Nonlinear Unknown Input Observer Design for Nonlinear Systems: A New Method

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Abstract: This paper provides new developments in the design of observers for nonlinear systems with unknown inputs. The proposed methods guarantee the error system stability and yield many additional degrees of freedom available to the designer. The algorithms for designing a nonlinear observer for nonlinear systems with unknown inputs is derived in detail. The proposed observers may be used for fault detection and isolation. A nonlinear mass-spring-damper model is given in order to highlight the efficiency of the proposed method.

Keywords: Nonlinear control systems; Nonlinear observer; Unknown input observer; Error stability analysis; Mass-spring-damper system;

1. INTRODUCTION

Observer design is an important problem that has various applications such as output feedback control, system monitoring, process identification and fault detection. The observer was first proposed and developed by Luenberger (1971) in the early 1960s. After the pioneering work of Luenberger on estimation problems, various types of estimators have been developed for both linear and nonlinear systems. An observer for nonlinear systems whose nonlinearity is globally Lipschitz has been developed by many researchers including Abbaszadeh and Marquez (2010). The full order Luenberger-like observer for nonlinear systems whose nonlinearity satisfy the well known Lipschitz condition has been proposed in recent years, see for example Raghavan and Hedrick (1994).

It is of importance to design observers for multivariable linear or nonlinear systems partially driven by unknown inputs. In many control systems, the measurement of all the input signals in the system is impossible. In some cases, the uncertainties of certain parameters of the system can be modelled as unknown inputs for designing a robust observer. For that type of system, an observer, which has the capability of estimating the states in the presence of unknown inputs, is required in order to design an appropriate control for the system. Such a problem arises in systems subject to disturbance or with inaccessible inputs and it may appear in many applications such as fault detection and isolation or parameter identification, see Mondal et al. (2009).

Design of observers for linear systems subject to unknown inputs has attracted considerable attention in the last three decades. Different approaches have been considered to design unknown input observers (UIO) for linear systems following the conventional Luenberger design procedure or using sliding mode control theory. For both approaches the existence conditions are exactly the same. In Floquet and Barbot (2005) the proposed observation algorithm allows the user to recover the state and the unknown inputs in finite time.

Under less restrictive conditions several approaches for designing reduced order observers have been proposed by Guan and Saif (1991), Hou and Müller (1992), and Mehta et al. (2010) and for full order observers many methods have been presented including Hou et al. (1999) and Darouach (2009). Achieving less restrictive existence conditions and more direct design procedures has been a challenge in this area.

Since the 1990s, many attempts have been made to extend the existing UIO design from linear systems to nonlinear systems. UIOs for bilinear systems were designed by many researchers including Zasadzinski et al. (1998), and Hamidi et al. (2008). UIOs for more general nonlinear systems were also proposed in Seliger and Frank (1991), Yang and Saif (1996), Ha et al. (2003), Barbot et al. (2007) and Barbot et al. (2009).

More recently, observer architectures utilising the concept of sliding mode control/observer for uncertain systems, see for example, Zak (2003), Koshkouei and Zinober (2004), and Barbot and Floquet (2009). Also a direct extension of the linear results to the nonlinear case was referred to as nonlinear unknown input observer (NUIO) and considered systems with nonlinearities that are functions of inputs and outputs.

An UIO for nonlinear systems using $H_{\infty}$ approach has been designed by Pertew et al. (2005) and a nonlinear observer for descriptive type of nonlinear systems with unknown inputs based on linear matrix inequality (LMI) approach has been presented by Koeing (2006), Chen and Saif (2006a), and Chen and Saif (2006b). This paper presents an alternative method to design the NUIOs for nonlinear systems under certain conditions. Section 2 presents a mathematical description of the nonlinear system. Design of the NUIO for the nonlinear system along with theorems and error dynamics and stability analysis are addressed in Sections 3 and 4, respectively.
2. SYSTEM DESCRIPTION

