Adaptive Consensus for a Class of Uncertain Nonlinear Multi-Agent Dynamical Systems *

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Abstract: This paper is concerned with adaptive consensus in multi-agent systems where each agent is dominated by an uncertain nonlinear dynamics, and adaptive controllers are presented for networks with and without time-delays, respectively. Consensus analysis is provided with Lyapunov functions for both cases, which shows that, with certain assumptions on the communication networks, consensus will be achieved asymptotically despite of parametric uncertainty in each agent. Without assuming that dynamics of each agent are passive as is in the literature, the proposed control strategy adaptively renders the system passive. Moreover, this approach does not require linearity in the unknown parameter as assumed in our previous work. It can be obtained for undirected networks analogously, while the analysis here focuses on directed networks.

1. INTRODUCTION

Numerous large-scale networked dynamical systems exist in both nature and practical scenarios such as flocking, multi-vehicle rendezvous, which consist of many individual interacted on each other, and the whole system is expected to work well for some special missions. Comparing to single agent system for a certain mission, large-scale systems composed by multiple agents obviously have greater effectiveness and operational capability. Applications of multi-agent systems include sensor networks, formation control of autonomous robotics, UAVs networks, and so on (see Fax and Murray [2004], Kingston et al. [2005], Lawton and Beard [2003], Spanos and Murray [2005], to name just a few). For example, an algorithm has been developed for the problem of cooperative timing for a team of UAVs in Kingston et al. [2005], and a formaion maneuvers problem is solved by using some decentralized approach for groups of mobile robots in Lawton and Beard [2003].

To enable such applications, consensus is important where groups of agents are operating in a coordinated fashion. Indeed, consensus problems have received significant attention in recent years and various results have been developed under various assumptions (for example, Ren et al. [2007], Olfati-Saber et al. [2007] and the references therein). However, most existing results assume the dynamics of the agents are known linear or nonlinear dynamics. Nevertheless, it is well known that almost all the physical plants contain nonlinearity and the environment may change from time to time. Therefore, nonlinear adaptive consensus problem is of great interests and vital necessity.

On the nonlinearity of dynamics, up to our knowledge, consensus for coupled nonlinear oscillators was addressed by partial contraction theory in Wang and Slotine [2005]. Output synchronization problem of nonlinear systems was studied in Chopra and Spong [2006] for both time delay free case and time delay case by a passivity-based control method. However, agents considered in Chopra and Spong [2006] do not contain uncertainties in the dynamics and is assumed to be passive.

On the other hand, an adaptive consensus problem was addressed in Kaizuka [2010] and Kaizuka and Tsumura [2010] for linear cases. An extension of these works for a nonlinear case was considered in our previous result Sumizaki et al. [2010] for a group of agents which are governed by a class of nonlinear dynamics, with the assumption that the unknown parameter is linearly parameterized.

The objective of this paper is to study an adaptive consensus problem of a group of agents with unknown nonlinear dynamics. In particular, the unknown parameter is allowed to be nonlinearly parameterized which cannot be handled by the previous works Kaizuka [2010], Kaizuka and Tsumura [2010], and Sumizaki et al. [2010]. Moreover, the distributed adaptive controllers proposed in these papers are all based on the model reference adaptive control method and thus, they require a model which is governed by a linear dynamics. Additionally, certain assumptions on the connection between the model and agents are needed. In fact, except the information flow from the model to the agents is directed, the communication between the agents is required to be bidirected.

In this paper, adaptive controllers are presented for networks with and without time-delays, respectively, and no
precise model is needed. Consensus analysis will be provided with Lyapunov functions for both cases. It will be shown that, with certain assumptions on the communication networks, consensus will be achieved asymptotically despite of parametric uncertainty in each agent. The analysis will focus on directed networks, while it can be applied to undirected networks analogously without any difficulty. Without assuming that dynamics of each agent are passive as in the literature, the control strategy we propose in this note adaptively renders the system passive.

Moreover, it is worth mentioning that the considered approach does not require linearity in the unknown parameter as assumed in the previous work Sumizaki et al. [2010]. As will be shown later, the control strategy for each agent can be viewed as a combination of two parts, that is, an adaptive part and a consensus part. In particular, the adaptive part only utilizes its own information of each agent, which guarantees the passivity of the agent. On the other hand, the consensus part makes use of the output information of the considered agent itself and its neighbors to guarantee the consensus performance. Therefore, it is clear that if each agent is passive, we only need to use the consensus part which will be defined as an external input to guarantee the consensus, and if we only consider one single agent, the adaptive controller part will adaptively render the agent passive.

