MIMO Controller Synthesis for LTI and LPV Systems Using Input-Output Models

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Abstract: This paper considers the problem of designing MIMO fixed-structure controllers based on a polynomial framework for both Linear Time-Invariant (LTI) and Linear Parameter-Varying (LPV) systems having input-output models. For SISO systems, a polynomial method has been proposed recently to solve such problems, where the a priori selection of a central polynomial is carried out heuristically, and for LPV systems the synthesis problem is solved using a sum-of-squares relaxation approach. To reduce the complexity of the design procedure, we propose a combined gradient and LMI-based algorithm for optimizing the choice of the central polynomial matrix in terms of the closed-loop performance. Moreover, for LPV systems, the computational complexity of the sum-of-squares approach is reduced by introducing polytopic input-output LPV representations. Quadratic stability and performance is guaranteed in the admissible scheduling range. The proposed methods are illustrated with simulation examples.

1. INTRODUCTION

Controller synthesis of MIMO Linear Time-Invariant (LTI) systems having input-output models has been an active field of research over last three decades. Based on a recent developed framework on positive polynomials a sufficient LMI condition for stability and fixed-structure controllers for scalar LTI systems in transfer function form has been proposed in [Henrion et al., 2003b]. However, the choice of a central polynomial to overcome the non-convexity of the original design problem plays a key role in that design technique. It affects the closed-loop performance and even the feasibility of the synthesis problem; this is the main limitation of that approach, [Yang et al., 2007]. Heuristic ways were proposed in [Henrion et al., 2003b; Yang et al., 2007] to choose such a polynomial. With a heuristic approach, the synthesis problem becomes however a real challenge in the case of MIMO systems where a central polynomial matrix needs to be selected.

An extension to a SISO Linear Parameter-Varying (LPV) gain-scheduling design with guaranteed $H_\infty$ performance over the whole parameter range has been recently proposed in [Gilbert et al., 2010]. The idea of the central polynomial has been used again. Two issues may prevent the synthesis procedure in [Gilbert et al., 2010] from being a practical tool: (1) the fact that the a priori selection of the central polynomial is still based on ad-hoc methods, and (2) the computational complexity associated with the sum-of-squares relaxation technique which has been employed to solve the synthesis problem. These two issues are much more involved in the MIMO case.

In this note, based on the results of [Gilbert et al., 2010] for SISO systems and employing a combined gradient-LMI algorithm of [Abbas et al., 2008] for solving BMIs in certain control problems, a systematic synthesis procedure for both LTI and LPV MIMO input-output models is proposed. In this design technique a low-complexity (low-order and fixed-structure) LTI or LPV controller is computed without a priori selection of any parameters other than the controller structure and the design specifications; the selection of the central polynomial matrix is optimized with respect to the achieved closed-loop performance. Furthermore, the computational complexity of the LPV synthesis is reduced by introducing a polytopic input-output LPV representation.

2. POLYNOMIAL-BASED MIMO LTI CONTROL

In this section the results of [Gilbert et al., 2010] on discrete-time MIMO LTI systems are briefly reviewed. Consider a MIMO LTI discrete-time system having input and output vectors $u(k) \in \mathbb{R}^{n_u}$ and $y(k) \in \mathbb{R}^{n_y}$, respectively, with matrix fraction description (MFD)

$$A(z)Y(z) = B(z)U(z),$$

where $Y(z)$, $U(z)$ are the z-transforms of the output and input signals, respectively. $B(z)$, $A(z)$ are the numerator and denominator polynomials matrices of degree $n$ and are represented as

$$B(z) = \sum_{i=0}^{n} B_i z^{-i}, \quad \bar{B} = [B_0 \ B_1 \ldots \ B_n],$$

$$A(z) = \sum_{i=0}^{n} A_i z^{-i}, \quad \bar{A} = [A_0 \ A_1 \ldots \ A_n],$$

where $A_i \in \mathbb{R}^{n_y \times n_y}$ and $B_i \in \mathbb{R}^{n_u \times n_u}$ are the $i^{th}$ coefficient matrices of the denominator and numerator matrix polynomial, respectively.