Consider the following nonlinear system

\[ \dot{x}(t) = Ax(t) + Bu(t) + D\mu(t) + Sg(x,u,t) + Kafo(t) \]
\[ y(t) = Cx(t) + Kafo(t) \]

(1)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) represent the state, input and output vectors respectively. \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, S \in \mathbb{R}^{p \times s}, Ka \in \mathbb{R}^{n \times r} \) and \( Ka \in \mathbb{R}^{p \times s} \) are known matrices. \( fo \) and \( f_{ao} \) are the actuator and sensor faults. \( D \in \mathbb{R}^{n \times q} \), is referred to as the known input disturbances distribution matrix and \( \mu(t) \in \mathbb{R}^q \) is an unknown bounded vector. The function \( \mu(t) \) describes the disturbances and any kind of modelling uncertainties such as noise, time-varying terms, and parameter variations. In system (1), the function \( g(x,u,t) \in \mathbb{R}^s \) represents the known nonlinearity function.

Prior to nonlinear observer design, the following assumption is made:

Assumption 1:

- There is no fault in the output, \( fo(t) = 0 \).
- \( CD \neq 0 \) is of full rank.
- \( D, S, \) and \( Ka \) are full column rank matrices.
- The output \( y(t) \) and its derivative \( \dot{y}(t) \) are available.

3. NONLINEAR UNKNOWN INPUT OBSERVER DESIGN (NUIO):

In this section the design procedure of an observer is considered. The conditions which eliminate some terms from the error system are given. These conditions are sufficient conditions for designing the observer.

3.1 Observer design

An unknown input observer (UIO) for the system (1) is designed such that its state estimation error vector \( e(t) \), approaches zero asymptotically, regardless of the presence of the unknown input term in the system.

3.2 NUIO Design

An full-order NUIO could be designed in the following form:

\[ \dot{z}(t) = Nz(t) + Ly(t) + Gu(t) + H^*Sg(\dot{x},u,t) + H^*D\hat{\mu}(t) + H^*Kafo(t) \]
\[ \dot{x}(t) = z(t) - Ey(t) \]

(2)

where \( z \in \mathbb{R}^n \) is the state observer, \( g(\dot{x},u,t) \), \( \hat{\mu}(t) \) and \( fo(t) \) are the estimate of \( g(x,u,t) \), \( \mu(t) \) and \( fo(t) \) respectively. Matrices \( N \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times p}, G \in \mathbb{R}^{n \times m}, H^* \in \mathbb{R}^{n \times n} \) with \( E \in \mathbb{R}^{n \times p} \), and \( \dot{x} \) is an estimate of \( x \). It is desired to design the observer such that \( \dot{x} \) eventually tends to \( x \). The error equation for system (1) and observer (2) is defined as follows:

\[ e(t) = x(t) - \dot{x}(t) = x(t) - z(t) + E\dot{y}(t) \]

(3)

By substituting the system output defined in (1) into the error equation (3), the time-derivative of the error equation (3) will have the following form:

\[ \dot{e}(t) = \dot{x}(t) - \dot{z}(t) + EC\dot{x}(t) \]

(4)

Then substituting (1) and (2) into (4), where \( H = I_n + EC \) and \( H = H^* \), yields

\[ \dot{e}(t) = Ne\dot{z}(t) + (HA - NH - LC)x(t) + (HB - G)\dot{u}(t) + HS(g(t,u,x) - g(\dot{x},u,t)) + HD(\mu(t) - \hat{\mu}(t)) + HKafo(t) - f_{ao}(t) \]

(5)

If the following conditions hold:

\[ HD = E_0 \]
\[ HB - G = 0 \]
\[ HA - NH - LC = 0 \]

(6)

Then the error equation (5) is presented as follows,

\[ \dot{e}(t) = Ne\dot{z}(t) + A_1HSe_\mu + A_0HD\dot{e}_\mu + A_1HK_aoe_\mu(t) \]

(9)

where \( e_\mu \) is an estimate of \( e_\mu \), \( e_\mu(t) = \mu(t) - \hat{\mu}(t) \) and \( e_\mu(t) = \mu(t) - \hat{\mu}(t) \) represent the nonlinearity, disturbance and actuator fault errors respectively. Matrices \( A_1, A_0, D, \) and \( A_1 \) are the design matrices.

