On input-to-state stability of delay difference equations

Rob H. Gielen ∗ Mircea Lazar ∗ Andrew R. Teel ∗∗

Abstract: Input-to-state stability (ISS) of delay difference equations (DDEs) subject to external disturbances is studied. Each delayed state of the DDE is considered as a subsystem of an interconnected system. Thus, it can be proven via a small-gain theorem for interconnected systems that a DDE is ISS if it admits an ISS-Lyapunov-Razumikhin function (ISS-LRF). As a by-product of this approach, an explicit construction of an ISS-Lyapunov-Krasovskii function is also obtained. Then, necessary conditions under which the Razumikhin method can be used to establish ISS are derived. An example, which establishes that not every DDE that is ISS admits an ISS-LRF, indicates the significance of the developed necessary conditions. Moreover, these conditions provide a non-trivial necessary condition for linear DDEs in particular.

Keywords: Time delay, difference equations, Lyapunov methods, interconnected systems.

1. INTRODUCTION

Systems affected by delay can be found in many applications within the control field, see the monograph (Kolmanovskii and Myshkis, 1999) for an excellent overview. As most modern controllers are implemented via a computer, difference equations have become an increasingly important modeling class, see, e.g., (Agarwal, 1992). As such, many systems affected by delay are modeled as a delay difference equation (DDE). In the context of Lyapunov theory, two different types of functions can be used to establish stability of DDEs. Firstly, the Lyapunov-Krasovskii function (LKF), see, e.g., (Hetel et al., 2008; Gielen et al., 2010), which is a function that depends only on the current state and the entire delayed state trajectory. Secondly, the Lyapunov-Razumikhin function (LRF) see, e.g., (Liu and Marquez, 2007; Gielen et al., 2010), which is a function that is required to decrease only if a certain condition on the delayed state trajectory and current state holds. Both methods have been used extensively for stability analysis and stabilizing controller synthesis for DDEs, see, e.g., (Fridman and Shaked, 2005; Gielen and Lazar, 2009) and the references therein.

Robust stability of DDEs subject to external disturbances has been scarcely studied so far. In (Chen et al., 2004) and (Zhang et al., 2010), based on the existence of a LKF, a controller with a finite ℓ2-gain guarantee was designed for switched linear DDEs. Another approach (Åström and Wittenmark, 1990) to study robust stability of DDEs subject to external disturbances is to augment the state vector with all delayed states that affect the current state, which yields a standard difference equation of higher dimension. Then, using the techniques presented in (Jiang and Wang, 2001) it can be shown that the augmented system is input-to-state stable (ISS) if it admits an ISS-Lyapunov function (ISS-LF). Similar to (Gielen et al., 2010, Lemma III.3), which addresses the nominal case, it then follows that if the augmented system is ISS, then the original DDE is also ISS. Moreover, the ISS-LF for the augmented system is an ISS-LKF for the DDE. The advantage of this approach is that the notion of ISS, in contrast to the ℓ2-gain analysis, allows for a nonlinear gain function from the disturbance to the system state. Unfortunately, as the ISS-LKF is a function of the current state and all delayed states, it becomes increasingly complex when the size of the delay increases. Moreover, the sublevel sets of the ISS-LKF do not provide an invariant set in the original state space but rather one in the higher dimensional state space corresponding to the augmented system. As an LRF is based on Lyapunov conditions that involve the DDE directly, a LRF does not suffer from the above-mentioned drawbacks, see (Gielen et al., 2010) for a detailed discussion. That a DDE is ISS if it admits an ISS-LRF was shown in (Liu and Hill, 2009).

In the present paper, each delayed state of a DDE is considered as a subsystem of an interconnected system. This leads to an alternative proof to the results presented in (Liu and Hill, 2009) and allows the derivation of several other relevant results as well. The above approach can be considered a counterpart for discrete-time systems to the results in (Teel, 1998), where the Razumikhin theorem for continuous-time systems, see, e.g., (Kolmanovskii and Myshkis, 1999), was proven using small-gain arguments. The contributions of the present paper are as follows. Firstly, a translation to the discrete-time do-

* R. H. Gielen and M. Lazar acknowledge the support of the Veni grant number 10230, awarded by the Dutch organizations STW and NWO. A. R. Teel acknowledges the support of the Air Force Office of Scientific Research grant FA9550-09-1-0203 and the National Science Foundation grants ECCS-0925637 and CNS-0720842.
main of a small-gain theorem for interconnected systems (Dashkovskiy et al., 2010) is used to prove that a DDE is ISS if it admits an ISS-LRF. The advantage of using the small-gain theorem is that an ISS-LKF, constructed from the ISS-LRF, is obtained as a by-product. Moreover, inspired by small-gain results for interconnected systems, an explicit construction of an ISS-LKF is also provided.