2.1 $D$-Stability

Let a region in the complex plane be defined as
\[ D := \{s : H_{11} + H_{12}s + H_{12}^*s^* + H_{22}ss^* < 0\}, \] (3)

where
\[
H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}, \]
(4)
\[ H \in \mathbb{C}^{2 \times 2}. \]

For the unit disk we have \( H_{11} = -1, H_{12} = 0, H_{22} = 1 \) and \( z = s \). We say that a polynomial matrix is \( D \)-stable if all roots of its determinant are inside \( D \). Now define the matrix \( \Pi \in \mathbb{R}^{2m \times (n+1)m} \) as
\[
\Pi = \begin{bmatrix} I_m & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m \end{bmatrix}, \]
(5)

where \( m \) is equal to \( n_y \). Next, a sufficient LMI condition for stability of the system (1) is given, [Henrion et al., 2003a].

**Theorem 1.** \( (D) \)-stability of LTI systems. Given a stable polynomial matrix \( C(z) \), the polynomial matrix \( A(z) \) is \( D \)-stable if there exists a symmetric matrix \( P \in \mathbb{R}^{m \times m} \) such that
\[
C^T \bar{A} + \bar{A}^T C - F(P) \geq 0, \]
(6)
where \( \bar{A} \) is as in (2b),
\[
F(P) = \Pi^T (H \otimes P) \Pi, \]
(7)
and \( H, \Pi \) are as in (4) and (5), respectively. \( \otimes \) stands for the Kronecker product.

**Proof.** See [Henrion et al., 2003a].

The polynomial
\[
C(z) = \sum_{i=0}^{n} C_i z^{-i}, \quad \bar{C} = [C_0 \ C_1 \ldots C_n], \]
(8)
is referred to as the central polynomial [Henrion et al., 2003a], and we introduce \( \bar{C} \in \mathbb{R}^{m \times m(n+1)} \) as the coefficient matrix of \( C(z) \).

### 2.2 \( H_\infty \) Performance

The result in [Gilbert et al., 2010] on the \( H_\infty \) performance of SISO LTI systems can be extended to MIMO systems as follows:

**Theorem 2.** Given a stable polynomial matrix \( C(z) \), the system with MFD in (1) is stable and satisfies the \( H_\infty \) performance constraint
\[
\|B(z)A^{-1}(z)\|_\infty < \gamma, \]
(9)
if there exists a symmetric matrix \( P \) and positive scalars \( \lambda, \gamma \) which satisfy
\[
\begin{bmatrix} C^T \bar{A} + \bar{A}^T C - F(P) - \lambda \bar{C}^T \bar{C} & \bar{B}^T \\ \bar{B} & \lambda^2 I_m \end{bmatrix} \geq 0, \]
(10)
where \( I_m \in \mathbb{R}^{m \times m} \) is the identity matrix.

**Proof.** The proof follows the same lines as for the SISO case in [Gilbert et al., 2010].

### 2.3 MIMO LTI Controller Design

Now suppose that it is required to design a controller for the MIMO system described in (1) with (2a-b). Let the controller have input-output representation
\[
A_K(z)U(z) = B_K(z)E(z), \]
(11)
where \( A_K(z) \) and \( B_K(z) \) are given in (16).
The two-loop gradient-LMI algorithm of [Abbas et al., 2008] for solving control problems with BMI constraints is utilized here to solve the above controller synthesis problem. The algorithm has two main loops: in the outer one a linear cost function is minimized, while in the inner loop a BMI constraint \( X < 0 \) is enforced by minimizing the spectral abscissa \( \alpha(X) \) of \( X \) to make it negative definite. In the case considered here, in the outer-loop the closed-loop performance \( \gamma \) is reduced, while in the inner-loop the feasibility problem \( \alpha(X(d_k, P, \lambda, C)) < 0 \) is solved over the decision variables \( d_k, P, \lambda, C \), which include the coefficients of the central polytopic matrix.