To obtain matrices \( N, E, H, G \) and the observer gain \( L \) the following steps should be followed:

- First substitute \( H = I_n + EC \) into (6), the matrix \( E \) may be presented as follows:

\[ E = (E_0 - D)(CD)^+ + \Gamma(I - (CD)(CD)^+) \]

(10)

where \( (CD)^+ \) is the pseudo-inverse of \( (CD) \), \( \Gamma \) is an arbitrary matrix.

- Then by substituting \( E \) into \( H = I_n - EC \), matrix \( H \) will be obtained.

- After that substitute \( H \) into (7) to obtain matrix \( G \).

- Assume that \( (HA,C) \) is an observable pair and \( P_0 \) is the symmetric positive definite (s.p.d.) solution of the following algebraic Riccati equation, then the gain \( K \) is selected such that the matrix \( N = HA - KC \) is stable.

\[ (HA)^TP_0 + P_0(HA) - P_0CTR^{-1}CP_0 = -Q_0 \]

(11)

where \( Q_0 \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{p \times p} \) are arbitrary s.p.d. matrices. By selecting \( K = P_0CTR^{-1} \) the matrix \( N \) will be an stable matrix.

- Finally by substituting \( N, K \) and (6) into (8) the observer gain \( L \) will be obtained:

\[ L = HAE + K(I_p - CE) \]

(12)
4. STABILITY ANALYSIS OF THE ERROR SYSTEM

In this Section the behaviour of the error system (9) is considered. The desire is to obtain the stability of $e_p(t)$, $e_q(t)$ and $e_r(x, u, t)$, to show that each of these errors are stable, then the stability of error (9) is guaranteed.

4.1 The disturbance error $e_{\mu}(t)$ stability analysis

The unknown input (disturbance) $\mu(t)$, could be estimated from the available known signals as follows:

$$\dot{\tilde{\mu}}(t) = M_{11}\tilde{y}(t) + M_{12}\tilde{u}(t) + M_{13}\tilde{x}(t) + M_{14}\tilde{x}(t)$$
$$ + M_{15}g(x, u, t) + M_{16}u(t)$$

(13)

where matrices $M_{11} \in \mathbb{R}^{1 \times p}$, $M_{12} \in \mathbb{R}^{1 \times p}$, $M_{13} \in \mathbb{R}^{1 \times n}$, $M_{14} \in \mathbb{R}^{1 \times n}$, $M_{15} \in \mathbb{R}^{1 \times n}$ and $M_{16} \in \mathbb{R}^{1 \times n}$ will be designed to obtain the observer $\tilde{\mu}(t)$.

$\mu(t)$ is obtained from equation (1) as follows:

$$\mu(t) = D^+\dot{x}(t) - D^+Az(t) - D^+Bu(t) - D^+sg(x, u, t)$$
$$ - D^+K_\alpha f_a(t)$$

(14)

Using (13) and (14), the disturbance error $e_{\mu}(t)$ can be rewritten as:

$$e_{\mu}(t) = \mu(t) - \dot{\tilde{\mu}}(t)$$
$$= -(D^+ A + M_{11} C)x(t) + (D^+ - M_{12} C)\dot{x}(t)$$
$$ - (D^+ B + M_{16})u(t) - (D^+ + M_{15})g(x, u, t)$$
$$ - M_{13}\dot{x}(t) - M_{14}\dot{x}(t) + D^+K_\alpha f_a(t)$$

(15)

