Abstract: This paper is concerned with control problems of multiple mobile robots such as wheeled mobile robots defined on $\mathbb{SE}(2)$ or free-flying robots defined on $\mathbb{SO}(3)$. If each robot was supposed to be controlled independently, the total number of control inputs would be the number of individual inputs times the number of robots. In this paper, we show that it is possible to reduce the number of control inputs where the whole system remains controllable by imposing artificial kinematic constraints between the robots, e.g., two wheeled mobile robots can be controllable with only 2 or 3 inputs. We also discuss what happens if we try to make the multiple robots track the reference trajectories.

Keywords: nonholonomic system, nonlinear system, mobile robot, controllability

Control problems of nonholonomic systems have been attracting a lot of interests of nonlinear control theorists since early 90’s. A typical platform of nonholonomic system is the kinematic model of mechanical systems with nonholonomic (non-integrable) constraints, described by

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i, \quad x \in M, u \in \mathbb{R}^m$$

where the state space $M$ is an $n$-dimensional smooth manifold. This class of nonholonomic systems are called driftless systems. For driftless systems, each state can be equilibrium by setting $u = 0$. Brockett (1983) showed that no equilibrium can be asymptotically stabilized by any continuous state feedback if $m < n$. Nevertheless, if the system is controllable in nonlinear sense (Nijmeijer and van der Schaft (1990)), it is still possible to reach any desirable equilibrium from any initial state. This fact encouraged us to develop advanced nonlinear controller such as non-smooth or non-static feedback law, and provided challenging issues to nonlinear control theory.

In this paper, we suppose to control $\ell$ copies of driftless systems, each of which is controllable with $n$ states and $m$ inputs. If we consider the whole system as an $\ell n$-dimensional system with $\ell m$ inputs, the control problem is trivial since the system is controllable with no doubt. In contrast, we show that the whole system can be controllable with control inputs less than $\ell m$, by adding appropriate kinematic constraints between the robots.

This paper is organized as follows. Section 1 gives a brief summary of notations on Lie group formalism and left-invariant systems. In section 2, we deal with control problem of multiple planar robots on $\mathbb{SE}(2)$. In section 3, we deal with control problem of multiple spatial robots on $\mathbb{SO}(3)$. In section 4, we discuss some physical interpretation of the obtained results and their potential applications. Section 5 concludes the paper.

1. PRELIMINARIES

1.1 Lie groups

Let $G$ be a Lie group, namely, $G$ is both a group and a smooth manifold where its group operation $\ast$ is smooth in terms of manifold mappings. For any $g \in G$, its left action $L_g$ is defined as a mapping $L_g : G \rightarrow G : h \mapsto g \ast h$. $L_g$ is a diffeomorphism of manifolds as well as an endomorphism of group. Moreover, $L : g \mapsto L_g$ is a homomorphism from the group $G$ to the set of endomorphisms on $G$.

Let $T_hG$ denote the tangent space of $G$ at $h \in G$, and $X(G)$ denote the set of all vector-fields defined on $G$. For a map $\phi : G \rightarrow G : h \mapsto \phi(h)$, the derivative of $\phi$ at $h$ is denoted by $d_h\phi : T_hG \rightarrow T_{\phi(h)}G$. If $X$ is $L_g$-invariant for all $g \in G$, i.e.,

$$(d_hL_g)(X_h) = X(L_g(h)), \quad \forall h \in G,$$

then $X$ is called left-invariant vector-field. The set of all left-invariant vector-fields on $G$ forms an Lie algebra $\mathfrak{g}$, which is called the Lie algebra of $G$. Operation of the Lie algebra $\mathfrak{g}$ (which satisfies multi-linearity, skew-symmetry and Jacobi identity) is consistent with the standard Lie bracket of vector-fields

$$[X_1(g), X_2(g)] = \frac{\partial X_2}{\partial g} X_1(g) - \frac{\partial X_1}{\partial g} X_2(g).$$

Also $\mathfrak{g}$ is isomorphic to the linear space $T_eG$; for $\xi \in T_eG$, there is a one-to-one correspondence to a left-invariant vector-field $X(g)$, i.e.,

$$\xi = X(e), \quad X(g) = d_eL_g(\xi).$$

We identify both of them throughout the paper.