Any gradient-based optimization algorithm can be used to minimize \( \alpha \) in the inner loop (here the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm has been used) if the gradient of \( \alpha \) with respect to the decision variables is available. To solve problems with large number of decision variables and to speed up the algorithm, the following idea is utilized. Collect the entries of the matrix \( C \) in a vector \( x_C \in \mathbb{R}^{n_C} \) and the entries of \( P, d_k, \lambda \) in another vector \( x_L \in \mathbb{R}^{n_L} \). The decision variables can then be gathered in the vector

\[
\begin{align*}
x &= \begin{bmatrix} x_C^T & x_L^T \end{bmatrix}^T, \quad (18)
\end{align*}
\]

where \( x \in \mathbb{R}^{n_x}, n_x = n_{xc} + n_{xl} \). Now, given \( x_C \), the BMI constraint \( X < 0 \) becomes an LMI constraint in \( x_L \), which can be solved as an LMI feasibility problem using LMI solvers. Note that the converse is not true, i.e., given \( x_L \) the BMI \( X < 0 \) does not turn into an LMI in \( x_C \), see (16). The problem solved in the inner loop can be decomposed as follows: given \( x_C \) solve the corresponding LMIs to find \( x_L \), which is then used to find the direction in which \( \alpha(X) \) is minimized with respect to \( x_C \) in order to update \( x_C \). In this case the calculation of the gradient of \( \alpha \) will be with respect to \( x_C \). The procedure can be summarized as Algorithm 1. Finally, the gradient of \( \alpha(X) \) w.r.t. \( C \) is required in the gradient search; the gradient computation can be found in [Ali et al., 2010].

4. GAIN-SCHEDULING LPV CONTROL

In this section the results of Section 2 are extended to MIMO LPV systems in input-output form. Polytopic input-output LPV models are introduced to avoid the computational complexity of the sum-of-squares approach proposed in [Hol and Scherer, 2004].

4.1 Quadratic-Stability of Input-Output LPV Systems

Consider a MIMO LPV system in input-output description

\[
A(\theta, q)y(k) = B(\theta, q)u(k),
\]

where

\[
\begin{align*}
B(\theta, q) &= \sum_{i=0}^n B_i(q)q^{-i}, \quad \bar{B}(\theta) = [B_0(\theta) B_1(\theta) \ldots B_n(\theta)] \\
A(\theta, q) &= \sum_{i=0}^n A_i(q)q^{-i}, \quad \bar{A}(\theta) = [A_0(\theta) A_1(\theta) \ldots A_n(\theta)]
\end{align*}
\]

(21a)

(22a)

In this model \( q \) represents the forward-shift time operator, i.e. \( qa(k) = u(k+1) \), which is used here since the \( z \)-transform is not relevant for LPV systems. The time-dependent parameter vector \( \theta_\ell(k) \) or \( \theta_h \) in (22) has the form

\[
\theta_\ell = [\theta_1(k) \theta_2(k) \ldots \theta_m(k)]^T \in \mathbb{R}^{n_\ell}
\]

and depends on a vector of measurable scheduling signals \( p(k) \in \mathbb{R}^{n_p} \) according to \( \theta_\ell = p(\rho(k)) \), where \( p \) is a continuous mapping.

We restrict the discussion to LPV systems where i) the parameter dependence is affine; that is, the coefficient matrices \( A_i \in \mathbb{R}^{n_m \times n_m}, B_i \in \mathbb{R}^{n_m \times n_u} \) in (22a-b) depend affinely on \( \theta_\ell \), ii) the time-varying parameter vector \( \theta_h \) varies in a polytope \( P_\theta \in \mathbb{R}^{n_\theta} \), which is assumed to be a compact set with vertices \( \theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n} \), that is

\[
\theta_h \in P_\theta := \text{conv}\{\theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n}\}.
\]

(23)

This description encompasses many practical situations; note that if required affine parameter dependence can be achieved by introducing new parameters. It is clear that 

\[
\bar{B}(\theta_h) \quad \text{and} \quad \bar{A}(\theta_h)
\]

in (22a-b) range in a polytope whose vertices are the images of the vertices \( \theta_{v_1}, \theta_{v_2}, \ldots, \theta_{v_n} \), according to

\[
\begin{align*}
\bar{B}(\theta_h) &= \text{conv}\{\bar{B}_i(\theta_h)\}, & i = 1, \ldots, n_v, \\
\bar{A}(\theta_h) &= \text{conv}\{\bar{A}_i(\theta_h)\}, & i = 1, \ldots, n_v
\end{align*}
\]