Since $e_\alpha(t) = x(t) - \dot{x}(t)$ then by substituting $\dot{x}(t)$ into (15), the following is obtained:

$$e_\mu(t) = -(D^+ A + M_{11} C - M_{13})x(t) + (D^+ - M_{12} C - M_{14})\dot{x}(t) - (D^+ B + M_{16})u(t) - (D^+ + M_{15})g(x, u, t)$$
$$ + M_{13}\dot{x}(t) + M_{14}\dot{x}(t) - D^+K_\alpha f_a(t)$$

(16)

Therefore, if the following conditions hold:

$$M_{14} = D^+ - M_{12} C$$
$$M_{13} = -D^+ A - M_{11} C$$
$$M_{16} = D^+ B$$
$$M_{15} = -D^+$$
$$D^+K_\alpha = 0$$

(17)

then (16) can be written as follows:

$$e_{\mu}(t) = -(D^+ A - M_{11} C)e_\alpha(t) + (D^+ - M_{12} C)e_\alpha(t)$$

(18)

Equation (18) shows that the disturbance error $e_{\mu}(t)$ is a function of $e_\alpha(t)$, and if $e_\alpha(t)$ is stable then $e_{\mu}(t)$ tends to zero.

4.2 The actuator fault error $e_\alpha(t)$ stability analysis

The actuator fault $f_a(t)$, could also be estimated from the available known signals as follows:

$$f_a(t) = F_{11}\tilde{y}(t) + F_{12}\tilde{u}(t) + F_{13}\tilde{x}(t) + F_{14}\tilde{x}(t)$$
$$ + F_{15}g(x, u, t) + F_{16}u(t)$$

(19)

where matrices $F_{11} \in \mathbb{R}^{r \times p}$, $F_{12} \in \mathbb{R}^{r \times p}$, $F_{13} \in \mathbb{R}^{r \times n}$, $F_{14} \in \mathbb{R}^{r \times n}$, $F_{15} \in \mathbb{R}^{r \times n}$ and $F_{16} \in \mathbb{R}^{r \times n}$ need to be designed. From (1), $f_a(t)$ is given by:

$$f_a(t) = K_\alpha^+\dot{x}(t) - K_\alpha^+ A\dot{x}(t) - K_\alpha^+ Bu(t) - K_\alpha^+ g(x, u, t)$$
$$ - K_\alpha^+ D_\mu(t)$$

(20)

Using (19) and (20), $e_\alpha(t)$, the actuator fault error is obtained:

$$e_\alpha(t) = f_a(t) - \dot{f}_a(t)$$
$$= -(K_\alpha^+ A + F_{11})x(t) + (K_\alpha^+ - F_{12})\dot{x}(t)$$
$$ - (K_\alpha^+ B + F_{16})u(t) - (K_\alpha^+ + F_{15})g(x, u, t)$$
$$ + F_{13}\dot{x}(t) - F_{14}\dot{x}(t) - K_\alpha^+ D_\mu(t)$$

(21)

From $e_\alpha(t) = x(t) - \dot{x}(t)$ substitute $\dot{x}(t)$ into (21), yields:

$$e_\alpha(t) = -(K_\alpha^+ A + F_{11} C - F_{13})x(t) + (K_\alpha^+ - F_{12} C - F_{14})\dot{x}(t)$$
$$ + F_{13}\dot{x}(t) + F_{14}\dot{x}(t) - K_\alpha^+ D_\mu(t)$$

(22)

Thus if the following conditions hold:

$$F_{14} = K_\alpha^+ - F_{12} C$$
$$F_{13} = -K_\alpha^+ A - F_{11} C$$
$$F_{16} = -K_\alpha^+ B$$
$$F_{15} = -K_\alpha^+$$
$$0 = K_\alpha^+ D$$

(23)

then the fault error (22) is presented as:

$$e_\alpha(t) = -(K_\alpha^+ A - F_{11} C)e_\alpha(t) + (K_\alpha^+ - F_{12} C)e_\alpha(t)$$

(24)

Equation (24) shows that the actuator fault error $e_\alpha(t)$ is a function of $e_\alpha(t)$, and tends to zero if $e_\alpha(t)$ goes to zero.