1.2 Left-invariant systems on Lie groups

Let us consider the following control system defined on a Lie group $G$:
\[ \dot{g} = \sum_{i=1}^{m} u_i X_i(g) = (d_e L_g) \sum_{i=1}^{m} u_i \xi_i \quad (2) \]

where \( u \in \mathbb{R}^m \) is the control inputs, \( X_1, \ldots, X_m \) are left-invariant vector-fields on \( G \). Let \( n := \dim(G) \). This is a special form of \( m \)-input \( n \)-state driftless system. The smallest Lie subalgebra generated by \( X_1, \ldots, X_m \in \mathfrak{g} \) closed under \( \mathfrak{g} \) bracketting is called controllability Lie algebra. This is identified with the involutive closure generated by the vector-fields \( X_1(g), \ldots, X_m(g) \). Despite that its instantaneous mobility is restricted in the \( m \)-dimensional subspace of \( T_g G \), there exists a control input \( u(t) \) which delivers \( g(0) \) to arbitrary target \( g_2 \in G \) if the system is controllable; the system is controllable if and only if the controllability Lie algebra coincides with \( \mathfrak{g} \) itself.

### 1.3 Planar Wheeled Mobile Robot

Fig. 1 shows a standard wheeled mobile robot on the plane. \((x, y)\) denotes the position of the robot and \( \theta \) denotes its orientation relative to the \( x \)-axis. We assume that each wheel does not slide sideways. Then the kinematic model of the wheeled mobile robot is given by the following driftless state equation:

\[ \dot{g} = X_1(g)u_1 + X_2(g)u_2 \quad (3) \]

where the state of the mobile robot is \( g = (x, y, \theta) \), which is an element of a Lie group \( \text{SE}(2) \). \( u_1 \) is the forwarding velocity and \( u_2 \) is the steering angular velocity. As mentioned above, we identify the element \((1, 0, 0)^T \in \mathfrak{se}(2)\) with the corresponding left-invariant vector-field \((\cos \theta, \sin \theta, 0)^T\), both denoted by \( i \).

The first-order Lie bracket generated by \( X_1, X_2 \)

\[ [X_1, X_2] = \mathbf{k}, \quad \mathbf{k} := [j, i] = (-\sin \theta, \cos \theta, 0)^T \quad (4) \]

implies the sideways motion of the robot, which is prohibited at each instant. Thus the system is controllable

\[ \text{span}\{X_1, X_2, \mathbf{k}\} = \mathfrak{se}(2) \quad (5) \]

i.e., \( i, j, k \) forms a basis of the Lie algebra \( \mathfrak{se}(2) \). Note that

\[ [j, k] = -i, \quad [i, k] = 0, \quad [i, j] = [j, i] = [k, k] = 0. \]

This system is called a first-order controllable system (Murray et al. (1994)), for the controllability Lie algebra is spanned by up to first-order Lie brackets.

![Fig. 1. Single wheeled mobile robot on \( \text{SE}(2) \)](image)

### 2. CONTROL OF MULTIPLE PLANAR MOBILE ROBOTS

Let us consider two copies of planar wheeled mobile robots. The state vector is

\[ g := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \text{SE}(2) \times \text{SE}(2). \]

where Lie group structure of \( \text{SE}(2) \times \text{SE}(2) \) is naturally extended to the direct product \( \text{SE}(2) \times \text{SE}(2) \).