The rest of the paper is organized as follows. In Section 2, we present the problem formulation. Section 3 proposes adaptive controllers for networks with and without time-delays, respectively, and the corresponding result for unknown linear systems. A numerical example is given in Section 4 to illustrate the proposed methods and concluding remarks is followed in Section 5. All the proofs of the main results presented in Section 3 are provided in Appendix, together with some graph theory notions which are used throughout the paper.

2. PROBLEM FORMULATION

Consider a group of agents described by the following nonlinear dynamics:

\[ \dot{z}_i = f_i(z_i, \theta_i) + h_i(z_i, y_i)u_i, \]
\[ \dot{y}_i = f_i(z_i, y_i, \theta_i) + g_i(z_i, y_i, \theta_i)u_i \quad (i = 1, \ldots, N), \]

where \( z_i \in \mathbb{R}^{n_i} \), \( y_i \in \mathbb{R}^m \) are the states of the ith agent, \( u_i \in \mathbb{R}^m \) is the input, \( y_i \in \mathbb{R}^m \) is the output, and \( \theta_i \in \Sigma_i \subset \mathbb{R}^l \) denotes an unknown agent parameter vector with \( \Sigma_i \) any compact subset of \( \mathbb{R}^l \). It is assumed that all the functions in the system above are sufficiently smooth satisfying \( f_i(0, \theta_i) = 0 \) and \( f_i(0, \theta_i) = 0 \) for all \( \theta_i \in \mathbb{R}^l \).

**Remark 2.1.** Systems described by (1) are a fairly general class of uncertain nonlinear systems. Indeed, consider the general uncertain nonlinear systems described by

\[ \dot{x}_i = \phi_i(x_i, \theta_i) + \kappa_i(x_i, \theta_i)u_i, \]
\[ y_i = \psi_i(x_i, \theta_i) \quad (i = 1, \ldots, N), \]

where \( x_i \in \mathbb{R}^n \) is the state of the ith agent, \( y_i \in \mathbb{R}^m \) is the output, \( u_i \) and \( \theta_i \) are defined as in system (1). It is also assumed that all the functions in the system are sufficiently smooth satisfying \( \phi_i(0, \theta_i) = 0 \) and \( \kappa_i(0, \theta_i) = 0 \) for all \( \theta_i \in \mathbb{R}^l \). Equation (1) characterizes the normal form of the above equations describing a nonlinear system with \( n \) inputs and \( m \) outputs, and having a vector relative degree \([1, 1, \ldots, 1] \) at \( x_i = 0 \) (Isidori [1995]).

For each agent \( i \), the class of adaptive controller we consider could be described as follows:

\[ u_i(z_i, y_i, \dot{\theta}_i) = \rho^T_i(z_i, y_i, \dot{\theta}_i) + v_i(y_i, u_i) \]
\[ \dot{\theta}^T_i = -\frac{\partial \rho_i(z_i, y_i, \dot{\theta}_i)}{\partial \theta_i}, \]

where \( v_i \) is an external input which will be specified later, and \( j \in \mathcal{N}_i, \mathcal{N}_i := \{ j \in \mathcal{N} | (j, i) \in \mathcal{E} \} \) is the set of agents whose output information can be obtained by agent \( i \), \( p_i \) is some \( C^1 \) function that will be defined later, \( \Gamma_i \) some positive definite matrix.

Then the adaptive consensus problem can be stated as follows: For each agent \( i, i = 1, \ldots, N \), design an adaptive control law of form (2), such that, for all \( z_i(0) \in \mathbb{R}^{n_i-m}, y_i(0) \in \mathbb{R}^m \), \( \theta_i \in \Sigma_i \subset \mathbb{R}^l \), the output of the closed-loop system composed of (1) and (2) achieves consensus in the following sense:

\[ \lim_{t \to \infty} \| y_i(t) - y_j(t) \| = 0, \quad \forall i, j. \]

To establish the solvability of the adaptive consensus problem for system (1), let us first make the following assumption.

