Control of LTI systems over digital erasure channels

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Abstract: This paper studies networked control systems closed over digital erasure channels, when a class of source coding schemes is employed. Building upon recent results, we show that it is possible to convert analysis and design problems involving digital erasure channels into problems involving two signal-to-noise ratio (SNR) constrained channels in series, one subject to an equality SNR constraint, and the second one subject to an inequality SNR constraint. This insight is then used to establish a sufficient condition on the channel average data-rate and dropout probability that allow one to achieve mean square stability. We also provide performance guarantees and a numerical example to illustrate our results.

Keywords: Networked control systems, average data-rate, data dropouts, signal-to-noise ratio.

1. INTRODUCTION

The analysis and design of networked control systems (NCSs) subject to one type of communication constraint has received much attention in the literature (see, e.g., Antsaklis and Baiiilieul (2007); Nair et al. (2007); Schenato et al. (2007); Hespanha et al. (2007); Silva et al. (2010)). Although many of the available results are fundamental in nature, they usually do not apply to realistic scenarios where communication links are subject to several communications constraints simultaneously. A notable exception to the above is the simultaneous consideration of delays and data dropouts, which naturally lends itself to a unified treatment; see, e.g., Gao et al. (2008). In this paper, we focus on NCSs closed over lossy channels where data dropouts are accompanied by average data-rate limitations. Relevant literature is reviewed below.

The work by Gao et al. (2008) studies the joint effect of quantization, random delays, and data dropouts in NCSs. To that end, the authors focus on logarithmic quantizers treated as a sector bound uncertainty (Fu and Xie (2005)), and present a linear matrix inequality (LMI) based procedure to design static state feedback controllers that achieve a prescribed performance level. A similar quantization model is used by Niu et al. (2009), where the stabilization of NCSs subject to quantization and data dropouts is studied for a specific output feedback architecture, again using LMIs. A related approach is presented in Che et al. (2009), where sufficient LMI conditions for the design of dynamic output feedback controllers are presented, when zooming quantizers are employed (Brockett and Liberzon (2000)).

Further insights are contained in Tsumura et al. (2009). That work presents an explicit characterization of the coarsest logarithmic quantizer that allows one to design static state feedback controllers that achieve mean square stability in control systems closed over erasure channels. No design methodologies are presented by Tsumura et al. (2009). Another relevant work is Bao et al. (2007), where a general networked control architecture is studied. However, the results in Bao et al. (2007) are presented in an implicit form, and only a suboptimal iterative design method is proposed.

The works referred to above focus on specific control and quantization architectures and, hence, do not provide insights valid in general cases. A more fundamental approach was adopted by Tatikonda and Mitter (2004). In that work, the authors present necessary and sufficient conditions for almost sure observability and stabilizability of noiseless plant models, when controlled or observed over digital erasure channels. Unfortunately, the results by Tatikonda and Mitter (2004) cannot be extended to the noisy plant case (Matveev and Savkin (2005)).

Data loss in digital channels can be seen as a form of time-varying channel behaviour. Such approach has been adopted by Minero et al. (2009), where channel characteristics are assumed constant for a given block of consecutive channel uses, and to change at random from block to block. At each channel use within these blocks, the channel is able to transmit, without errors, a given number of bits that varies randomly from block to block. For such a channel, Minero et al. (2009) present necessary and sufficient conditions for the mean square stabilization of noisy scalar LTI plant models. For general plants, Minero et al. (2009) present only necessary conditions. Martins et al. (2006) also study necessary and sufficient conditions for stabilization over variable rate digital channels, albeit for the scalar plant case only.

The recent work by You and Xie (2010) considers the control of fully observed noiseless LTI systems over digital channels, where data dropouts occur in an i.i.d. fashion and a finite (and fixed) number of bits is available at each channel use. For single input plants, You and Xie...
(2010) present necessary and sufficient conditions for mean square stabilization. In the multiple-input plant case, only sufficient conditions are derived.

This paper considers NCSs closed over digital erasure channels. In particular, we focus on LTI dynamic output feedback controllers for noisy single-input single-output (SISO) LTI plants. As a first contribution, we show that the design and analysis of such a NCS can be carried out by converting the corresponding control problem into a control problem over two additive noise channels connected in series, one subject to an equality SNR constraint, and the second one subject to an inequality SNR constraint. To do so, we use a specific source coding scheme based on entropy coded dithered quantizers (Zamir and Feder (1995)), and build upon the results by Silva et al. (2011); Silva and Pulgar (2011). This insight is then used to establish a sufficient condition on the channel average data-rate and data dropout probability that allows one to achieve mean square stability. We also provide performance guarantees.