(24)

Because of this property, we shall refer in the sequel to such LPV models as polytopic input-output LPV representations. Now the result in Theorem 1 can be extended to the MIMO input-output LPV systems as follows: Given a stable polynomial matrix \( C(z) \), a polynomial matrix \( \bar{A}(\theta_h) \), as in (22b), is stable for all \( \theta \in P_\theta \) if there exists a symmetric matrix \( P \) such that:

\[
C^T \bar{A}(\theta_h) + \bar{A}(\theta_h)C = F(P) \geq 0, \quad \forall \theta_h \in P_\theta,
\]

(25)

where \( F(P) \), \( \bar{A}(\theta_h) \), and \( C \) are given in (7), (22b) and (5), respectively. With the polytopic representation of the

**Algorithm 1 Gradient-LMI Algorithm for Solving BMIs**

**Step 0.** Choose an initial value \( \gamma^0 \) and initial step size 0 < \( \rho^0 < 1 \). Set \( j = 1 \) and repeat the following.

**Step 1.** Generate \( m_1 \) initial vectors \( x_{C_i}^0, i \in \{1, \ldots, m_1\} \), and set \( j = j + 1 \).

**Step 2.** For each \( i \): Use gradient search to solve

\[
\min \alpha(x_{C_i}^j).
\]

(19)

At the \( k \)-th iteration of the gradient search, the required \( x_{C_i}^j \) is calculated for given \( \gamma^j-1 \) and \( x_{C_i}^{j-1} \) by solving the LMI problem

\[
\min \alpha(x_{C_i}^{j,k}).
\]

(20)

Then the gradient of \( \alpha(x_{C_i}^{j,k}) \) is computed with respect to \( x_{C_i}^{j,k-1} \).

**Step 3.** Find

\[
\alpha_{\min}^j = \min_{i \in \{1, \ldots, m_1\}} \alpha(x_{C_i}^j).
\]

**Step 4.** If \( \alpha_{\min}^j < 0 \), increase the step size by \( \gamma^j = \gamma^j-1 \eta \) such that \( \gamma^j < 1, \eta > 1 \) and decrease \( \gamma \) by \( \gamma^j = \gamma^j-1(1-\gamma^j) \); otherwise

- decrease the step size by \( \gamma^j = \gamma^j-1(1-\gamma^j) \); otherwise

**Step 5.** If the difference between \( \gamma^j \) and \( \gamma_j-1 \) is less than a specified value \( \text{exit} \); otherwise go to **Step 1**.
input-output LPV system (21), the condition (25), which imposes an infinite number of constraints, can be reduced to a finite set of LMI$s. Using affine dependence on $\theta_k$, it can be easily shown that (25) will hold for all $A(\theta_k), B(\theta_k)$ if and only if it holds at the vertices $\bar{A}_i, \bar{B}_i$, $i = 1, \ldots, n_v$. The following result formalizes this fact.

**Theorem 3.** (Quadratic-stability, polytopic form). Consider a polytopic LPV system described by the input-output representation in (21) with (24), then given a stable central polynomial matrix $C(z)$, this LPV system is quadratically stable if there exists a symmetric matrix $P$ satisfying the system of LMI$s

$$
C^T \bar{A}(\theta_k) + \bar{A}(\theta_k)C - F(P) \geq 0, \quad i = 1, \ldots, n_v. \tag{26}
$$