**Assumption 2.1.** The matrix \( g_i(z_i, y_i, \theta_i) \) in (1) is parameter independent, i.e., \( g_i(z_i, y_i, \theta_i) = g_i(z_i, y_i) \), and nonsingular for all \( z_i \in \mathbb{R}^{n_i-m} \), and \( y_i \in \mathbb{R}^m \).

Under Assumption 2.1, for the sake of convenience, assume \( g_i(z_i, y_i) \) is an identity matrix. Then system (1) can be written in the following form:

\[ \dot{z}_i = f_i(z_i, \theta_i) + h_i(z_i, y_i, \theta_i)y_i \]
\[ \dot{y}_i = f_i(z_i, y_i, \theta_i) + u_i \quad (i = 1, \ldots, N). \]

**Remark 2.2.** The corresponding linear system of system (4) that satisfies Assumption 2.1 can be described in the following form:

\[ \dot{z} = A_0 z + B_0 y \]
\[ \dot{y} = C_0 z + D_0 y + u_i \quad (i = 1, \ldots, N), \]

where \( A_0, B_0, C_0, \) and \( D_0 \) are some matrices of unknown parameter \( \theta_i \).

Therefore, our problem is to find an adaptive controller of form (2) that guarantees output consensus (3) for system (4). We further introduce the following assumption to achieve this.

**Assumption 2.2.** For each \( i (i = 1, \ldots, N) \), there exists a \( C^1 \) positive definite function \( W_i(z_i, \theta_i) \), such that, for all \( z_i \in \mathbb{R}^{n_i-m}, \theta_i \in \Sigma_i \),

\[ \frac{\partial W_i(z_i, \theta_i)}{\partial z_i} f_i(z_i, \theta_i) \leq -s_i(z_i, \theta_i), \]

where \( s_i(z_i, \theta_i) \) is some smooth positive definite function.
Remark 2.3. It can be seen that, under Assumption 2.2, the zero dynamics \( \dot{z}_i = f_0(z_i, \theta_i) \) is stable and thus is minimum phase. For linear case, i.e., system (5), Assumption 2.2 can be simplified by assuming that matrix \( A_{\theta_i} \) is Hurwitz for all \( \theta_i \in \Sigma_i \).

3. MAIN RESULT

In this section, we present our main result for the consensus problem without and with time-delays, respectively.

3.1 Adaptive Consensus without Time Delay

Consider the controller

\[
u_i(z_i, y_i, y_j, \hat{\theta}_i) = \rho_i^T(z_i, y_i, \hat{\theta}_i) + \sum_{j \in N_i} a_{ij}(y_j - y_i),
\]

\[
\dot{\hat{\theta}}_i^T = -\frac{\partial \rho_i(z_i, y_i, \hat{\theta}_i) y_i}{\partial \theta_i} \Gamma_i,
\]

where \( a_{ij} \) is the element of the adjacency matrix, \( N_i := \{ j \in N \mid (j, i) \in E \} \) the set of agents whose output information can be obtained by agent \( i \), \( \Gamma_i \) some positive definite matrix, and \( \rho_i \) is defined as

\[
\rho_i(z_i, y_i, \hat{\theta}_i) = -f_i^T(z_i, y_i, \hat{\theta}_i) - \frac{\partial W_i(z_i, \hat{\theta}_i)}{\partial z_i} h_i(z_i, y_i, \hat{\theta}_i) \tag{7}
\]

and satisfies the following convexity condition:

\[
\rho_i(z_i, y_i, \theta_1 y_i - \rho_i(z_i, y_i, \theta_2) y_i, \\
\frac{\partial \rho_i(z_i, y_i, \theta_2) y_i}{\partial \theta_i} (\theta_1 - \theta_2) \tag{9}
\]

for all \( \theta_1, \theta_2 \in \Sigma_i \).

Now we are ready to state our main result for the case where no time delay exists in the communication network.

**Theorem 3.1.** Under Assumptions 2.1-2.2, controller (7) solves the adaptive consensus problem of system (4) if the communication graph is strongly connected and balanced, and there exists a unique path between any two distinct nodes.

The proof is provided in the Appendix.