Our results are derived for a specific control and source coding architecture, and focus on average data-rate constraints. Nevertheless, our results recover, within a modest penalty, the sufficient conditions for stabilization over digital erasure channels derived by You and Xie (2010) and, in the scalar plant case, those in Minero et al. (2009). A distinguishing feature of our approach is that it enables one to address problems beyond stabilization, providing a starting point for addressing optimal control problems over digital erasure channels.

The paper is organized as follows: Section 2 describes the considered setup. Section 3 presents an auxiliary NCS whose study is instrumental for the remainder of the paper. Section 4 shows how to use the results of Section 3 to deal with digital erasure channels. A numerical example is provided in Section 5. Section 6 draws conclusions.

Notation: \( \mathbb{R} \) denotes the reals, \( \mathbb{R}^+ \) the (strictly) positive reals, and \( \mathbb{N}_0 \) the non-negative integers. \( \mathbb{R}_p \) denotes the set of all proper real-rational transfer functions and \( \mathbb{RH}_2 \) stands for the set of all stable and strictly proper real-rational transfer functions. \( P\{\cdot\} \) and \( \mathbb{E}\{\cdot\} \) denote probability and expectation, respectively. If \( x \) is a (asymptotically) wide sense stationary process, then \( \sigma^2_x \) denotes its (stationary) variance. The space \( \mathcal{L}_2 \) is defined as usual, and its norm is denoted by \( ||\cdot||_2 \) (Zhou et al. (1996)).

2. PROBLEM SETUP

Consider the following channel model:

**Definition 1.** A (scalar) digital erasure channel is a device with scalar input \( v_c \) and scalar output \( w_c \) related via

\[
w_c(k) = \theta(k)v_c(k),
\]

where \( \theta \) is an i.i.d. sequence, \( \theta(k) \in \{0, 1\}, P\{\theta(k) = 1\} = p, \) for some \( p \in (0, 1), v_c(k) \) is constrained to belong to a countable alphabet of binary words, and the average length of \( v_c(k) \) (measured in nats per sample) \( ^1 \) upper bounded by a given constant \( \varpi_{\max} \).

In this paper, we focus on the NCS of Figure 1, where \( G \) models a SISO LTI plant, \( K \) is an LTI controller that has
to be designed, \( d \) is a disturbance, \( y \) is the measurable plant output, and \( u \) is the control input. In Figure 1, the feedback path comprises a digital erasure channel, an encoder, and a decoder (\( E \) and \( D \), respectively). As made explicit in Figure 1, we also assume that there exists one-step-delayed feedback around the channel (this amounts to assuming the existence of packet acknowledgements, as in TCP-like protocols; Schenato et al. (2007)).

We will work under the following assumption:

**Assumption 1.**

(a) \( G \) is a strictly proper SISO LTI system, and \( K \) is a proper LTI system.
(b) The initial state of \( G \) and \( K \), collectively referred to as \( x_o \), is a second order random variable.
(c) The signal \( d \) is a second order wide sense stationary process, uncorrelated with \( x_o \), and having power spectral density \( S_d = |\Omega_d|^2 \).
(d) \( x_o, d \) are independent of \( \theta \).

This paper aims at characterizing the interplay between communication constraints and the stability and performance of the NCS of Figure 1. To do so, Section 3 starts by focusing on an auxiliary situation involving an SNR constrained analog erasure channel. Later, in Section 4, we make a specific choice for the encoder \( E \) and the decoder \( D \) in Figure 1, and show how to use the results of Section 3 to deal with digital erasure channels.

3. SNR CONSTRAINED ERASURE CHANNELS

3.1 Definitions and preliminary results

We introduce the following auxiliary channel type:

**Definition 2.** A (scalar) SNR constrained erasure channel is a device with scalar input \( v \) and scalar output \( w \) related via

\[
w(k) = \theta(k)v(k), \quad w(k) = \nu(k) + q(k),
\]

where \( \theta \) is as in Definition 1, \( q \) is a zero mean white noise sequence, independent of \( (\theta, x_o, d) \), and whose variance \( \sigma^2_q \) can be chosen in \( \mathbb{R}^+ \) subject to the stationary SNR constraint

\[
\gamma \triangleq \frac{\sigma^2_q}{\sigma^2_{\nu}} \leq \Gamma,
\]

where \( \sigma^2_{\nu} \) denotes the stationary variance of \( v \) (assumed to exist), and \( \Gamma \in \mathbb{R}^+ \) is a given constant.