**Proof.** Immediate.

### 4.2 Quadratic $\mathcal{H}_\infty$ Performance

The $\mathcal{H}_\infty$ performance measure used for SISO input-output LPV systems in [Gilbert et al., 2010] can be extended to the MIMO case as follows: Given a stable central polynomial matrix $C(z)$, the MIMO LPV system described in input-output form as in (21) has quadratic $\mathcal{H}_\infty$ performance if there exists a single symmetric matrix $P$ such that

$$
C^T \bar{A}(\theta_k) + \bar{A}(\theta_k)C - F(P) - \lambda C^T \bar{C} B(\theta_k) \lambda \geq 0 \tag{27}
$$

for all admissible values of the parameter vector $\theta_k$. In the special case of polytopic input-output LPV systems, given a stable central polynomial matrix $C(z)$ and using the variable transformation $\chi = \lambda \gamma^2$, the condition (27) can be reduced to a finite set of LMI$s. Again using the affine dependence on $\theta_k$, (27) will hold for all $A(\theta_k)$ and $B(\theta_k)$ if and only if it holds at the vertices $\bar{A}_i$ and $\bar{B}_i$, $i = 1, \ldots, n_v$.

### 4.3 Gain-Scheduled LPV Controller Synthesis

Now suppose that it is required to design a low-complexity gain-scheduled MIMO controller in input-output form for the LPV system described in (21). Let the controller depend on the same scheduling parameter $\theta_k$ as the LPV system, and let it have a polytopic input-output representation

$$
A_K(\theta_k, q) u(k) = B_K(\theta_k, q) e(k), \tag{28}
$$

where

$$
B_K(\theta_k, q) = \sum_{i=0}^{n_K} B_{K,i}(\theta_k) q^{-i}, \tag{29a}
$$

$$
B_K(\theta_k) = [B_{K,0}(\theta_k) B_{K,1}(\theta_k) \ldots B_{K,n_K}(\theta_k)],
$$

$$
A_K(\theta_k, q) = \sum_{i=0}^{n_K} A_{K,i}(\theta_k) q^{-i}, \tag{29b}
$$

$$
A_K(\theta_k) = [A_{K,0}(\theta_k) A_{K,1}(\theta_k) \ldots A_{K,n_K}(\theta_k)].
$$

As shown in Fig. 2, the closed-loop system from the reference signal vector $r(k)$ to the output signal vector $y(k)$ can be described also by a polytopic input-output LPV representation, where its parameter-dependent coefficient vectors range in a polytope of vertices as follows

$$
\begin{bmatrix}
B_{cl}(\psi_k) \\
A_{cl}(\psi_k)
\end{bmatrix} \in \text{conv} \left\{ \begin{bmatrix}
B_{cl,1} \\
A_{cl,1}
\end{bmatrix}, \begin{bmatrix}
B_{cl,1} \\
A_{cl,1}
\end{bmatrix} : i = 1, \ldots, n_{vcl} \right\}. \tag{30}
$$

![Fig. 2. Negative feedback configuration in LPV case.](image)

where a new time-dependent vector

$$
\psi_k = \psi(k) = [\psi_1(k) \psi_2(k) \ldots \psi_{n_v}(k)]^T \in \mathbb{R}^{n_v}
$$

has been defined such that the closed-loop parameter-dependent coefficients have affine dependence on $\psi_k$ and vary in a polytope $P_\psi$ of vertices $\psi_{v_1}, \psi_{v_2}, \ldots, \psi_{v_{n_{vcl}}}$. The elements of the vector $\psi_k$ are functions of the original scheduling parameters since the plant as well as the controller depend on the same parameter vector $\theta_k$. Note that at this point it is only required to compute the vertices $\psi_{v_1}, \psi_{v_2}, \ldots, \psi_{v_{n_{vcl}}}$ and the corresponding closed-loop matrices as shown in (30).

In a similar way the sensitivity function of the feedback system from $r(k)$ to $e(k)$ and the control sensitivity function of the feedback system from $r(k)$ to the control input signal $u(k)$ can be represented as affine functions of $\psi_k$.

Note that in any case, the closed-loop input-output representation can be described by two polynomial matrices which are linear in the controller variables. Defining $\psi_k$ to insure affine dependence is not unique. Collect the controller coefficients in a vector $d_k \in \mathbb{R}^{n_{vcl}}$. Then, the controller synthesis problem can be stated as: given a stable central polynomial matrix $C(z)$, find $d_k, P, \lambda, \chi$ such that the inequality

$$
C^T \bar{A}_c(\psi_k) + \bar{A}_c(\psi_k)C - F(P) - \lambda C^T \bar{C} B_{cl}(\psi_k) \chi \geq 0 \tag{31}
$$

holds at each point inside $P_\psi$. With the polytopic input-output LPV representation of the closed-loop system, the above condition is a finite set of LMI$s and the gain-scheduled controller synthesis problem can be formulated as follows.