4.3 The nonlinearity error $e_\beta(t)$ stability analysis

The nonlinearity $g(x, u, t)$, could be estimated from the available known signals as follows:

$$g(x, u, t) = G_{11}\tilde{y}(t) + G_{12}\tilde{u}(t) + G_{13}\tilde{x}(t) + G_{14}\dot{x}(t)$$
$$ + G_{15}u(t) + G_{16}\dot{f}_a(t)$$

(25)

where matrices $G_{11} \in \mathbb{R}^{q \times p}$, $G_{12} \in \mathbb{R}^{q \times p}$, $G_{13} \in \mathbb{R}^{q \times n}$, $G_{14} \in \mathbb{R}^{q \times n}$, $G_{15} \in \mathbb{R}^{q \times n}$ and $G_{16} \in \mathbb{R}^{q \times q}$ need to be designed. From equation (1), $g(x, u, t)$ is obtained as:

$$g(x, u, t) = S^+\dot{x}(t) - S^+ A\dot{x}(t) - S^+ Bu(t) - S^+ K_\alpha f_a(t)$$
$$ - S^+ D_\mu(t)$$

(26)

Using (25) and (26), $e_\beta(x, u, t)$, the nonlinearity error is giving by:
Then if the following conditions hold:

\[ G_{13} = -S^+ A - G_{11} C \]
\[ G_{14} = S^+ - G_{12} C \]
\[ G_{15} = -S^+ B \]
\[ G_{16} = -S^+ K_a \]
\[ S^+ D = 0 \]

then the nonlinear error estimation \( e_g(x, u, t) \) is giving by:

\[
e_g(x, u, t) = -(S^+ A + G_{11} C) e_x(t) + (S^+ - G_{12} C) e_x'(t) + G_{16} e_a(t) + G_{16} e_a(t) - S^+ D e_\mu(t) \quad (29)
\]

Now by substituting \( e_a \) into (30), the error \( e_g \) is giving by:

\[
e_g(x, u, t) = -(S^+ A + G_{11} C - G_{16} F_{12} C) e_x(t) + (S^+ - G_{12} C - G_{16} F_{12} C) e_x'(t) + G_{16} e_a(t) - S^+ D e_\mu(t) \quad (31)
\]

Equation (31) shows that the nonlinearity error \( e_g(x, u, t) \) is a function of \( e_x(t) \), so it tends to zero if \( e_x(t) \) goes to zero.

Substituting (18), (24) and (31) into equation (9) yields:

\[
\dot{M}e_x(t) = \tilde{G}e_x(t) \quad (32)
\]

with

\[
\tilde{G} = N + A_d H K_a (K^+ A - F_{11} C) + A_d H D (S^+ + G_{13} C + G_{14} K^+ A - G_{16} F_{12} C) \quad (33)
\]

where \( I_n \) is an identity matrix of size \( n \), and,

\[
\tilde{G} = N + A_d H K_a (K^+ A - F_{11} C) + A_d H D (S^+ + G_{13} C + G_{14} K^+ A - G_{16} F_{12} C) \quad (34)
\]

Since \( \tilde{M} \) is nonsingular, \( A_g \in \mathbb{R}^{n \times n}, A_a \in \mathbb{R}^{n \times n} \) and \( A_d \in \mathbb{R}^{n \times n} \) should be selected to make \( \tilde{M}^{-1} \tilde{G} \) Hurwitz. If \( \tilde{M} \) was not singular and \( \tilde{M}^{-1} \) does not exist, then the singular value decomposition (SVD) technique needs to be used. Hence the state error equation (32) is represented as:

\[
\dot{e}_x(t) = M^{-1} \tilde{G} e_x(t) \quad (35)
\]

which satisfies the asymptotic stability of the state error estimation (9). In this method sufficient design degrees of freedom \( E_0, M_{ij}, F_{ij} \) and \( G_{ij} \) and nonuniqueness of \( A_g, A_d \) and \( A_a \) also provides extra design degrees of freedom.