Then, secondly, necessary conditions for the existence of an ISS-LRF are derived. An example, which establishes that not every DDE that is ISS admits an ISS-LRF, indicates the significance of the developed necessary conditions. Moreover, these conditions provide a non-trivial necessary condition for linear DDEs in particular.

2. PRELIMINARIES

Let \( \mathbb{R}, \mathbb{R}^+, \mathbb{Z} \) and \( \mathbb{Z}_+ \) denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively. For every \( c \in \mathbb{R} \) and \( \Pi \subseteq \mathbb{R} \), define \( \Pi_{\geq c} := \{ k \in \Pi \mid k \geq c \} \) and similarly \( \Pi_{< c} \). Furthermore, \( \Pi_1 := \Pi \) and \( \mathbb{Z}_1 := \mathbb{Z} \cap \Pi_1 \).

For a vector \( x \in \mathbb{R}^n \), let \( |x|_1 \), \( i \in \mathbb{Z}_{[1,n]} \) denote the \( i \)-th component of \( x \) and let \( ||x|| \) denote an arbitrary norm. For a matrix \( A \in \mathbb{R}^{n \times n}, \) let \( |A| := \max \{ |Ax| \mid x \in \mathbb{R}^n, ||x|| \leq 1 \} \) denote its induced norm and let \( \lambda_{\max}(A) \) denote the spectral radius of \( A \).

Let \( x := \{ x(l) \}_{l \in \mathbb{Z}_+} \), with \( x(l) \in \mathbb{R}^n \), \( l \in \mathbb{Z}_+ \), denote an arbitrary sequence and define \( |x|_1 := \sup \{|x(l)|_1 \mid l \in \mathbb{Z}_+\} \). Furthermore, \( x_{[\cdot, i]} := \{ x(l) \}_{l \in \mathbb{Z}_1}, \) with \( c_1, c_2 \in \mathbb{Z}_+ \), denotes a sequence which is ordered monotonically with respect to the index \( l \in \mathbb{Z}_{[1,c_2]} \). Let \( S_\mathbb{R} := S \times \mathbb{R} \) for \( S \subseteq \mathbb{R} \).

2.1 Delay difference equations

Consider the DDE

\[
x(k+1) = f(x_{[k-h,k]), u(k)), \quad k \in \mathbb{Z}_+,
\]

where \( x_{[k-h,k]} \in (\mathbb{R}^n)^{h+1}, h \in \mathbb{Z}_+ \) is the maximal delay and \( u(k) \in \mathbb{R}^m \) is a disturbance input. Furthermore, \( f : (\mathbb{R}^n)^{h+1} \times \mathbb{R}^m \to \mathbb{R}^n \) is a function with the origin as equilibrium point, i.e., \( f(0_{(h+1)}, 0) = 0 \).

\[
x(k, x_{[k-h,k]), u_{[0,k-1]}])\quad \text{for} \quad k \geq h+1
\]

with initial condition \( x_{[-h,0]} \in (\mathbb{R}^n)^h \) and all \( u_{[0,k-1]} := \{ u(i) \}_{i \in \mathbb{Z}_{[0,k-1]}}, u(i) \in \mathbb{R}^m \).

Definition 1. The DDE (1) is called input-to-state stable (ISS) if there exist a \( \beta \in \mathcal{KL} \) and a \( \gamma \in \mathcal{KL} \), such that

\[
\max \{ \beta(||x_{[-h,0]}, u_{[0,k-1]}||), \gamma(||u_{[0,k-1]}||) \},
\]

for all \( k \in \mathbb{Z}_{\geq 1}, x_{[-h,0]} \in (\mathbb{R}^n)^h \) and all \( u_{[0,k-1]} := \{ u(i) \}_{i \in \mathbb{Z}_{[0,k-1]}}, u(i) \in \mathbb{R}^m \).

Note that ISS (as it is defined in Definition 1) is a global property which is often referred to as global ISS.