If we suppose to use \( 2 \times 2 = 4 \) control inputs, i.e., to control each robots independently, state equation of the whole system is given by

\[ \dot{g} = X_1(g)u_1 + X_2(g)u_2 + X_3(g)u_3 + X_4(g)u_4 \quad (6) \]

\[ X_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} j \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ i \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 \\ j \end{pmatrix} \quad (7) \]

where \( u_1, u_2 \) are control inputs for the first robot, while \( u_3, u_4 \) are those for the second robot (see Fig. 2). It is obvious to see that \( X_1, X_2, X_3, X_4 \) are left-invariant vector-fields on \( \text{SE}(2) \times \text{SE}(2) \). The system is almost trivially controllable in this case, for its controllability Lie algebra is

\[ \text{span}\{X_1, X_2, X_3, X_4\} = \mathfrak{se}(2) \times \mathfrak{se}(2) \]

which implies it is also a first-order controllable system.

![Fig. 2. Control of 2 robots on \( \text{SE}(2) \) with 4 inputs](image)

In the rest of this section, we suggest two types of techniques to control multiple robots using less control inputs.

#### 2.1 Common generator assignment

**2 robots case.** Suppose the forwarding velocity of the both robots is supplied by a common input \( u_1 \), while the steering angular velocity is given independently by \( u_2 \) and \( u_3 \) (Fig. 3). Then the corresponding state equation is given by

\[ \dot{g} = X_1(g)u_1 + X_2(g)u_1 + X_3(g)u_3 \quad (8) \]

\[ X_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} j \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ j \end{pmatrix} \]

where \( X_1, X_2, X_3 \) are also left-invariant vector-fields on \( \text{SE}(2) \times \text{SE}(2) \). Hence (8) is a 3-input left-invariant system on \( \text{SE}(2) \times \text{SE}(2) \).

Let us verify controllability of the system (8). Note that \( j \)-direction of each robot can be independently controlled by \( X_2 \) and \( X_3 \). The first-order Lie brackets generated by \( X_1, X_2, X_3 \) are

\[ [X_2, X_3] = \begin{pmatrix} ij \\ 0 \end{pmatrix}, \quad [X_2, X_1] = \begin{pmatrix} ij \\ 0 \end{pmatrix}, \quad [X_3, X_1] = \begin{pmatrix} 0 \\ jk \end{pmatrix} \]
Now the independency with respect to $k$ is achieved by $X_2; X_1$ and $X_3; X_1$. Finally, considering the second-order Lie brackets
\[
X_2; X_1 = \begin{pmatrix} j & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} jk & 0 \\ 0 & -i \end{pmatrix}
\]
would suffice to guarantee the controllability. Thus the controllability Lie algebra
\[
\text{span}\{X_1, X_2, X_3, [X_2, X_1], [X_3, X_1], [X_2, [X_2, X_1]]\}
\]
spans the entire Lie algebra. The last basis can be replaced by $[X_3, [X_3, X_1]]$, namely,
\[
\text{span}\{X_1, X_2, X_3, [X_2, X_1], [X_3, X_1], [X_3, [X_3, X_1]]\} = \mathfrak{se}(2) \times \mathfrak{se}(2).
\]
Therefore the system is a second-order controllable system, while the original wheeled mobile robot is a first-order one.

It is notable that the controllability Lie algebra is structurally different from that of chained systems. The chained structure should contain a specific vector-field, called single generator, with which all the Lie brackets in the controllability Lie algebra are generated (such as $\{X_1, [X_1, X_2], [X_1, [X_1, X_2]], \cdots \}$). This is not the case for the system (8) due to the presence of $[X_2, [X_2, X_1]]$ or $[X_3, [X_3, X_1]]$. Instead, this can be considered as a two generator system with the generator $X_2$ and $X_3$. Therefore the control strategy proposed for chained systems are not applicable, which makes the control problem interesting and it is quite interesting that such a simple problem setting leads us to a non-chained system example. For more details on control of multi-generator nonholonomic systems, see Ishikawa (2002).

From this point of view, we call this manner of assigning the control inputs the common generator assignment.

$\ell$ robots case. The formentioned result on 2 robots can be generalized to the case of $\ell$ robots.