It is noted that for each agent \( i \) \((i = 1, \cdots, N)\), controller (7) can be viewed as a combination of two parts; that is, an adaptive part and an consensus part. In particular, when there is no external input \( v_i \), the controller is an adaptive controller that only utilizes its own information of agent \( i \). This part guarantees the passivity of the agent. On the other hand, the external input defined as

\[
v_i = \sum_{j \in N_i} a_{ij}(y_j - y_i), \tag{10}
\]

which uses the output information of agent \( i \) itself and its neighbors, guarantees the consensus. Therefore, it is clear that if the agent itself is passive, we only need to use the consensus controller defined as an external input above, while if we only consider one single agent, the adaptive controller with \( v_i = 0 \) will adaptively render the system passive.

3.2 Adaptive Consensus with Time Delay

In this section, we assume there are time delay in the communication topology. Assume that the time delay affects only the information that is being transmitted, and the delay is constant and bounded. We say the agents achieve consensus if

\[
\lim_{t \to \infty} \| y_i(t - \tau_{ji}) - y_j(t) \| = 0, \forall i, j \neq i \tag{11}
\]

where \( \tau_{ji} \) is the time delay for information communicated from agent \( i \) to reach agent \( j \).

Consider the following controller:

\[
u_i(z_i, y_i, y_j, \hat{\theta}_i) = \rho_i^T(z_i, y_i, \hat{\theta}_i) + \sum_{j \in N_i} a_{ij}(y_j(t - \tau_{ij}) - y_i)
\]

\[
\dot{\hat{\theta}}_i^T = -\frac{\partial \rho_i(z_i, y_i, \hat{\theta}_i) y_i}{\partial \theta_i} \Gamma_i, \tag{12}
\]

where \( a_{ij}, N_i, \Gamma_i, \) and \( \rho_i \) are defined as in controller (7), and \( \tau_{ij} \) is the time delay for information communicated from agent \( j \) to reach agent \( i \).

The main result for the case where there exists communication delays in the communication network can be stated in the following theorem.

**Theorem 3.2.** Under Assumptions 2.1-2.2, controller (12) solves the adaptive consensus problem of system (4) in the sense of (11) if the communication graph is strongly connected and balanced, and there exists a unique path between any two distinct nodes.

The proof is attached in the Appendix.

3.3 Adaptive Consensus for Unknown Linear System

This subsection presents the adaptive consensus result for the unknown linear system described by system (5).

Consider controller (7) with function \( \rho_i \) defined as

\[
\rho_i(z_i, y_i, \theta, \hat{\theta}_i) = -z_i^T C_{\theta_i} - y_i^T D_{\theta_i} - \frac{\partial W_i(z_i, \theta, \hat{\theta}_i)}{\partial z_i} B_{\hat{\theta}_i} \tag{13}
\]

and satisfies the convexity condition described by inequality (9) for all \( \theta_1, \theta_2 \in \Sigma_i \).

As stated in Remark 2.3, assume matrix \( A_{\theta_i} \) is Hurwitz for all \( \theta_i \in \Sigma_i \). Then there exists a symmetric positive definite matrix \( P_{\theta_i} \) satisfying

\[
P_{\theta_i} A_{\theta_i} + A_{\theta_i}^T P_{\theta_i} < 0 \quad (i = 1, \cdots, N). \tag{14}
\]

Letting \( W_i(z_i, \theta) = z_i^T P_{\theta_i} z_i \) implies \( \frac{\partial W_i(z_i, \theta, \hat{\theta}_i)}{\partial z_i} A_{\theta_i} < 0 \).

Then the result for the adaptive consensus problem for unknown linear system (5) without time delay existing in the communication can be stated in the following corollary.

**Corollary 3.1.** Under Assumptions 2.1, assume matrix \( A_{\theta_i} \) is Hurwitz for all \( \theta_i \in \Sigma_i \), then controller (7) with function \( \rho_i \) defined by (13) solves the adaptive consensus problem of the unknown linear system (5) if the communication graph is strongly connected and balanced.
The proof can be established by following the proof of Theorem 3.1, which is presented in the Appendix.

The result for the case where there exists time delay in the communication network can be obtained similarly and thus is omitted due to the space limit.