2.2 A small-gain theorem for interconnected systems

Consider a set of \( N \in \mathbb{Z}_{\geq 2} \) interconnected systems. The dynamics of the \( i \)-th subsystem, \( i \in \mathbb{Z}_{[1,N]} \), is given by

\[
z_i(k+1) = g_i(z_i(k), \ldots, z_N(k), u(k)), \quad k \in \mathbb{Z}_+,
\]

where \( z_i(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^m \) is a disturbance input and \( g_i : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \) \( i \in \mathbb{Z}_{[1,N]} \), is a function with the origin as equilibrium point, i.e., \( g_i(0, \ldots, 0) = 0 \).

Note that it can be assumed without any restriction to the generality of the results that all systems have the same input. The interconnected system is described via the state vector \( z := [z_1^T, \ldots, z_N^T]^T \in \mathbb{R}^n \), which yields

\[
z(k+1) = g(z(k), u(k)), \quad k \in \mathbb{Z}_+,
\]

where \( n := \sum_{i=1}^N n_i \) and \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is obtained from the functions \( g_i, i \in \mathbb{Z}_{[1,N]} \).

Let \( \lambda_1, \lambda_2 \in \mathbb{K} \), \( \mu_i \in \mathbb{K} \cup \{0\} \), for all \( i \in \mathbb{Z}_{[1,N]} \), and let \( \gamma_{ij} \in \mathbb{K} \cup \{0\} \) for all \( i, j \in \mathbb{Z}_{[1,N]} \). Then, consider a set of functions \( W_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+ \), \( j \in \mathbb{Z}_{[1,N]} \), that satisfy

\[
\lambda_1 \gamma_{ij} ||z_j||_1 \leq W_j(z_j) \leq \lambda_2 \gamma_{ij} \{ ||z_j||_1 \}, \quad \forall z_j \in \mathbb{R}^n.
\]

Definition 2. Let \( \gamma_{ij}(s) < s \) for all \( s \in \mathbb{R}_{=0} \). A function \( W_i, i \in \mathbb{Z}_{[1,N]} \), that satisfies

\[
W_i(g(z_1, \ldots, z_N, u)) \leq \max_{j \in \mathbb{Z}_{[1,N]}} \{ \max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{ij}(W_j(z_j)) \mu_i ||u||_1 \},
\]

for all \( z_j \in \mathbb{R}^n, j \in \mathbb{Z}_{[1,N]} \), and all \( u \in \mathbb{R}^m \) is called an ISS-Lyapunov function (ISS-LF) for subsystem (2).

Remark 2. Requires not that \( W_i \) is an ISS-LF for system (2), i.e., for \( z_j = 0 \) for all \( j \neq i \), but also that the effect of the other subsystems on subsystem (2) can be bounded via \( \gamma_{ij} \). Next, a nonlinear small-gain theorem for interconnected difference equations is established.

Theorem 3. Suppose that all subsystems \( (2), i \in \mathbb{Z}_{[1,N]} \), admit an ISS-LF. Furthermore, suppose that for all \( y \in \mathbb{R}^N \setminus \{0\} \), there exists a \( d(y) \in \mathbb{Z}_{[1,N]} \) such that

\[
\max_{j \in \mathbb{Z}_{[1,N]}} \gamma_{ij}(d(y)) < |y|_1.
\]

Then, the following claims hold:

(i) There exist \( \sigma_i \in \mathbb{K} \), \( i \in \mathbb{Z}_{[1,N]} \), such that

\[
W(z) = \max_{i \in \mathbb{Z}_{[1,N]}} \sigma_i^{-1}(W_i(z_i))
\]

is an ISS-LF for system (3);

(ii) System (3) is ISS.

The proof of Theorem 3 can be obtained mutatis mutandis from the proof of Corollary 5.7 in (Dashkovskiy et al., 2010) and is omitted here for brevity. Alternative small-gain theorems for interconnected difference equations, which also parallel the continuous-time results in (Dashkovskiy et al., 2010), can be found in, e.g., (Jiang et al., 2008; Liu et al., 2010).