We will refer to \( \Gamma \) in (2) as the maximum allowable SNR of the SNR constrained erasure channel.

\[ 1 \] nat equals \( \ln 2 \) bits.
The NCS of Figure 1, when an SNR constrained erasure channel is used as the link between \( v \) and \( w \), can be modeled as a Markov jump linear system (MJLS; Costa et al. (2005)). We will thus introduce the following notion of stability (Costa et al. (2005)):

**Definition 3.** Consider the NCS of Figure 1, where the link between \( v \) and \( w \) is given by an SNR constrained erasure channel, and suppose that Assumption 1 holds. Denote the state of the resulting dynamic system by \( x \). We say that the NCS is mean square stable (MSS\(^2\)) if and only if there exists finite \( \mu \) and \( M \), both not depending on \( (x_0, d, q) \), such that \( \lim_{k \to \infty} \mathbb{E}\{x(k)\} = \mu \) and \( \lim_{k \to \infty} \mathbb{E}\{x(k)x(k)^T\} = M \).

We next summarize a key enabling result that allows one to easily analyze NCSs subject to data dropouts:

**Theorem 1.** (Silva and Pulgar (2011)). Consider the NCS of Figure 1, where the link between \( v \) and \( w \) is given by an SNR constrained erasure channel. Also consider the auxiliary situation of Figure 2, where the link between \( \tilde{v} \) and \( \tilde{w} \) (see (1)) has been replaced by a gain equal to the successful transmission probability \( p \) and a source of zero mean additive white noise, denoted by \( q_0 \), that is uncorrelated with \( (x_0, d, q) \). If Assumption 1 holds, \( \sigma_q^2 \in \mathbb{R}^+ \) is fixed, and the SNR constraint in (2) is disregarded, then:

1. The NCS of Figure 1 is MSS if and only if (i) the LTI system of Figure 2 is internally stable and well posed, and (ii) there exists a choice for the variance of \( q_0 \), say \( \sigma_q^2 \), such that the steady state variance of \( \tilde{w} \) in Figure 2 satisfies \( \sigma_q^2 = p(1-p)\sigma_q^2 \).

2. If the LTI system of Figure 2 is internally stable and well posed, and the variance of \( q_0 \) is chosen so as to satisfy \( \sigma_q^2 = p(1-p)\sigma_q^2 \), then the power spectral densities of all signals of Figure 2 are equal to those of the corresponding signals in the NCS of Figure 1.

It follows from Theorem 1 that, when the SNR constraint in (2) is removed, the MSS of the resulting NCS is equivalent to the internal stability of an auxiliary LTI system and a stationary equality SNR constraint. If these two conditions are satisfied, then the stationary variance of any signal in the auxiliary LTI system is equal to the stationary variance of the corresponding signal in the NCS of interest.

### 3.2 Mean square stabilization

We now use Theorem 1 to characterize the interplay between MSS and both the successful transmission probability \( p \) and the maximum admissible SNR \( \Gamma \), in NCSs that use SNR constrained erasure channels. To do so, recall that in our setup the designer has the freedom to choose both the LTI controller \( K \) and the noise variance \( \sigma_q^2 \).

**Theorem 2.** Consider the NCS of Figure 1, where the link between \( v \) and \( w \) is given by an SNR constrained erasure channel. If Assumption 1 holds, then there exists \( K \in \mathbb{R}_p \) and \( \sigma_q^2 \in \mathbb{R}^+ \) such that the resulting NCS is MSS if and only if the maximum allowable SNR \( \Gamma \) satisfy

\[
\frac{p\Gamma}{\Gamma(1-p) + 1} > \left( \prod_{i=1}^{p^n} |p_i|^2 \right) - 1, \tag{3}
\]

where \( p_1, \ldots, p_{p^n} \) denote the unstable poles of the plant \( G \) (counting multiplicities).

**Proof.** Given Theorem 1, it suffices to show that (3) is equivalent to the existence of \( K \in \mathbb{R}_p \) and noise variances \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \) such that the auxiliary LTI system of Figure 2 is internally stable, well posed, and

\[
\sigma_q^2 + p = \Gamma \sigma_q^2, \quad \sigma_q^2 = p(1-p)\sigma_q^2 \tag{4}
\]

for some \( p \geq 0 \).