**Theorem 1.** Suppose $\ell$ copies of the planar wheeled mobile robots defined on
\[
\text{(SE}(2)\right)^\ell := \text{SE}(2) \times \text{SE}(2) \times \cdots \times \text{SE}(2)
\]
Then the robots are controllable using $\ell + 1$ control inputs under the following common generator assignment:
\[
\dot{g} = X_1(g)u_1 + \cdots + X_{\ell+1}(g)u_{\ell+1}
\]
where
\[
X_1 = \begin{pmatrix} i \\ j \\ \cdots \\ i \end{pmatrix}, X_2 = \begin{pmatrix} j \\ 0 \\ \cdots \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ j \\ \cdots \\ 0 \end{pmatrix}, \cdots X_{\ell+1} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ j \end{pmatrix}
\]

**Proof.** For $i \in \{2, \cdots, \ell + 1\}$, we have
\[
[X_i, X_1] = \begin{pmatrix} 0 \\ \cdots \\ k \end{pmatrix}, [X_i, [X_i, X_1]] = \begin{pmatrix} -i \\ \cdots \\ 0 \end{pmatrix},
\]
where $k$ appears in the $i - 1$-th component of $[X_i, X_1]$ and $-i$ appears in the $i$-1-th component of $[X_i, [X_i, X_1]]$. This concludes the system is controllable with up to second-order Lie brackets. 

2.2 Coupled generator assignment

2 robots case. Let us further reduce the number of control inputs. Now, suppose the common forwarding velocity is given by the sum of $u_1$ and $u_2$, while the angular velocity of the first robot is $u_1$ and the angular velocity of the second robot is $u_2$ (Fig. 4). In other words, $u_1$ will cause forwarding motion of the both robots and rotation of the first robot at the same time, while $u_2$ will cause forwarding motion of the both robots and rotation of the second robot at the same time. Then the corresponding state equation is given by
\[
\dot{g} = X_1(g)u_1 + X_2(g)u_2
\]
\[
X_1 = \begin{pmatrix} i + j \end{pmatrix}, X_2 = \begin{pmatrix} i + j \end{pmatrix}
\]
where $X_1$ and $X_2$ are left-invariant vector-fields on SE(2) $\times$ SE(2). Hence (12) is a 2-input left-invariant system on SE(2) $\times$ SE(2).

Though this assignment may look restrictive as the motions of the both robots are strongly coupled to each other, it is still possible to show the controllability of the whole system. The only first-order Lie bracket is
\[
[X_1, X_2] = \begin{pmatrix} i + j, i \\ i, i + j \end{pmatrix} = \begin{pmatrix} j, i \\ i, j \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix}.
\]
Let $X_{12}$ denote $[X_1, X_2]$ to save the space. Then the second-order Lie brackets are
\[
[X_1, X_{12}] = \begin{pmatrix} i + j, k \\ i, -k \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix},
\]
\[
[X_2, X_{12}] = \begin{pmatrix} i, k \\ i + j, -k \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}.
\]
Finally, the third-order Lie brackets are
\[
[X_1, [X_1, X_{12}]] = \begin{pmatrix} i + j, -i \\ i, 0 \end{pmatrix} = \begin{pmatrix} -k \\ 0 \end{pmatrix},
\]
\[
[X_2, [X_2, X_{12}]] = \begin{pmatrix} i, 0 \\ i + j, i \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}.
\]
In summary, this system is controllable by considering up to third-order Lie brackets,
\[
\text{span}\{X_1, X_2, X_{12}, [X_1, X_{12}], [X_2, X_{12}], [X_1, [X_1, X_{12}]]\}
\]
\[
= \text{im}\left(\begin{pmatrix} i + j \\ i + j \\ k \end{pmatrix} \begin{pmatrix} -i \\ 0 \\ -k \end{pmatrix} \begin{pmatrix} i \\ i + j \\ -k \end{pmatrix} \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}\right) = \mathfrak{se}(2) \times \mathfrak{se}(2).
\]
The last basis can be replaced by \[ X_2; [X_2; X_{12}] \]. We call this manner of assigning control inputs the \textit{coupled generator assignment}. Now we succeeded to reduce the control inputs without loss of controllability, though the cost we have paid is the complexity of controllability structure.