3. MAIN RESULT

In this section, firstly, ISS of DDEs is studied using the Razumikhin method and Theorem 3. Secondly, necessary conditions for the existence of an ISS-LRF are presented.
The DDE (1) can be considered as an interconnection of difference equations such as (3). Therein, each delayed state is the state of one of the subsystems (2). Thus, conditions similar to those in Theorem 3 can be used to establish stability of the DDE (1). To this end, we define the concept of an ISS-LRF and ISS-LKF. Let \( \alpha_1, \alpha_2 \in \mathcal{K}_{\infty} \) and let \( \nu \in \mathcal{K} \).

**Definition 4.** A function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) which admits a \( \rho \in \mathcal{K}_{\infty} \) such that
\[
\rho(s) < s, \quad \forall s \in \mathbb{R}_{>0},
\]
\[
\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||), \quad \forall x \in \mathbb{R}^n,
\]
\[
V(f(x_{i-0}, u)) \leq \max\{ \max_{\theta \in [\alpha_i, h]} \rho \circ V(x(\theta)), \nu(||u||) \},
\]
follows that, indeed, \( V \) is an ISS-LF for all subsystems (2), \( i \in \mathbb{Z}_{[1, h+1]} \). Furthermore, consider any \( y \in \mathbb{R}_+^n \setminus \{0\} \).

\[ \text{If} \quad [y_1] \geq \max_{i \in \mathbb{Z}_{[2, h+1]}} |y_i|, \text{then} \quad (6a) \text{yields that} \]
\[ \max_{i \in \mathbb{Z}_{[2, h+1]}} \rho([y_1]) < [y_1]. \]

Moreover, if \( [y_1] < \max_{i \in \mathbb{Z}_{[2, h+1]}} |y_i| \), then there exists a \( i(y) \in \mathbb{Z}_{[2, h+1]} \) such that
\[ [y_{i(y)}] < [y_1]. \]

As such (4) holds and hence the hypothesis of Theorem 3 is satisfied. It then follows from claim (ii) of Theorem 3 that the DDE (1) is ISS, which establishes claim (i).

Furthermore, it follows from claim (i) of Theorem 3 that there exist \( \sigma_i \in \mathcal{K}_{\infty}, \alpha, \bar{\alpha} \in \mathbb{Z}_{[1, h+1]} \), such that the function \( (8) \) is an ISS-LF for the interconnected system (3) with augmented-state vector \( z \) and hence the function \( (8) \) is an ISS-LKF for the DDE (1) with
\[
\bar{\rho}(s) := \max_{i \in \mathbb{Z}_{[1, h+1]}} \max_{j \in \mathbb{Z}_{[1, h+1]}} \sigma_{i,j} \circ \gamma_{i,j} \circ \sigma_j(s),
\]
\[
\bar{\nu}(s) := \sigma_{1,1}^{-1} \circ \nu(s),
\]
\[
\bar{\alpha}_1(s) := \min_{i \in \mathbb{Z}_{[1, h+1]}} \sigma_{1,1}^{-1} \circ \alpha_1(s),
\]
\[
\bar{\alpha}_2(s) := \max_{i \in \mathbb{Z}_{[1, h+1]}} \sigma_{1,1}^{-1} \circ \alpha_2(s),
\]
which completes the proof.

\[ \text{The proof of Theorem 6 shows that the Razumikhin method for DDEs is an exact application of the small-gain theorem for interconnected difference equations. Therein,} \]
\[ (6a) \text{guarantees fulfillment of the small-gain condition (4).} \]

**Remark 7.** A generalization of Theorem 6 to delay difference inclusions rather than DDEs is possible under the assumption that \( (6c) \) holds for all trajectories of the delay difference inclusion.

**Remark 8.** If a DDE is ISS, it is also asymptotically stable (AS), i.e., for \( u(k) = 0 \) for all \( k \in \mathbb{Z}_+ \). Therefore, Theorem 6 also provides a different proof for the results in (Gielen et al., 2010, Theorem III.8) and (Liu and Marquez, 2007, Theorem 3.2). Moreover, the results therein are generalized in the sense that \( \rho \) is a nonlinear function rather than a linear function. Further results in this paper have similar implications when AS is of concern.

**How to obtain the functions \( \sigma_i \in \mathcal{K}_{\infty}, \alpha, \bar{\alpha} \in \mathbb{Z}_{[1, h+1]} \), in Theorem 3 is a nontrivial problem that was discussed in (Dashkovskiy et al., 2010, Section 8). As the construction of these functions, although complicated, solvable, Theorem 6 is constructive. Next, inspired by (Dashkovskiy et al., 2010, Section 8.4), a simpler and explicit construction of an ISS-LKF is presented.