Consider the scheme of Figure 2 and assume that \( K \in \mathcal{S} = \{ K \in \mathbb{R}_p : \text{the LTI of Figure 2 is internally stable and well posed} \} \). Then, for any \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \),

\[
\sigma_q^2 = \|T_{dv}O_d\|^2_2 + \sigma_q^2 \|T_{dy}w\|^2_2 + \sigma_q^2 \|T_{qw}v\|^2_2, \tag{5a}
\]

\[
\sigma_q^2 = \|T_{dv}O_d\|^2_2 + \sigma_q^2 \|T_{dy}w\|^2_2 + \sigma_q^2 \|T_{qw}v\|^2_2. \tag{5b}
\]

Hence, there exist \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \) satisfying (4) if and only if

\[
- \left( \|T_{dv}O_d\|^2_2 + \rho \right) = \sigma_q^2 \left( \|T_{qw}v\|^2_2 - \Gamma \right) + \sigma_q^2 \|T_{qw}v\|^2_2 \tag{6a}
\]

\[
- \|T_{dv}O_d\|^2_2 = \sigma_q^2 \|T_{qw}v\|^2_2 + \sigma_q^2 \left( \|T_{qw}v\|^2_2 - \frac{1}{p(1-p)} \right) \tag{6b}
\]

admit a solution where \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \). To proceed, we note that a necessary condition for (6) to hold for some \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \) is that

\[
\|T_{qw}v\|^2_2 - \frac{1}{p(1-p)} < 0, \quad \|T_{qw}v\|^2_2 - \Gamma < 0. \tag{7}
\]

Thus, by solving (6) for \( \sigma_q^2, \sigma_q^2 \) and using (7), one easily concludes that, if there exist \( \sigma_q^2, \sigma_q^2 \in \mathbb{R}^+ \) satisfying (4), then

\[
0 > \|T_{qw}v\|^2_2 \|T_{qw}v\|^2_2 - \left( \|T_{qw}v\|^2_2 - \Gamma \right) \left( \|T_{qw}v\|^2_2 - \frac{1}{p(1-p)} \right). \tag{8}
\]

Now, we note that \( T_{qw} = T_{qw} + 1, T_{qw} = T_{qw} = p^{-1} T_{qv} \) and that, since \( K \in \mathcal{S}, T_{qv} \in \mathbb{R}H_2 \) and thus \( \|T_{qv} + 1\|^2_2 =

\[\text{Here, and in Proposition 3 below, } T_{xy} \text{ denotes the closed loop transfer function form } x \text{ to } y \text{ in the feedback system of Figure 2.}\]
1 + \|Tqv\|^2_2. Using these facts, a straightforward manipulation shows that (8) is equivalent to
\[
\frac{p}{\Gamma(1 - p) + 1} > \frac{\|Tqv\|^2_2}{\Gamma(1 - p) + 1} = \inf_{K \in S} \frac{\|Tqv\|^2_2}{\Gamma(1 - p) + 1} \geq \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1,
\]
where the last equality follows from Theorem 17 in Silva et al. (2010).

We now prove the converse. If (3) holds, then the result invoked in the last equality in (9) implies that there exists \(K \in S\) such that \(\|Tqv\|^2_2 < p / \Gamma(1 - p) + 1\). Similarly, since \(p \in (0, 1), 0 < \Gamma < \infty\), and the relationship between \(Tqv\) and \(Tqv, Tqv, Tqv\) noted above holds, we also conclude that, if (3) holds, then (7) also holds. The above facts imply that the noise variances that solve (6) are nonzero.

Theorem 2 gives an explicit necessary and sufficient condition on the successful transmission probability \(p\) and the maximum allowable SNR \(\Gamma\) of an SNR constrained erasure channel, for being able to find a controller \(K\) and a noise variance \(\sigma^2_y\) that guarantee that the resulting NCS is MSS and that the SNR constraint in (2) is satisfied.

To further explore the result of Theorem 2, we now focus on two special cases:

**Corollary 1.** Consider the setup and assumptions of Theorem 2. Define
\[
p_{\text{inf}} \triangleq 1 - \frac{1}{\prod_{i=1}^{n_p} |p_i|^2}, \quad \gamma_{\text{inf}} \triangleq \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1.
\]

(1) A necessary condition for the existence of \(K \in \mathcal{R}_p\) and \(\sigma^2_y \in \mathbb{R}^+\) such that the resulting NCS is MSS is that \(p > p_{\text{inf}}\). For any given \(p > p_{\text{inf}}\), there exists \(K \in \mathcal{R}_p\) and \(\sigma^2_y \in \mathbb{R}^+\) such that the resulting NCS is MSS if and only if
\[
\Gamma > \frac{\left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1}{1 - (1 - p) \prod_{i=1}^{n_p} |p_i|^2}.
\]

(2) A necessary condition for the existence of \(K \in \mathcal{R}_p\) and \(\sigma^2_y \in \mathbb{R}^+\) such that the resulting NCS is MSS is that \(\Gamma > \gamma_{\text{inf}}\). For any given \(\Gamma > \gamma_{\text{inf}}\), there exists \(K \in \mathcal{R}_p\) and \(\sigma^2_y \in \mathbb{R}^+\) such that the resulting NCS is MSS if and only if
\[
p > \left(1 + \frac{1}{\Gamma}\right) \left(1 - \frac{1}{\prod_{i=1}^{n_p} |p_i|^2}\right).
\]