\textbf{\ell\ robots case}. This result can also be generalized to the case of \ell\ robots.

\textbf{Theorem 2}. Suppose \ell\ copies of the planar wheeled mobile robots defined on \((\mathbb{SE}(2))^\ell\). Then the robots are controllable using \ell\ control inputs under the following coupled generator assignment:

\[
\dot{y} = X_1(g)u_1 + \cdots + X_{\ell}(g)u_{\ell}
\]

where

\[
X_1 = \begin{pmatrix} i+j \\ i \\ \vdots \end{pmatrix}, \quad X_2 = \begin{pmatrix} i+j \\ i \\ \vdots \end{pmatrix}, \quad \cdots \quad X_{\ell} = \begin{pmatrix} i \\ i \\ \vdots \end{pmatrix}
\]

\textbf{Proof}. First-order Lie brackets are given by

\[
[X_i, X_j] = \begin{pmatrix} k \\ \vdots \\ -k \\ \vdots \end{pmatrix}, \quad i, j \in \{1, \cdots, \ell\}, i < j,
\]

where \(k\) appears in the \(i\)-th component, \(-k\) appears in the \(j\)-th component and the rest components are all 0. Clearly they are all linearly independent to each other. By skew symmetry, the number of the first-order Lie brackets is \(\ell(\ell - 1)/2\), which is not less than \(\ell\) if \(\ell \geq 3\). Recalling \([j, k] = -i\), we have

\[
[X_i, [X_i, X_j]] = \begin{pmatrix} 0 \\ \vdots \\ -i \\ 0 \end{pmatrix}, \quad i, j \in \{1, \cdots, \ell\}, i < j,
\]

where \(-i\) appears in the \(i\)-th component. Collecting all these Lie brackets, the system is controllable with up to second-order Lie brackets. \(\square\)

\textbf{Remark 3}. It is interesting to note that the system is second-order controllable for \(\ell \geq 3\), despite it is third-order controllable for \(\ell = 2\). This is mainly because we cannot find two independent vector-fields from the first-order Lie bracket due the skew symmetry \([X_1, X_2] = -[X_2, X_1]\). In this sense, it is easy to control 3 robots with 3 inputs rather than controlling 2 robots with 2 inputs. \(\bullet\)

\textbf{Remark 4}. As far as we consider the control input assignment as linear combination of the forwarding velocity \(i\) and the angular velocity \(j\), the least number of control inputs is \(\ell\) for \(\ell\) robots to make them controllable. This is because the \(j\) component cannot be generated as a result of any Lie bracket, in contrast to the fact that \(i\) can be produced by \([k, j]\) and \(k\) can be produced by \([j, i]\). Therefore the input vector-fields \(X_1, \cdots, X_{\ell}\) must contain the full basis corresponding to \(j\)-components, so \(m \geq \ell\) should hold. \(\bullet\)

2.3 Control example—forwarding with different speeds

We have shown controllability of \(\ell\) mobile robots using \(\ell\) or \(\ell + 1\) control inputs. Since the state equations are given in the form of left-invariant systems on Lie groups, now it is possible to make use of several control methods developed in the last two decades, such as those by (Sarti et al. (1993); Struemper (1998); Leonard and Krishnaprasad (1995); Bullo (2000)). A promising technique among them is the transverse function approach proposed by Morin and Samson (2003), which enables us systematic design of time-varying feedback controllers ensuring practical stability (ultimate boundedness) of the closed-loop systems. This approach can deal with not only the point-to-point control problem and also the tracking control to reference trajectory, and is also capable of controlling non-chained systems (Ishikawa et al. (2009)).

Instead of going into detail of a specific design method, let us discuss the following illustrative situation. Recall that the input assignments can be considered as \textit{kinematic constraints} between the robots. In the first example of the common generator assignment (8), a constraint that the two robots must have the common forwarding velocity \(u_1\) is imposed. Therefore, it is apparently impossible for the two robots to follow the parallel straight lines with \textit{different speeds}, say \(2v\) and \(v\) (see Fig. 5, left).