**Theorem 9.** Suppose that the DDE (1) admits an ISS-LRF, i.e., \( V \). Then, an ISS-LKF for (1) is given by
\[
\tilde{V}(x_{i-0}) := \max_{\theta \in [\alpha_i, h]} \rho_{i+1} \circ V(x(\theta)),
\]
where \( \rho_i(s) := \frac{\rho(s) + \rho_{i+1}}{i+1}, \alpha, \bar{\alpha} \in \mathbb{Z}_{[1, h]} \), and \( \rho_{h+1}(s) = s \).

**Proof.** Note that, by definition, \( \rho_i \in \mathcal{K}_\infty \) for all \( i \in \mathbb{Z}_{[1, h+1]} \). Next, it is established that
\[ \rho(s) < \rho_1(s) < \ldots < \rho_s(s) < \rho_{h+1}(s) = s. \]

As \( \rho(s) < s \) it holds that
\[ \rho(s) < s = (i+1)^2s - (i+2)is, \]
for all $s \in \mathbb{R}_{>0}$. The above is equivalent to

$$(i + 2)(\rho(s) + is) < (i + 1)(\rho(s) + (i + 1)s),$$

which implies that $\rho_i(s) < \rho_{i-1}(s)$, for all $i \in \mathbb{Z}_{[1,\infty)}$ and all $s \in \mathbb{R}_{>0}$. Obviously, $\rho(s) < \frac{\alpha_1 s}{\alpha_1 +1}$, which establishes that (10) holds. Next, let $\pi_i(s) := \rho_{i-1} \circ \rho_i^{-1}(s)$, $i \in \mathbb{Z}_{[1,\infty)}$, for all $s \in \mathbb{R}_{>0}$ with $\rho(s) = \rho(s)$. Then, as $\rho_{i-1}(s) < \rho(s)$ it follows that $\pi_i(s) = \rho_{i-1} \circ \rho_i^{-1}(s) < \rho_i \circ \rho_i^{-1}(s) = s$. Letting $\pi(s) := \max_{i \in \mathbb{Z}_{[1,\infty)}} \pi_i(s)$, yields that $\pi(s) < s$ for all $s \in \mathbb{R}_{>0}$.

Consider any $x_{-h,0} \in (\mathbb{R}^n)^{h+1}$ and any $u \in \mathbb{R}^m$. Then,

$$V_-(x_{-h,0}, f(x_{-h,0}, u)) = \max_{\theta \in \mathbb{Z}_{[-h,0]}} \max_{\theta \in \mathbb{Z}_{[-h,0]}} \rho_{h+\theta} \circ V(x(\theta)),$$

which implies that $\rho_{i}(s) < \rho_{i}(s)$ for all $i \in \mathbb{Z}_{[1,\infty)}$, all $z(0) \in \mathbb{R}^n$, all $u(0,k-1) := \{u(i)\}_{i \in \mathbb{Z}_{[0,k-1]}}$, $u(i) \in \mathbb{R}^m$, and all $k \in \mathbb{Z}_{\geq 1}$. Letting $\gamma_u(s) := \alpha^{-1}_{2} \circ \nu(s)$ it follows that

$$\rho_i(s) := \beta(r, s) := 2cK^s, \quad \gamma_u(s) := 2c\|B\|s = \frac{2c\|B\|s}{1 - \kappa}.$$

As for all $(x(0), x(-1)) \in \mathbb{R} \times \mathbb{R}$, it holds that $\|x(0)\| \leq \|x(-1)\|$ and there exists a $c_1 \in \mathbb{R}_{>0}$ such that $\|x(0)\| \leq c_1 \|x_{-1,0}\|$. It follows that the DDE (11) is ISS with $\beta(r, s) := 2cK^s$ and $\gamma_u(s) := 2c\|B\|s = \frac{2c\|B\|s}{1 - \kappa}$.

Next, suppose that the DDE (11) admits an ISS-LRF $V : \mathbb{R} \to \mathbb{R}$ with corresponding functions $\nu$ and $\rho$. Let $x(0) = 1, x(-1) = 0$ and $u(0) = 0$. From (11) it follows that $x(1) = 1$. As $V$ is an ISS-LRF, (6c) yields that $V(x(1)) = V(1) \leq \max(\rho \circ V(x(0)), \rho \circ V(x(-1)), \nu(0)) = \rho \circ V(1)$.