**Proof.** Immediate from Theorem 2 and the fact that \(p \in (0, 1)\) and \(\Gamma > 0\).

We know from Silva et al. (2010) that \(\gamma_{\text{inf}}\) corresponds to the minimal SNR compatible with MSS, when no data dropouts are present in the considered NCS (i.e., when \(p = 1\)). Similarly, Silva and Pulgar (2011) showed that \(p_{\text{inf}}\) is the minimal successful transmission probability compatible with MSS, when no SNR constraints are present in the considered NCS (i.e., when \(\Gamma \rightarrow \infty\)). We thus see that Theorem 2 is consistent with known results that apply to situations where, in isolation, either SNR constraints or data dropouts are present.

As expected, our results show that the presence of, e.g., SNR constraints, makes the requirements on the minimal successful transmission probability \(p\) compatible with MSS more stringent that in the case where only data dropouts are present. Indeed, from (11), we see that, for any given \(\Gamma > \gamma_{\text{inf}}\), the minimal value of \(p\) compatible with MSS increases (with respect to \(p_{\text{inf}}\)) in an amount \(\Delta p\) given by
\[
\Delta p = \Gamma \left(1 - \frac{1}{\prod_{i=1}^{n_p} |p_i|^2}\right).
\]

An analogous statement can be made when \(p > p_{\text{inf}}\) is assumed to be given. The above discussion shows that, when MSS is the sole control objective, it is a simple matter to tradeoff channel reliability (i.e., dropout probability) for channel SNR, and vice versa.

### 3.3 Optimal designs

In this section, we suggest an approach for the optimal design of the controller \(K\) and the noise variance \(\sigma^2_y\) that defines an SNR constrained erasure channel.

Consider the NCS of Figure 1, where the link between \(v\) and \(w\) is given by an SNR constrained erasure channel. For this NCS, we measure performance by using the stationary variance \(\sigma^2_y\) of the plant output \(y\). Define
\[
\bar{\sigma}^2 = \inf_{\gamma \in \mathcal{S}} \sigma^2_y.
\]

where the optimization is carried out with respect to all controllers \(K \in \mathcal{R}_p\) and all noise variances \(\sigma^2_y \in \mathbb{R}^+\) such that the resulting NCS is MSS.

We introduce a simplifying assumption:

**Assumption 2.** If the optimization problem in (12) is feasible, then \(T_{dv} \Omega_d \neq 0\) at the optimum.

Assumption 2 is sensible. If it were not satisfied, then optimal performance would be achieved without sending information on the disturbance \(d\) over the channel.

**Theorem 3.** Consider the NCS of Figure 1, where the link between \(v\) and \(w\) is given by an SNR constrained erasure channel. If Assumptions 1 and 2 hold, and \(p\) and \(\Gamma\) satisfy (3), then
\[
\bar{\sigma}^2 = \inf_{\gamma \in \mathcal{S}} \inf_{(\sigma^2_y, \sigma^2_x) \in (\mathbb{R}^+)^2} \sigma^2_y \geq \inf_{\gamma \in \mathcal{S}} \inf_{(\sigma^2_y, \sigma^2_x) \in (\mathbb{R}^+)^2} \sigma^2_y
\]

where \(\sigma^2_x\) and \(\sigma^2_y\) are the stationary variances of \(v\) and \(\hat{w}\) in the LTI system of Figure 2, and \(S = \{K \in \mathcal{R}_p : \text{the LTI of Figure 2 is internally stable and well posed}\}\).

**Proof.** Given Theorems 1, 2, and their proofs, it suffices to show that the second inequality constraint, namely \(\bar{\sigma}^2 \leq \sigma^2_x \leq (1 - p)^{-1} \sigma^2_y\), is active at the optimum (or can be made active without compromising optimality). To see this, recall that \(T_{qw} = 1 + T_{qv}\) and that \(T_{qv} \in \mathcal{R} T\) for any \(K \in \mathcal{S}\). Also note that, since \(\Gamma \leq \infty\) and \(p \in (0, 1)\), Assumption 2 guarantees that both \(\sigma^2_y\) and \(\sigma^2_x\) are nonzero at the optimum (see (5)). Hence, at the optimum, the second inequality constraint can be written as
Thus, if the second inequality constraint is not active at the optimum, then one can always decrease \( \sigma_{\theta}^2 \) so as to make the constraint active. This necessarily decreases both \( \sigma_{\phi}^2 \) and \( \sigma_{v}^2 \) (or keeps them unchanged). Hence, the second inequality constraint must be active at the optimum and the result follows. □

For a given successful transmission probability \( p \) and maximum allowable SNR \( \Gamma \), Proposition 3 shows that the problem of finding the minimum stationary plant output variance that is achievable in the considered NCS is equivalent to a nested optimization problem. In that formulation the inner problem is a standard quadratic optimal control problem that can be solved by using, e.g., LMs (Boyd et al. (1994)). For simple problems, one can use a grid for \((\sigma_{\theta}^2, \sigma_{\phi}^2)\) and then select the pair that makes the inner problem attain the smallest optimal value.