**Lemma 4.** (Input-output LPV controller). Given a stable central polynomial matrix $C(z)$ and a polytopic input-output LPV representation of the closed-loop system with $\psi_k \in P_\psi := \text{conv} \{\psi_{v_1}, \ldots, \psi_{v_{n_{vcl}}}\}$ and (30). There exists a polytopic input-output LPV controller

$$
\begin{bmatrix}
B_{cl}(\theta_k) \\
A_{cl}(\theta_k)
\end{bmatrix} := \sum_{i=1}^{n_v} \beta_i \begin{bmatrix}
B_{K,i} \\
A_{K,i}
\end{bmatrix}, \tag{32}
$$

which guarantees quadratic stability and quadratic $\mathcal{H}_\infty$ performance of the closed-loop system along all parameter trajectories in the polytope

$$
P_\theta = \{ \sum_{i=1}^{n_v} \beta_i \theta_{v_i} \mid \beta_i \geq 0, \sum_{i=1}^{n_v} \beta_i = 1 \}
$$

if there exist a vector $d_k \in \mathbb{R}^{n_{vcl}}$, a symmetric matrix $P \in \mathbb{R}^{m(n+K_m) \times m(n+K_m)}$ and positive scalars $\lambda, \chi$ satisfying the system of $n_{vcl}$ LMI$s

$$
\begin{bmatrix}
C^T \bar{A}_c(\psi_{v_i}) + \bar{A}_c(\psi_{v_i})C - F(P) - \lambda C^T \bar{C} B_{cl}(\psi_{v_i}) \\
B_{cl}(\psi_{v_i}) \chi
\end{bmatrix} \geq 0,
$$

\forall i = 1, \ldots, n_{vcl}. \tag{33}

**Proof.** Quadratic $\mathcal{H}_\infty$ performance $\gamma$ of the closed-loop system is guaranteed if there exists a stable central poly-
nominal controller $C(z)$, positive scalars $\lambda, \chi$ and a symmetric matrix $P \in \mathbb{R}^{n(n+n_k) \times m(n+n_k)}$ such that for all $\psi_k \in \mathbb{P}_\psi$, (31) holds, see Subsection 4.3.

Suppose that (33) holds at all $\psi_k \in \mathbb{P}_\psi$ for some $d_k$, symmetric matrix $P$, $\lambda > 0$ and $\chi > 0$, then the closed-loop system with the controller (32) has a quadratic $\mathcal{H}_\infty$ performance less than $\gamma$. Since $\psi_k \in \mathbb{P}_\psi$ implies $\theta_k \in \mathbb{P}_\theta$, where $\mathbb{P}_\theta \subseteq \mathbb{P}_\psi$, (33) holds for all $\theta_k \in \mathbb{P}_\theta$, from which the result follows readily.

Now, if the stable central polynomial matrix in (33) is unknown, the synthesis condition in (33) turns into a BMI (34)

\[ \min_{P,d_k,C}\gamma, \quad \text{s.t.} \quad X_i < 0, \forall i = 1, \ldots, n_{\text{vel}}. \]

Consequently, Algorithm 1 can be adopted to solve this synthesis problem. In order to use the algorithm we collect the BMI matrices at the vertices of the polytope (30) in one BMI as

\[ X = \text{diag}(X_1, X_2, \ldots, X_{n_{\text{vel}}}). \]

Again the decision variable vector $x$ is defined as in (18).