Obviously, (6a) then implies that a contradiction is reached. As the functions $V$ and $\rho$ were chosen, under the assumption that $\rho(s) < s$ for all $s \in \mathbb{R}_{>0}$, arbitrarily, it follows that the DDE (11) does not admit an ISS-LRF.

Proposition 12 shows that not every DDE that is ISS admits an ISS-LRF. Hence, it makes sense to search for further necessary conditions for the existence of an ISS-LRF. In order to apply the small-gain theorem when studying stability of the interconnected system (3), all subsystems (2) are required to be ISS. Indeed, this is the first assumption in Theorem 3. In light of this observation and as Theorem 6 was proven directly via Theorem 3, a similar requirement is to be expected for the DDE (1). Therefore, let $T := \{0,1\}$ and $S := \{-1,0,1\}$. Then, consider the following family of systems

$$z(k+1) = h_3(z(k), u(k)), \quad k \in Z_+,$$

where $z(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$,

$$h_3(z, u) := f([\delta_1 z, \ldots, [\delta_{h+1} z, u)],$$

and $\delta \in T^{h+1}$ or $\delta \in S^{h+1}$. To illustrate the family of systems (14), consider the DDE (11). Then, $h_3(z, u) = 1.5z + u$ and $h_3_3(z, u) = -0.5z + u$ for $\delta_1 = [-1,1]$ and $\delta_3 = [1,0]^T$, respectively. Next, necessary conditions for the existence of an ISS-LRF are established. To the best of the authors’ knowledge this is the first time that necessary conditions, apart from the obvious condition that the DDE (1) is ISS, for the existence of an ISS-LRF are presented. Note that these necessary conditions are of considerable value for DDEs without disturbances as well.

Proposition 13. Suppose that the DDE (1) admits an ISS-LRF. Then, the family of systems (14) obtained from the DDE (1) is ISS for all $\delta \in T^{h+1}$.

Proof. Consider any $\delta \in T^{h+1}$. As (6c) holds for all $x_{[-h,0]} \in (\mathbb{R}^n)^{h+1}$, it also holds for $x_{[-h,0]} := \{[\delta_1 z, \ldots, [\delta_{h+1} z, u] \} \forall z \in \mathbb{R}^n$. Therefore, it follows from (6c) that $V(h_3(z, u)) = V(f(x_{-h,0}, u)) \leq \max(\rho \circ V(z), \mu(\|u\|))$.

Next, let $\{z(k, 0), u_{[0,k-1]}\}_{k \in \mathbb{Z}_{\geq 1}}$ denote a trajectory of system (14) from initial condition $z(0) \in \mathbb{R}^n$ with disturbance $u_{[0,k-1]} := \{u(i)\}_{i \in \mathbb{Z}_{[0,k-1]}} u(i) \in \mathbb{R}^m$. Applying the above inequality recursively and using (6b) that $\|z(k, 0), u_{[0,k-1]}\| \leq \max(\alpha_{1} \circ \rho \circ \alpha ||z(0)||, \alpha_{1} \circ \nu(\|u_{[0,k-1]}\|))$, for all $z(0) \in \mathbb{R}^n$, all $u_{[0,k-1]} := \{u(i)\}_{i \in \mathbb{Z}_{[0,k-1]}} u(i) \in \mathbb{R}^m$, and all $k \in \mathbb{Z}_{\geq 1}$. Letting $\gamma_u(s) := \alpha^{-1}_{1} \circ \nu(s)$ and $\gamma_u(s) := 2c\|B\|s = \frac{2c\|B\|s}{1 - \kappa}$.
\[ \beta(r, s) := \alpha^{-1} \circ \rho^s \circ \alpha_2(r) \] it follows, since \( \rho(s) < s \) for all \( s \in \mathbb{R}_{>0} \), that \( \beta \in KL \) and, furthermore, that \( \gamma_u \in K \).

Therefore, system (14) is ISS with \( \beta \in KL \) and \( \gamma_u \in K \). \( \square \)

Under an additional assumption the result of Proposition 13 can be strengthened.

**Assumption 14.** The equality \( h_3(-z, u) = -h_3(z, u) \) or \( h_3(z, -u) = -h_3(z, u) \) holds for all \( \delta \in S^{h+1}, z \in \mathbb{R}^n \) and all \( u \in \mathbb{R}^m \).