The key point in Proposition 3 is that the optimal design of \( K \) and \( \sigma_{\phi}^2 \) can be turned into an optimal control problem subject to SNR constraints where, besides the controller \( K \), one has the freedom to choose the variances of the noises defining such constraints (i.e., both \( \sigma_{\theta}^2 \) and \( \sigma_{\phi}^2 \)). (See also Pulgar et al. (2011).)

4. DIGITAL ERASURE CHANNELS

We now show how to use the results of Section 3 to address control problems over digital erasure channels.

When a digital erasure channel is used in the NCS of Figure 1, the quantization of \( v \) becomes mandatory. To quantize \( v \) we will use an entropy coded dithered quantizer (Zamir and Feder (1995)), slightly modified so as to take data dropouts into account.

At the encoder side, the channel input \( v \) (a binary word) is generated via

\[
v_k(k) = H_k(s(k), d_h(k)), \quad s(k) = Q(v(k) + d_h(k)),
\]

where \( d_h \) is a real-valued dither signal assumed to be available at both the encoder and decoder sides, \( Q: \mathbb{R} \rightarrow \mathcal{A} \equiv \{1, \cdots, \Delta; i \in \mathbb{Z} \} \) corresponds to a uniform quantizer with step size \( \Delta > 0 \), \( \mathcal{H}_k : \mathcal{A} \times \mathbb{R} \rightarrow \mathcal{A}_c(k) \) corresponds to the mapping describing an entropy-coder (EC; also called loss-less encoder (Cover and Thomas, 2006, Ch.5)) whose output symbol is chosen according to the conditional distribution of \( s(k) \), given \( d_h(k) \), and the set \( \mathcal{A}_c(k) \) is, for every \( k \), a set of prefix-free binary words (Cover and Thomas (2006)).

At the decoder side, \( w \) is obtained via

\[
w_k(k) = \begin{cases} 0, & \text{if } \theta(k) = 0, \\ \hat{s}(k) - d_h(k), & \text{if } \theta(k) = 1, \end{cases}
\]

where, whenever \( \theta(k) = 1 \),

\[
\hat{s}(k) = \mathcal{H}_k^{-1}(w_v(k), d_h(k)),
\]

and \( \mathcal{H}_k^{-1} : \mathcal{A}_c(k) \times \mathbb{R} \rightarrow \mathcal{A} \) corresponds to the mapping describing the entropy-decoder (ED) that is complementary to the EC at the encoder side; i.e., \( \mathcal{H}_k^{-1}(\mathcal{H}_k(s(k), d_h(k)), d_h(k)) = s(k) \), for every \( k \in \mathbb{N}_0 \). Note that instantaneous knowledge of \( \theta(k) \) at the decoder side is assumed to be available. It is also worth noting that the ED does not rely on the history of neither \( w_v \) nor \( d_h \) to construct \( \hat{s}(k) \). It uses only the current values of such signals, which become available at the decoder side when successful transmission takes place. Also, since the channel is noiseless and EC-ED pairs are lossless (Cover and Thomas (2006)), we also have that, if \( \theta(k) = 1 \), then \( \hat{s}(k) = s(k) \).

We denote the expected length of the symbol \( v_k(k) \) (in nats) by \( R(i) \) and define the average data-rate of the source coding scheme as

\[
R \triangleq \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i).
\]

If a digital erasure channel is used together with the encoder-decoder pair described in (14)-(15), then the link between \( v \) and \( w \) in the NCS of Figure 1 can be equivalently characterized by

\[
w(k) = \theta(k)v(k), \quad \hat{w}(k) = v(k) + q(k),
\]

where

\[
q(k) \triangleq Q(v(k) + d_h(k)) - (v(k) + d_h(k))
\]

corresponds to quantization errors. A key property of the sequence \( q \) is presented next:

**Lemma 1.** Consider the NCS of Figure 1, where the link between \( v \) and \( w \) is given by (16) with \( q \) as in (17), and \( \theta \) is as in Definition 1. If Assumption 1 holds, \( \Delta < \infty \) and the dither \( d_h(k) \) is independent of \( x_o, d, h, \Delta_1^{-1} \) and uniformly distributed on \((-\Delta/2, \Delta/2)\), then \( q(k) \) is independent of \( (x_o, d, h, \Delta_1^{-1}) \) and uniformly distributed on \((-\Delta/2, \Delta/2)\).