![Fig. 5. Practical tracking of infeasible trajectory](image)

Perfect tracking is infeasible

Practical tracking is still possible

However, once we compromise with a slight inaccuracy, it is still possible to practically track the references (see Fig. 5, right). The robot 2 is moving in a slower average speed by making a small detour from the reference, while the robot is moving faster.

A numerical example is shown in Fig. 6–Fig. 8. Show we kept the constant forwarding speed \(u_1 = 1\). The robot 1 starts from the origin and is tracking a horizontal line at \(y = 0.1\) using a simple state feedback \(u_2 = -k_1(y_1 - 0.1) - k_2\theta_1\). On the other hand, \(u_3\) is designed so that the robot 2 tracks an winding curve \(y = A\sin\omega x + 0.5\), where \(A = 0.4\) and \(\omega = 4\pi\). Fig. 6 shows the trajectories of both robots on the \(x\)-\(y\) plane. From the time response of \(x\) in Fig. 7, we can see that the speed of the robot 2 in the \(x\)-direction is indeed slower than the robot 1.
3. EXTENSION TO SPATIAL ROBOTS

It is not difficult to extend these results to spatial mobile robots defined on Lie group $SO(3)$ or $SE(3)$. If we only consider pure rotation of a robot, its kinematics is described as a left-invariant system on $SO(3)$:

$$SO(3) := \{ R \in \mathbb{R}^{3 \times 3} \mid RR^T = I, \det R = +1 \}$$

which is a 3-dimensional smooth manifold. Its Lie algebra $so(3)$ is isomorphic to the set of $3 \times 3$ skew symmetric matrices, spanned by

$$i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

each of which represents an infinitesimal rotation about a principal axis $x, y$ and $z$ (see Fig.9).

With the fundamental algebraic rules

$$[i, j] = k, \quad [j, k] = i, \quad [k, i] = j,$$

it is easy to see that a single robot on $SO(3)$ is controllable with two inputs, i.e.,

$$\dot{g} = X_1(g)u_1 + X_2(g)u_2, \quad X_1 = i, X_2 = j$$

is controllable.

**2 robots case.** For two robots on $SO(3)$, suppose the following input assignment

$$\dot{g} = X_1(g)u_1 + X_2(g)u_2 + X_3(g)u_3, \quad g \in SO(3) \times SO(3)$$

$$X_1 := \begin{pmatrix} i \\ i \end{pmatrix}, X_2 := \begin{pmatrix} j \\ 0 \end{pmatrix}, X_3 := \begin{pmatrix} 0 \\ k \end{pmatrix}$$

which is a 3-input 6-state left-invariant system on $SO(3) \times SO(3)$. $u_1$ gives the common angular velocity about the body-fixed $x$-axes, while $u_2$ and $u_3$ give the angular velocities about the body-fixed $y$-axes of each robot independently.

This system is indeed controllable with up to second-order Lie brackets, shown as follows.

- First-order Lie brackets

$$[X_1, X_2] = \begin{pmatrix} k \\ 0 \end{pmatrix}, \quad [X_1, X_3] = \begin{pmatrix} 0 \\ k \end{pmatrix}$$

- Second-order Lie brackets

$$[X_2, [X_1, X_2]] = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$\text{span}\{X_1, X_2, [X_1, X_2], [X_1, X_3], [X_2, [X_1, X_2]]\} = \text{im}\begin{pmatrix} i & j & 0 & k & 0 & i \\ i & 0 & j & k & 0 & 0 \end{pmatrix} = so(3) \times so(3).$$

