we set \( G^- = 0 \) and \( G^+ = 0 \) in Theorem 1 and Theorem 2, the four terms \( V_2(k), V_3(k), V_2(k) \) and \( V_3(k) \) in \( V(k, P) \) all vanish. In this case, we can omit \( V_4(k) \) by adding the following term

\[
\sigma(k) = 2(D_{2v}v(k+1) - f(v(k+1)))^T \times S(f(v(k+1)) - D_{1v}v(k+1))
\]

to the forward difference of \( V_1(k) \) along the solution of the closed-loop system (13). As a result, from the perspective of this observation, we have in fact used a simple quadratic Lyapunov function and the sector condition (5) as in Fu and Xie (2005). This obviously indicates that the corresponding criterion in Fu and Xie (2005) is more conservative than Theorem 1.

Remark 3. In Zhou et al. (2010), when \( G^- = 0 \) and \( G^+ = 0 \), for the quantized state feedback stabilization problem, we have presented connections between the results in Fu and Xie (2005) and those obtained from the absolute stability approach. When \( G^- = 0 \) and \( G^+ = 0 \), it is expected that similar connections exist between the quantized \( \mathcal{H}_\infty \) control results in Fu and Xie (2005) and the proposed results in Theorem 1 and Theorem 2. This is left for future work.

Remark 4. By comparing (17) and (29), we can easily conclude that Theorem 1 renders less conservative results than Theorem 2. However, the results obtained from Theorem 2 are generally less conservative results than those in Fu and Xie (2005) and Gao and Chen (2008), which can be seen from the numerical example in Section 4. For more details about the relationship between (17) and (29), one can refer to Proposition 9 of Zhou et al. (2010).

4. ILLUSTRATIVE EXAMPLE

To show the effectiveness of the proposed result, in this section we cite an example from Gao and Chen (2008) in which the system matrices of (12) are given as below:

\[
A = \begin{bmatrix}
0.8 & -0.25 & 0 & 1 \\
1 & 0 & 0 & 0 \\
-0.8 & 0.5 & 0.2 & -1.03 \\
0 & 0 & 1 & 0
\end{bmatrix},
B = \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
F = 0,
\]

\[
C = [0.2 -0.2 0 -0.2],
D = 0.5,
E = \begin{bmatrix}
0.5 \\
-0.2 \\
0.3 \\
0.1
\end{bmatrix}.
\]

One has \( \sigma(A) = (-0.3388, 0.4277 \pm 1.1389, 0.4835) \), thus this is an unstable system. In addition, it is easy to see that the pair \((A, B)\) is controllable. As in Gao and Chen (2008), we suppose the state-feedback in (8) is given by

\[
K = [0.8 -0.5 0 1]
\]

and the quantizer \( f(\cdot) \) in (12) is logarithmic as defined in (4). Following the procedure in Section 2.2, we can get the augmented system (13).

Our purpose here is to determine, for a given quantization density, the minimum values of the guaranteed \( \mathcal{H}_\infty \) performance. The results by different approaches are listed in Table 1. From the table, we can see that, for a given quantization density, the minimum guaranteed \( \mathcal{H}_\infty \) performance obtained by the absolute stability approach proposed in this paper is much less conservative than that obtained by the sector bound approach and quantization-dependent approach.

Moreover, it can be clear seen that though results obtained from Theorem 2 are a little more conservative than those from Theorem 1, they are much more conservative than those from Fu and Xie (2005) and Gao and Chen (2008).

Table 1: Minimum guaranteed \( \mathcal{H}_\infty \) disturbance attenuation levels by different methods

<table>
<thead>
<tr>
<th>Method</th>
<th>( \rho )</th>
<th>( \rho )</th>
<th>( \rho )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fu and Xie (2005)</td>
<td>2.1604</td>
<td>2.7882</td>
<td>3.8691</td>
<td>6.1798</td>
</tr>
<tr>
<td>Gao and Chen (2008)</td>
<td>1.8715</td>
<td>2.2983</td>
<td>2.9607</td>
<td>4.0978</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.4212</td>
<td>0.4582</td>
<td>0.5028</td>
<td>0.5593</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.3956</td>
<td>0.4239</td>
<td>0.4580</td>
<td>0.5069</td>
</tr>
</tbody>
</table>

5. CONCLUSION

This paper revisits the quantized \( \mathcal{H}_\infty \) control problem from the absolute stability point of view. By using some geometric properties of the logarithmic quantizer and the sector bound conditions, less conservative results are obtained through constructing Tsypkin-type Lyapunov functions. The effectiveness and merits of the proposed method are shown through a numerical example. Current work under consideration is to explore connections between the quantized \( \mathcal{H}_\infty \) control results in Fu and Xie (2005) and the proposed results in Theorem 1 and Theorem 2. Other topics worth consideration are to extend the absolute stability approach to output feedback control for linear systems with input quantization (Haddad and Kapila (1996)), and quantized feedback control of systems with uncertainties (Fu and Xie (2010), Haddad and Bernstein (1994), Haddad and Bernstein (1995) and Molchanov and Liu (2002)).
REFERENCES