4. NUMERICAL EXAMPLE

Consider a networked system composed by $N$ agents with each agent described by the following differential equations:

$$
\begin{align*}
\dot{z}_{i1} &= -z_{i1} + \theta_{i1}^2 z_{i1} y_i^2 \\
\dot{z}_{i2} &= -z_{i2} + \theta_{i2} y_i \\
\dot{y}_i &= u_i, \quad (i = 1, \ldots, N).
\end{align*}
$$

(15)

It can be seen that system (15) is in the form of (1) with states $z_i = [z_{i1} \ z_{i2}]^T \in \mathbb{R}^2$, $y_i \in \mathbb{R}$, input $u_i \in \mathbb{R}$, output $y_i \in \mathbb{R}$, and unknown parameter vector $\theta_i = [\theta_{i1} \ \theta_{i2}]^T \in \mathbb{R}^2$.

Assume the agents are connected by a graph shown in Figure 1 with adjacency matrix element $a_{ij} = 1$ for all $i, j$. It can be verified that all the assumptions for solving the adaptive consensus problem of this networked system are satisfied. In particular, the zero dynamics is stable and we choose the Lyapunov function $W_i(z_i) = \frac{1}{2}(z_{i1}^2 + z_{i2}^2)$. Therefore, there exists a controller of the form (2) and (B.6) as follows:

$$
\begin{align*}
\dot{u}_i(z_i, y_i, \dot{\theta}_i) &= -\theta_{i1} z_{i1} y_i - \theta_{i2} y_i + \sum_{j \in N_i} (y_j - y_i), \\
\dot{\theta}_i^T &= \Gamma_i \left(2\theta_{i1} z_{i1} y_i^2 + z_{i2} y_i^2\right)
\end{align*}
$$

with $\Gamma_i$ some positive definite matrix.

Computer simulation is conducted to show the consensus performance. Let $\Gamma_i = I$, $i = 1, \ldots, 6$, and initial conditions be randomly given. The real value of the unknown parameters are given as $\theta_{11} = -1$ and $\theta_{12} = 2$, for $i = 1, \ldots, 6$. The consensus performance of the networked system and the profiles of parameter estimates are shown in Figure 2 and Figure 3-4, respectively. It can be seen that consensus of the group of agents is achieved despite of parametric uncertainty in each agent.

5. CONCLUSION

In this paper, we considered the adaptive consensus problem in multi-agent systems where each agent is dominated by an uncertain nonlinear dynamics. In particular, the unknown parameter is allowed to be nonlinearly parameterized, which cannot be handled by the previous works Kaizuka [2010], Kaizuka and Tsumura [2010], and Sumizaki et al. [2010]. Adaptive controllers are proposed for networks with and without time-delays, respectively. Analysis of the stability and consensus of the system are provided by using Lyapunov functions. The result is expected to extend to handle directed networks with switching topology in the future.

REFERENCES


Communication between agents can be represented by a graph.

A directed graph $G$ consists of a node set $\mathcal{N}$ and an edge set $\mathcal{E}$, where $\mathcal{N} = \{1, 2, \ldots , N\}$ is a finite nonempty set, and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of ordered pairs of nodes, called edges.

An edge $(i, j) \in \mathcal{E}$ in a directed graph denotes that node $j$ can obtain information from node $i$, and node $i$ is a neighbor of node $j$ accordingly.

A directed path is a sequence of edges in a directed graph of the form $(i_1, i_2), (i_2, i_3), \cdots$. An undirected path in an undirected graph is defined analogously.

A direct graph is strongly connected if there is a directed path from every node to every other node.

An undirect graph is connected if there is an undirected path between every pair of distinct nodes.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ of a directed graph with node set $\mathcal{N} = \{1, 2, \ldots , N\}$ is defined such that $a_{ij}$ is a positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. If the weights are not relevant, then $a_{ij} = 1$ for all $(j, i) \in \mathcal{E}$.

A graph is balanced if $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ for all $i$.

Every undirected graph is balanced.

As you seen, throughout the paper, the analysis of consensus protocols is presented for directed graph, while the result for the case where the communication between agents are undirected can be easily obtained analogously.

Appendix B. PROOF OF THEOREM 3.1

To establish the proof of Theorem 3.1, let us first introduce the following lemma.

**Lemma B.1.** Consider the controller (2) with $v_i$ an external input, $\Gamma_i$ some definite matrix, and $\rho_i$ defined in equation (8) satisfying (9).