**Proposition 15.** Suppose that Assumption 14 holds. Furthermore, suppose that the DDE (1) admits an ISS-LRF. Then, the family of systems (14) obtained from the DDE (1) is ISS for all \( \delta \in S^{h+1} \).

**Proof.** Consider any \( \delta \in S^{h+1} \). As (6c) holds for any \( x_{\cdot \cdot \cdot \cdot 0} \in \mathbb{R}^{n+1} \), it also holds for \( x_{\cdot \cdot \cdot \cdot 0} := \{ [\delta_1 z, \ldots, [\delta_{h+1} z] \} \) for any \( z \in \mathbb{R}^n \). Therefore, it follows from (6c) and Assumption 14 that

\[
V(h_3(z, u)) \leq \max \{ \rho \circ \max \{ V(z) \} + \mu(\|u\|) \}
\]

Applying the above inequalities recursively yields

\[
V(z(k), z(0), u_{\cdot \cdot \cdot \cdot 0}) \leq \max \{ \rho^k \circ \max \{ V(z(0)) \} + \mu(\|u_{\cdot \cdot \cdot \cdot 0}\|) \}
\]

for all \( k \in Z_{\geq 1} \). Then, using (6b) yields that

\[
\|z(k) - z(0), u_{\cdot \cdot \cdot \cdot 0}\| \leq \max \{ \alpha^{-1} \circ \rho \circ \alpha_2(\|z\|), \alpha^{-1} \circ \nu(\|u_{\cdot \cdot \cdot \cdot 0}\|) \}
\]

for all \( z(0) \in \mathbb{R}^n \), all \( u_{\cdot \cdot \cdot \cdot 0} := \{ (u(i))_{1 \leq i \leq h, u_{\cdot \cdot \cdot \cdot 0}, u(i) \in \mathbb{R}^m, and all k \in Z_{\geq 1} \}. Letting \( \gamma_u \in KL \) and, furthermore, that \( \gamma_u \in K \).

Therefore, system (14) is ISS with \( \beta \in KL \) and \( \gamma_u \in K \).

Assumption 14 holds for, among many others, linear and cubic functions. Therefore, for linear DDEs, i.e., \( x(k+1) = \sum_{\theta=0}^{\theta_h} A_\theta x(k + \theta) + u(k) \), it follows from Proposition 15 that the linear DDE admits an ISS-LRF only if

\[
\lambda_{\max} \left( \sum_{\theta_h=0}^{\theta_h} [\delta_{\theta_h+h} A_\theta] \right) < 1, \quad \forall \delta \in S^{h+1}
\]

Indeed, for the DDE (11), \( \lambda_{\max}(A_0 - A_{-1}) = 1.5 \) and hence it can also be concluded from Proposition 15 that the DDE (11) does not admit an ISS-LRF.

Many methods to establish stability of standard difference equations are available, see, e.g., (Agarwal, 1992). As such, before pursuing the construction of an ISS-LRF, the necessary conditions provided by Proposition 13 or Proposition 15 should be checked. Only if these tests succeed, it makes sense to employ constructive methods to obtain an ISS-LRF.

4. ILLUSTRATIVE EXAMPLE

To illustrate the application of the derived results, consider the following DDE, which is inspired by Example 6.1 in (Liu and Hill, 2009), i.e.,

\[
x(k+1) = Ax(k) + \tilde{f}(x(k), x(k-1)) + u(k), \quad (15)
\]

where \( \{x(k), x(k-1)\} \in \mathbb{R}^3 \times \mathbb{R}^3, u(k) \in \mathbb{R}^3, and k \in Z_+ \).