5. ILLUSTRATIVE EXAMPLES

5.1 Example 1: MIMO LTI Control

This example is taken from [Hjalmarsson, 1999]. The LV100 gas turbine engine has five states, two inputs and two outputs. The states of the system are the gas generator speed, fuel output, temperature, fuel flow actuator level and variable area turbine nozzle actuator level. The gas generator speed and the temperature are the outputs while the inputs are the fuel flow and the variable area turbine nozzle. The model of the system is discretized using the Tustin approximation with a sampling time of 0.1s. The model is in matrix fraction form as in (1) and (2) with $n = 5$ and the constant coefficient matrices are

\[ A_0 = I_2, \quad A_1 = -2.21 \cdot I_2, \quad A_2 = 1.284 \cdot I_2, \quad A_3 = 0.1044 \cdot I_2, \quad A_4 = -0.1607 \cdot I_2, \quad A_5 = -0.017 \cdot I_2, \]

\[ B_0 = \begin{bmatrix} 0.044566 & 0.005526 \\ 0.271071 & -0.005458 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.024350 & 0.002259 \\ 0.078480 & -0.004846 \\ -0.072482 & -0.009207 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.031879 & -0.002648 \\ -0.143357 & 0.004876 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.007872 & 0.000433 \\ 0.065111 & -0.000141 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0.028255 & 0.0037208 \\ 0.257810 & -0.000807 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0.028255 & 0.0037208 \\ 0.257810 & -0.000807 \end{bmatrix}, \]

A first order MIMO controller with PI structure is to be synthesized. It is expressed in the form (11) with (12a-b), where

\[ A_{K0} = I_2, \quad A_{K1} = -I_2, \quad B_{K0} = \begin{bmatrix} K_0 & K_0 \\ K_0 & K_0 \end{bmatrix}, \quad B_{K1} = \begin{bmatrix} K_1 & K_1 \\ K_2 & K_2 \end{bmatrix}. \]

The technique described in section 3 is used to design the controller based on the mixed-sensitivity loop shaping approach, see Fig 4. The weighting filter to shape the sensitivity function is tuned to be $W_S = I_2 \otimes \frac{0.00014z+0.00002}{1-0.9992z}$. With the above PI controller and $W_S$, a 2×2 stable central polynomial matrix of 7th degree is required with $C \in \mathbb{R}^{2\times16}$, note that its dimension and degree should be the same as those of the closed-loop denominator polynomial matrix. Clearly, it is difficult to choose heuristically the coefficients of this central polynomial matrix. Therefore, Algorithm 1 is adopted to minimize the quadratic $\mathcal{H}_\infty$ performance in the outer-loop of the algorithm and minimizing $\alpha(X)$ over the unknown controller parameters $d_k \in \mathbb{R}^4$, the symmetric matrix $P \in \mathbb{R}^{4\times4}$, the positive scalar $\lambda$, and the central polynomial matrix $C(z)$ of degree 7 in the inner-loop of the algorithm. The algorithm 1 converges after 13 feasible iterations for this example.

For comparison, a full order (7th-order) controller has been designed, based on a state-space representation of the model and using the same weighting functions above, with the help of the robust control toolbox of MATLAB [Mathworks, 2005]. The achieved performance $\gamma$ is 0.0014 and 0.0016 for the MIMO PI and the full-order controllers, respectively. Simulation results of the closed-loop system with both controllers are shown in Fig. 3. The tracking capability of the two outputs of the system is almost the same with the 7th and 2nd-order controllers; however the MIMO PI controller almost eliminates the coupling effect in comparison with the full order one.

5.2 Example 2: MIMO LPV Control

The presented approach in section 4 is illustrated here through a MIMO LPV academic example constructed for this purpose. Consider a polytopic input-output LPV system of the form (21) with (22a-b), where $n = 2$, $\theta_k \in \mathbb{P}_\theta$, $\mathbb{P}_\theta = [0, 0.5]$ and the parameter dependent coefficient matrices are given as follows:

\[ A_0(\theta_k) = I_2, \quad A_1(\theta_k) = (1 - 0.5\theta_k) \cdot I_2, \quad A_2(\theta_k) = (0.5 - 0.7\theta_k) \cdot I_2, \quad B_1(\theta_k) = 0_2, \quad B_2(\theta_k) = \begin{bmatrix} 0.5 - 0.4\theta_k & 0.2 - 0.1\theta_k \\ 0.6 - 0.2\theta_k & 0.1 - 0.4\theta_k \end{bmatrix}. \]