Let $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$. Assume that Assumptions 2.1-2.2 hold, then there exists a $C^1$ function $V_i(z_i, \theta_i, \tilde{\theta}_i) \geq 0$ such that, along the trajectories of system composed of (4) and (2), for all $\theta_i \in \Sigma_i$,

$$V_i(z_i, \theta_i, \tilde{\theta}_i) \leq -s_i(z_i, \theta_i) + v_i^T y_i. \quad (B.1)$$

**Proof:**

Let

$$V_i(z_i, y_i, \tilde{\theta}_i) = W_i(z_i, \theta_i) + \frac{1}{2} \theta_i^T y_i + \frac{1}{2} \tilde{\theta}_i^T \Gamma^{-1} \tilde{\theta}_i,$$

which is $C^1$ and satisfies $V_i(z_i, y_i, \tilde{\theta}_i) \geq 0$. The time derivative of $V_i(z_i, y_i, \tilde{\theta}_i)$ along the trajectory of the system composed of (4) and (2) is given by

$$\dot{V}_i = \frac{\partial W_i(z_i, \theta_i)}{\partial z_i} (f_0(z_i, \theta_i) + h_i(z_i, y_i, \theta_i)y_i) + (f_i(z_i, y_i, \theta_i) + p_i(z_i, y_i, \tilde{\theta}_i) + v_i)^T y_i - \frac{\partial [\rho_i(z_i, y_i, \tilde{\theta}_i)y_i]}{\partial \tilde{\theta}_i}, \quad (B.2)$$

Appendix A. GRAPH THEORY NOTIONS

This section is devoted to reviewing some basic concepts and notations in graph theory (see e.g. Godsil and Royle [2001]) that is sufficient to follow the result of the paper.


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Note that from (8) we have
\[
\frac{\partial W_i(z_i, \theta_i)}{\partial z_i} h_i(z_i, y_i, \theta_i) y_i = -f_i^T(z_i, y_i, \theta_i) y_i - \rho_i(z_i, y_i, \theta_i) y_i, \quad (B.3)
\]
then using (9) with \( \theta_1 = \theta_i, \theta_2 = \hat{\theta}_i \) gives
\[
\frac{\partial [\rho_i(z_i, \theta_i, \hat{\theta}_i) y_i]}{\partial \theta_i} \leq \rho_i(z_i, y_i, \theta_i) y_i - \rho_i(z_i, y_i, \hat{\theta}_i) y_i. \quad (B.4)
\]
Therefore, using (6), (B.3) and (B.4) gives
\[
\dot{V}_i \leq -s_i(z_i, \theta_i) + v_i^T y_i. \quad (B.5)
\]
The proof is thus completed.

**Remark B.1.** Lemma B.1 is motivated by Theorems 1 and 2 in Seron et al. [1995]. It can be viewed as a variation but also a global version of the result in Seron et al. [1995]. The proof here is self-contained and we say that controller (2) adaptively renders the closed-loop system composed of (4) and (2) partially state strictly passive since function \( s_i(z_i, \theta_i) \) does not depend on state variable \( y_i \). Function \( V_i(z_i, y_i, \theta_i) \) is called the storage function.

Now we are ready to establish our proof of Theorem 3.1. To obtain controller (7), let the external input \( v_i \) in controller (2) be defined as follows:
\[
v_i = \sum_{j \in \mathcal{N}_i} a_{ij} (y_j - y_i), \quad (B.6)
\]
where \( a_{ij} \) and \( \mathcal{N}_i \) are defined as in equation (7). To prove Theorem 3.1, let
\[
V = 2 \sum_{i=1}^{N} V_i.
\]
Noting that \( a_{ij} = 0 \) if \( j \notin \mathcal{N}_i \) gives \( \sum_{j \in \mathcal{N}_i} a_{ij} = \sum_{j=1}^{N} a_{ij} \). For convenience, let \( \sum_{i=1}^{N}, \sum_{j=1}^{N} \) denote \( \sum_{i=1}^{N}, \sum_{j=1}^{N} \) respectively.