Since error (32) is a function of \( e_x(t) \), the error is eventually stable and tends to zero.

\section{5. Example System}

Consider the mass-spring-damper (M-S-D) system in Figure 1 where two masses, springs and dampers are connected together serially (Roch-Cozalt et al. (2005)). \( x_1 \) and \( x_2 \) are the position and velocity of the first mass and \( x_3 \) and \( x_4 \) are the position and velocity of the second mass, respectively. \( A_{nl} \) is a nonsingular damping device whose damping force is \( F_{A_{nl}} = C_{nl} \text{sign}(x_2) \ln(1 + |x_2|) \), where \( C_{nl} \geq 0 \). On the other hand, an arbitrary and unknown force \( \mu(t) \) is applied on the second mass. \( u_1 \) and \( u_2 \) are the known input forces applied to masses 1 and 2, respectively.

The state equations of the system are presented as

\[
\dot{x} = \begin{pmatrix}
-k_1 + k_2 & 1 + b_2 & 0 & 0 \\
0 & m_1 & m_1 & m_1 \\
0 & 0 & m_2 & m_2 \\
0 & 0 & 0 & m_2
\end{pmatrix} u \\
+ \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} (g(x_2)) + \begin{pmatrix}
0.02 \\
0 \\
0 \\
0
\end{pmatrix} f_a(t) + \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} \mu(t)
\]

The actuator fault \( f_a(t) \) is any unwanted change in the length of the springs. The outputs of the system are \( y_1 = x_3 \) and \( y_2 = x_4 \). Using the following values for parameters: \( m_1 = 5 \text{kg}, m_2 = 1 \text{kg}, k_1 = 30 \text{N/m}, k_2 = 10 \text{N/m}, b_1 = 4 \text{Ns/m}, b_2 = 2 \text{Ns/m}, C_{nl} = 5 \text{N}, \) and \( \mu(t) = 0.04 \sin(t) + 2 \text{N} \), will give:

\[
A = \begin{pmatrix}
0 & 1.0 & 0 & 0 \\
0 & -8.0 & -1.2 & 2.0 & 0.4 \\
10.0 & 2.0 & -10.0 & -2.0 \\
0 & 0 & 0 & 1.0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 \\
0 & 0.20 & 0 \\
10.0 & 0.2 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \quad S = \begin{pmatrix}
-0.2 \\
0
\end{pmatrix}
\]

\[
K_a = \begin{pmatrix}
0.2 \\
0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & 0 & 0
\end{pmatrix}, \quad E_0 = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Fig. 1. The mass-spring-damper system.
The design matrices $A_d$, $A_a$ and $A_g$ are selected as follows to make $\tilde{M}^{-1}\tilde{G}$ hurwitz:

$$
A_d = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
A_a = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
A_g = \begin{pmatrix}
-0.20 & 0 & 0 & 0 \\
0 & -0.20 & 0 & 0 \\
0 & 0 & -0.20 & 0 \\
0 & 0 & 0 & -0.20
\end{pmatrix}.
$$

The robust nonlinear UIO has been designed for nonlinear systems and the stability of the error systems have been demonstrated. Sufficient existence conditions were derived for the nonlinear UIO. In this method sufficient design degrees of freedom ($E_0$, $M_{1i}$, $F_i$ and $G_{1i}$ for $i = 1, \ldots, n + 2$) and nonuniqueness of $A_g$, $A_f$ and $A_a$ also provides extra design degrees of freedom. This observer may be useful for state estimation and fault diagnosis for nonlinear systems. The effectiveness of the observer is shown with a numerical example. The simulation results show that the designed NUIO guarantees the asymptotically convergence of the state to zero in the presence of the unknown inputs.
Fig. 4. The behaviour of error estimation and the disturbance.

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