**Proof.** The result follows upon mimicking the proof of Theorem 1 in Zamir and Feder (1995). □

Provided the dither signal \( d_h \) is properly chosen, Lemma 1 guarantees that the NCS of Figure 1, when a digital erasure channel and the encoder-decoder pair in (14)-(15) is used as the link between \( v \) and \( w \), can be represented as a MJLS. Hence, the notion of stability introduced in Definition 3 makes sense in the present situation.

**Theorem 4.** (Silva et al. (2011)). Consider the setup and assumptions of Lemma 1. If the NCS is MSS, then there exists a choice for the EC-ED pair such that the average data-rate of the coding scheme satisfies

\[
R < \frac{1}{2} \ln (1 + \gamma) + \frac{1}{2} \ln \left( \frac{2\pi e}{12} \right) + \ln 2,
\]

where

\[
\gamma \triangleq \frac{\sigma_{\theta}^2}{\Delta^2/12}
\]

and \( \sigma_{\theta}^2 \) is the stationary variance of \( v \) in Figure 1. ■

**Remark 1.** The gap between \( R \) and \( \frac{1}{2} \ln (1 + \gamma) \) in (18), namely \( \frac{1}{2} \ln \left( \frac{2\pi e}{12} \right) + \ln 2 \) nats/sample (\( \approx 1.254 \) bits/sample), can be interpreted as a rate penalty arising from the fact that we use an entropy coded dithered quantizer. Further details can be found in Silva et al. (2011). ■

Assume that one is interested in a situation where the average data-rate \( R \) must satisfy \( R \leq R_{\max} \). In order to

\[x^{k-1} \text{ stands for } x(0), \cdots, x(k-1).\]
satisfy the latter constraint, it is sufficient that \( \gamma \) in (18) satisfies
\[
\gamma \leq \Gamma R_{\max} = e^{2(R_{\max} - 1)^2 \ln(2\pi e)} - 1.
\]
(19)

Also, since the quantization step \( \Delta \) is a design variable, the variance of the equivalent noise \( q \) in (17), i.e., \( \sigma_q^2 = \Delta^2/12 \), also becomes a decision variable that can be chosen in \( \mathbb{R}_+ \) subject to (19). We thus see that, by virtue of Lemma 1 and Theorem 4, when a digital erasure channel together with the encoder-decoder pair of (14)-(15) is used, the link between \( v \) and \( w \) in the NCS of Figure 1 becomes an SNR constrained erasure channel.

The above leads to the following immediate consequence of the results of Section 3:

**Corollary 2.** Consider the NCS of Figure 1, where the channel is a digital erasure channel and the encoder-decoder pair is as in (14)-(15). If Assumption 1 holds, \( \Delta \in \mathbb{R}_+^+ \) and the dither \( d_{th}(k) \) is independent of \( (x_{\alpha}, d, \theta, d_{th}^{-1}) \) and uniformly distributed on \((-\Delta/2, \Delta/2)\), then:

1. A sufficient condition for the existence of \( K \in \mathcal{R}_p \), \( \Delta \in \mathbb{R}_+^+ \), and an EC-ED pair such that the resulting NCS is MSS, is that the successful transmission probability \( p \) and the maximum average data-rate \( R_{\max} \) satisfy
\[
\frac{p \Gamma_{R_{\max}}}{\Gamma_{R_{\max}}(1-p)+1} > \left( \prod_{i=1}^{n_p} |p_i|^2 \right) - 1,
\]
(20)

where \( \Gamma_{R_{\max}} \) is as in (19), and \( p_i \) is as in Theorem 2.

2. If \( p \) and \( \Gamma_{R_{\max}} \) satisfy (20), then an upper bound on the minimum stationary plant output variance \( \sigma_q^2 \) that is achievable in the considered NCS, while satisfying the average data-rate constraint \( \mathcal{R} \leq R_{\max} \), is given by \( \sigma_q^2 |_{\Gamma_{R_{\max}}} \) (see (13)).