Since the graph is balanced, we have \( \sum_{j} a_{ij} = \sum_{j} a_{ji}, \forall i \), which yields
\[
\sum_{i} y_i^T \sum_{j} a_{ij} (y_j - y_i) = \sum_{i} y_i^T \sum_{j} a_{ji} (y_j - y_i). \quad (B.7)
\]
Interchanging index variables in the right hand side of equation (B.7) gives
\[
\sum_{i} y_i^T \sum_{j} a_{ji} (y_j - y_i) = \sum_{j} y_j^T \sum_{i} a_{ij} (y_j - y_i). \quad (B.8)
\]
From equations (B.7) and (B.8), we have
\[
\sum_{i} y_i^T \sum_{j} a_{ij} (y_j - y_i) = -\sum_{j} y_j^T \sum_{i} a_{ij} (y_j - y_i).
\]
Therefore, using the above equation together with (B.1) and (B.6) gives,
\[
\dot{V} \leq -2 \sum_{i} s_i(z_i, \theta_i) - \sum_{i} \sum_{j} (y_j - y_i)^T a_{ij} (y_j - y_i) \leq 0. \quad (B.9)
\]
Thus, the system is stable and all signals are bounded. Using Lasalle’s invariance principle gives \( \lim_{t \to \infty} \| y_i(t) - y_j(t) \| = 0, \forall j \in \mathcal{N}_i (i = 1, \cdots, N) \). Furthermore, since the communication graph is strongly connected, we can conclude that \( \lim_{t \to \infty} \| y_i(t) - y_j(t) \| = 0, \forall i, j \). The proof is thus completed.

**Appendix C. PROOF OF THEOREM 3.2**

To obtain controller (12), let the external input \( v_i \) in controller (2) be defined as follows:
\[
v_i = \sum_{j \in \mathcal{N}_i} a_{ij} (y_j(t - \tau_{ij}) - y_i), \quad (C.1)
\]
where \( a_{ij} \) and \( \mathcal{N}_i \) are defined as in equation (7). To prove Theorem 3.2, let
\[
V = \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij} \int_{t-\tau_{ij}}^{t} y_j^T(\sigma) y_j(\sigma) d\sigma + 2 \sum_{i=1}^{N} V_i. \quad (C.2)
\]
Using (B.1) and (C.1) gives,
\[
\dot{V} \leq \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij} (y_j^T(y_j - y_i) - y_j(t - \tau_{ij})^T y_j(t - \tau_{ij})) - 2 \sum_{i=1}^{N} s_i(z_i, \theta_i) + 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} y_j^T a_{ij} (y_j(t - \tau_{ij}) - y_i).
\]
For convenience, starting from here, we use the short notation \( \sum_{i} \) and \( \sum_{j} \) again. As the graph is balanced, we have \( \sum_{j} a_{ij} = \sum_{j} a_{ji}, \forall i \), which yields
\[
\sum_{i} \sum_{j} a_{ij} y_j^T y_j = \sum_{j} \sum_{i} a_{ij} y_j^T y_i = \sum_{i} \sum_{j} a_{ij} y_i^T y_j. \quad (C.3)
\]
Therefore,
\[
\dot{V} \leq -2 \sum_{i} s_i(z_i, \theta_i) - \sum_{j} \sum_{i} a_{ij} (y_j(t - \tau_{ij}) - y_i)^T (y_j(t - \tau_{ij}) - y_i) \leq 0, \quad (C.4)
\]
which shows that \( V(t) \) has a finite limit as \( t \to \infty \). By assumption that all the functions are sufficiently smooth, \( \dot{V} \) exists and is continuous. On the other hand, the ness of \( \dot{V} \) and inequality (C.4) imply that \( z_i, y_i, \theta_i \) are all bounded for all \( t \geq 0 \). Using the definition of \( \hat{\theta}_i \) shows \( \hat{\theta}_i \) is also bounded. Thus, \( u_i, v_i \), as well as \( \dot{z}_i, \dot{y}_i, \dot{\theta}_i (i = 1, \cdots, N) \), are bounded. Therefore, all the signals as well as their derivatives of the closed-loop system composed of (4) and (12) are bounded, which implies that \( \dot{V} \) is bounded. As a result, \( \ddot{V} \) is uniformly continuous. By Barbalat’s lemma, \( \dot{V} \) approaches zero as \( t \to \infty \), which implies that
\[
\lim_{t \to \infty} \| y_j(t - \tau_{ij}) - y_i(t) \| = 0, \forall j \in \mathcal{N}_i (i = 1, \cdots, N).
\]
Finally, since the communication graph is strongly connected, we can conclude that consensus is achieved in the sense of (11). The proof is thus completed.