**3 robots case.** Three robots on $SO(3)$ can also be controllable with the following input assignment:

$$\dot{g} = X_1u_1 + X_2u_2 + X_3u_3, \quad g \in SO(3) \times SO(3) \times SO(3)$$

$$X_1 := \begin{pmatrix} i \\ 0 \\ j \end{pmatrix}, X_2 := \begin{pmatrix} j \\ i \\ 0 \end{pmatrix}, X_3 := \begin{pmatrix} 0 \\ j \\ i \end{pmatrix}$$

which is a 3-input 9-state left-invariant system on $(SO(3))^3$. Now $u_1$ causes the rotation of the first robot about its $x$-axis, and the rotation of the third robot about its $y$-axis at the same time. Similar interpretation applies to $u_2$ and $u_3$. The Lie brackets generated from $X_1, X_2, X_3$ are:

- First-order Lie brackets
\[
X_4 := [X_1, X_2] = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad X_5 := [X_2, X_3] = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, \\
X_6 := [X_1, X_3] = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}
\]

- Second-order Lie brackets
\[
X_7 := [X_1, X_4] = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad X_8 := [X_2, X_5] = \begin{pmatrix} 0 & -j \\ 0 & 0 \end{pmatrix}, \\
X_9 := [X_3, X_6] = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}
\]

Thus the system is controllable since
\[
\text{span}\{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9\} = \text{im} \begin{pmatrix} i & j & 0 & 0 & 0 & -j & 0 & 0 \\ 0 & i & 0 & j & 0 & 0 & -j & 0 \\ j & 0 & i & 0 & 0 & k & 0 & 0 & -j \end{pmatrix} = \mathfrak{s}(3) \times \mathfrak{s}(3) \times \mathfrak{s}(3).
\]

4. DISCUSSION

4.1 Non-nilpotency of the controllability Lie algebra

In our process to guarantee the controllability with less control inputs, nilpotency of the controllability Lie algebra plays a crucial role. An Lie algebra is said to be nilpotent if there exists a finite natural number \( k \) such that all the Lie brackets higher than order \( k \) are zero. \( \mathfrak{SE}(2) \) is not nilpotent for there exists non-zero Lie bracket of any order, e.g.,
\[
[j, j, \cdots, [j, i] \cdots].
\]
Neither \( \mathfrak{se}(3) \) nor \( \mathfrak{SE}(3) \) is nilpotent. The nilpotency is the source of ‘redundant’ controllability that we made full use of. On the other hand, it is well known that single wheeled mobile robot on \( \mathfrak{SE}(2) \) is locally controllable to the following nilpotent form ( chained form )
\[
\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mu_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mu_2
\]
under appropriate coordinate and input transformation. This form is often preferred for control design, but the multiple copies of this nilpotent form cannot be controllable using less control inputs any more ( \( \ell \) copies require 2\( \ell \) inputs).

4.2 Trade-off between control simplicity and control channel economy

The less control inputs we use to guarantee the system controllability, the higher the order of controllability Lie algebra tends to be. Since higher-order systems are generally difficult to control than lower-order ones, there is a trade-off between reducing the control inputs and simplifying the controller. For example, the two planar mobile robots with 4 inputs (eq. (6)) is first-order controllable, while the reduced input cases (eqs. (8)-(12)) require higher-order Lie brackets.

Controlling higher-order nonholonomic systems often require fast switching or highly oscillatory inputs. Therefore this trade-off can be considered as interchanging the temporal resolution (i.e., bit rate of the control inputs with respect to time) and the spatial resolution (density of the control channels), as shown in Fig. 10. From this point of view, our aim in this paper was to reduce the bus size by increasing the temporal resolution.

5. CONCLUSION

In this paper, we analyzed the nonlinear controllability of multiple mobile robots using less control inputs. We showed that the number of control inputs can be cut down so that the whole system is controllable, by imposing appropriate constraints between the velocities of the robots. For example, \( \ell \) planar mobile robots can be controllable with \( \ell \) control inputs, being less than 2\( \ell \) as expected. This result is also interesting in the sense that we can produce a lot of challenging examples (i.e., with complicated controllability structure) of nonholonomic systems. We note that the results can also be extended to the case of \( \mathfrak{SE}(3) \) in a similar way.

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