3. If \( p \) and \( \Gamma_{R_{\max}} \) satisfy (20), and \( K_0 \) and \( \sigma_q^2 \) are the values of \( K \) and \( \sigma_q^2 \) that solve the optimization problem in (13) when \( \Gamma = \Gamma_{R_{\max}} \), then there exists an EC-ED pair such that, when \( K = K_0 \) and \( \Delta = \sqrt{12\sigma_q^2} \),
\[
\sigma_q^2 = \left[ \sigma_{q,\text{EC}}^2 \right]_{R_{\max}}(p, \Gamma_{R_{\max}}),
\]
while satisfying \( \mathcal{R} < \Gamma_{R_{\max}} \).

Corollary 2 provides a characterization of the interplay between communication constraints and control objectives when digital erasure channels are employed. Our results are based upon Theorem 4 which gives only an upper bound on \( \mathcal{R} \). Therefore, the results of Corollary 2 can only be stated in terms of upper bounds and sufficient conditions.

The bounds provided by Corollary 2 may be conservative when compared to the fundamental limits arising when general control and coding architectures are employed. The following suggests, at least for the mean square stabilization case, that this is not the case: The definition of \( \Gamma_{R_{\max}} \) and the fact that \( p \in (0,1) \) allows one to conclude that (20) is equivalent to \( p > \rho_{\text{inf}} \) and
\[
R_{\max} > \sum_{i=1}^{n_p} \ln |p_i| + \frac{1}{2} \ln \left( 1 - \frac{p}{\prod_{i=1}^{n_p} |p_i|^2} \right) \ln \left( \frac{2\pi e}{12} \right) + \ln 2.
\]

Thus, even though we focus on average data-rates and a class of source coding schemes, we recover, within a modest data-rate penalty, the sufficient condition established by You and Xie (2010) for single-input plants. In the scalar plant case, our results recover the sufficient conditions by Minero et al. (2009).

We finally note that the discussion at the end of Sections 3.2 and 3.3 also applies, *mutatis mutandis*, to the present case.

5. AN EXAMPLE

Assume that the plant is given by \( G(z) = (z - 1.5)^{-1} \), and that the disturbance \( d \) is unit variance white noise.

We first consider that \( G \) is controlled over an SNR constrained erasure channel. For this plant, \( \gamma_{\text{inf}} = 1.25 \) and \( p_{\text{inf}} = 0.556 \). We used the design guidelines in Theorem 3 to calculate the best achievable performance as a function of \( \Gamma \) and \( p \). Figure 3 shows the results for several values of \( p \in (p_{\text{inf}}, 1). \) As expected, the performance deteriorates as \( \Gamma \) approaches the right hand side of (10), and improves as \( \Gamma \to \infty \). When both \( p \to 1 \) and \( \Gamma \to \infty \), we recover the best non-networked performance (namely, \( \sigma_y^2 = 1 \)).

We now consider that \( G \) is controlled over a digital erasure channels. Assume that \( p = 0.75 \) and \( R_{\max} = 3 \) bit/sample. It is easy to check that these parameters satisfy (20). By using Corollary 2, we conclude that the minimal stationary output variance that is achievable in the considered situation is upper bounded by 3.4796, and that there exists a control and source coding scheme, within the proposed class, such that \( \sigma_y^2 = 3.4796 \) while \( R < 3 \). To verify these claims, we performed simulations using an actual entropy coded dithered quantizer. The quantizer was simulated by adding uniform random numbers to \( v \) prior to quantization, and then subtracting them after decoding. However, the EC-ED pair worked conditioned upon only a 4-level uniformly quantized version of the true dither values. The simulated performance \( \sigma_y^2 \) and the corresponding average data-rate \( R_{\text{sim}} \) are given by \( \sigma_{y,\text{sim}}^2 = 3.4638 \) and \( R_{\text{sim}} = 2.3151 < 3 \) bits/sample. These results are in agreement with Corollary 2.

6. CONCLUSIONS

This paper has studied NCSs closed over digital erasure channels, when a specific class of source coding schemes based on entropy coded dithered quantizers is employed. By doing so, we were able to convert analysis and design problems involving digital erasure channels into problems involving (analog) SNR constrained erasure channels. This fact allowed us to find a sufficient condition on the channel average data-rate and successful transmission probability that guarantees the existence of mean square stabilizing LTI controllers. We have also proposed a possible method for carrying out the design of that controller having performance guarantees in mind.

Future work should focus on the multiple channel case, and on optimal methods for controller design.

Note that, in You and Xie (2010), \( p \) stands for erasure probability, not successful transmission probability as in this paper.

The results correspond to averages over one hundred simulations, each one \( 10^4 \) samples long. The measured rate corresponds to the actual average of the number of bits sent through the channel.

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Fig. 3. Optimal performance as a function of the maximum admissible SNR $\Gamma$ for $p \in \{0.56, 0.565, 0.57, 0.58, 0.6, 0.62, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.999\}$. 

REFERENCES