A gain-scheduled MIMO controller with PI structure is sought; it is described as in (28) with (29) where $n_\theta = 1$. The controller depends on the same scheduling parameter.
as the plant, and its parameter dependent coefficient matrices are given as $A_{K0}(\theta_k) = I_2$, $A_{K1}(\theta_k) = -I_2$,

\[
B_{K0}(\theta_k) = \begin{bmatrix} K_{11}^1 + K_{12}^1 \theta_k & K_{12}^2 + K_{12}^1 \theta_k \\ K_{21}^1 + K_{22}^1 \theta_k & K_{22}^2 + K_{22}^1 \theta_k \end{bmatrix},
\]

\[
B_{K1}(\theta_k) = \begin{bmatrix} K_{11}^2 + K_{12}^2 \theta_k & K_{12}^3 + K_{12}^2 \theta_k \\ K_{21}^2 + K_{22}^2 \theta_k & K_{22}^3 + K_{22}^2 \theta_k \end{bmatrix}.
\]

Here we shape the sensitivity function $S$, to achieve desired properties (small rise time and good tracking) of the closed-loop system from the reference input $r(k)$ to the fictitious output $z(k)$, see Fig. 4. The design is carried out by choosing a suitable weighting filter $W_S$ such that $H_\infty$ norm of $W_S S$ is less than some prescribed value $\gamma > 0$. The closed-loop system, Fig. 4, includes the filter $W_S$ and the unknown controller parameter matrices $K^1, \ldots, K^4$. It turns out that the coefficients of the closed-loop system depend on $\theta_k$, which yields a non-polytopic description for the closed-loop system. In order to utilize the results in Lemma 4, a new parameter vector $\psi(k)$ is introduced as

\[
\psi_k = \psi(k) = [\psi_1(k) \; \psi_2(k)]^\top = [\theta_k \; \theta_k^2]^\top,
\]

which gives a polytopic input-output LPV representation for the closed-loop system. According to the dimension and the degree of the denominator polynomial matrix a $4^\text{th}$ order stable central polynomial matrix with $C \in \mathbb{R}^{2 \times 10}$ is required. Then the proposed algorithm is adopted to minimize the quadratic $H_\infty$ performance over the unknown controller parameters $\delta_k \in \mathbb{R}^{16}$, the symmetric matrix $P \in \mathbb{R}^{8 \times 8}$, the positive scalar $\lambda$, and the central polynomial matrix $C(z)$. The quadratic $H_\infty$ performance is obtained as 0.073, the controller parameters are

\[
K^1 = \begin{bmatrix} -0.3905 & 1.1277 \\ 1.8798 & -1.8205 \end{bmatrix}, K^2 = \begin{bmatrix} 0.9510 & 0.4504 \\ 0.5872 & -0.2052 \end{bmatrix},
\]

\[
K^3 = \begin{bmatrix} -0.6209 & 1.4066 \\ 0.9761 & 0.1107 \end{bmatrix}, K^4 = \begin{bmatrix} 0.0264 & 0.7869; \\ -0.8570 & -0.3617 \end{bmatrix},
\]

and the scalar $\lambda = 0.02679$. The closed-loop simulation of the LPV system is shown in Fig. 5, which shows good tracking at several levels of the operating region. The algorithm 1 converges after 19 feasible iterations for this example.

6. CONCLUSIONS

Based on a recently developed polynomial method, a synthesis approach has been presented that provides a straightforward way of designing low-complexity controllers for both MIMO LTI and LPV systems in input-output representation. Stability and quadratic $H_\infty$ performance conditions for polytopic input-output LPV models have been used for this purpose. An algorithm has been proposed to optimize the choice of the central polynomial

![Fig. 5. Results of Example 2](image)

matrix in terms of the closed-loop $H_\infty$ performance. In addition, for LPV systems the computational complexity of the sum-of-squares approach proposed in [Gilbert et al., 2010] is reduced by representing the closed-loop system in a polytopic form.

REFERENCES