Furthermore, \( A := \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0 & 0.5 \end{bmatrix} \) and

\[ \tilde{f}(x(k), x(k-1)) := \begin{bmatrix} 0.4(x(k-1)) \\ 0.4(x(k-1)) \sin(x(k))\frac{1}{1+\varepsilon} \\ 0.4(x(k-1)) \cos(x(k))\frac{1}{1+\varepsilon} \end{bmatrix}. \]

As observed in Section 3.2, before pursuing the construction of an ISS-LRF for the DDE (15), one should first establish if the difference equation (14) is ISS for all \( \delta \in T^2 \). Therefore, consider the following functions

\[
\delta_1(z, u) = u, \quad \delta_2(z, u) = Az + u, \quad \delta_3(z, u) = \begin{bmatrix} 1 \sin(z) \\ z \cos(z) \end{bmatrix}, \quad \delta_4(z, u) = \begin{bmatrix} 1 + \varepsilon \end{bmatrix} z, \quad \delta_5(z, u) = \begin{bmatrix} 1 \sin(z) \\ z \cos(z) \end{bmatrix} + u,
\]

where \( \delta_1 = [0 \ 0]^T, \delta_2 = [1 \ 1]^T, \delta_3 = [0 \ 1]^T, \delta_4 = [1 \ 1]^T. First, the system (14) for \( \delta_1 \) obviously is ISS. Secondly, as \( \lambda_{\max}(A) < 1, \) ISS of system (14) for \( \delta_2 \) can be established, using a reasoning similar to the one used in Example 11. Thirdly, consider the candidate ISS-LF \( V(z) = \|z\|^2 \) for the system corresponding to \( \delta_3 \). As \( 1 + \sin(2z)^2 + \|z\|^2 \geq 1 \) for all \( z \in \mathbb{R}^3 \) and using that \( r + s \leq \max \{ (1 + \varepsilon)r, (1 + \varepsilon)s \} \), for some constant \( \varepsilon \in \mathbb{R}_{>0} \) and all \( r, s \in \mathbb{R}_+ \), it follows that

\[
V(z(k+1)) \leq \left( \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \right) \|z\|^2 + \|u(k)\|^2 \leq \left( \begin{bmatrix} 1 + \varepsilon \end{bmatrix} \right) \|u(k)\|^2.
\]

Thus, \( \delta_3 \), consider the candidate ISS-LF \( V(z) = \|z\|^2 \). Using the same reasoning as above, it follows that

\[
V(z(k+1)) \leq \max \{ (\|A\|^2 + 0.4)(\|z\|^2 + \|u(k)\|^2 \leq \max \{ kV(z(k)), 1 + \varepsilon \|u(k)\|^2 \}
\]

where \( k = (\|A\|^2 + 0.4)(1 + \varepsilon) \) Again, for any \( \varepsilon \) small enough it follows that \( k \in \mathbb{R}_{(0,1)} \) and hence the function \( V \) is an ISS-LF for the system corresponding to \( \delta_3 \), which establishes that the system is ISS. Similarly, \( \delta_3 \), consider the candidate ISS-LF \( V(z) = \|z\|^2 \). Using the same reasoning as above, it follows that

\[
V(z(k+1)) \leq \max \{ kV(z(k)), 1 + \varepsilon \|u(k)\|^2 \}
\]

where \( k = (\|A\|^2 + 0.4)(1 + \varepsilon) \) Again, for any \( \varepsilon \) small enough it follows that \( k \in \mathbb{R}_{(0,1)} \) and hence the function \( V \) is an ISS-LF for the system corresponding to \( \delta_4 \), which establishes that the system is ISS. As \( V(z) = \|z\|^2 \) is an ISS-LF for the family of systems (14) for all \( \varepsilon \in T^2 \), it is a good idea to investigate if \( V(x) = \|x\|^2 \) is an ISS-LF for the DDE (15). Using that \( r + s \leq \max \{ (1 + \varepsilon)r, (1 + \varepsilon)s \} \), for some \( \varepsilon \in \mathbb{R}_{>0} \) and all \( r, s \in \mathbb{R}_+ \), yields

\[
x(k+1) \leq \sum_{\theta=0}^{\theta_h} A_\theta x(k + \theta) + \tilde{f}(x(k), x(k-1)) + u(k) \leq \max \{ kV(z(k)), 1 + \varepsilon \|u(k)\|^2 \}
\]

where \( k = (\|A\|^2 + 0.4)(1 + \varepsilon) \) Therefore, it follows that for any \( \varepsilon \in \mathbb{R}_{(0,1)} \), \( V(x) = \|x\|^2 \) is an ISS-LF for the DDE (15) with \( \rho(s) = \kappa s \) and \( \nu(s) = \frac{1+\varepsilon}{s} \), for all \( s \in \mathbb{R}_+ \), and hence the DDE (15) is ISS. Moreover, it also
Fig. 1. The values of the ISS-LRF (——) and the ISS-LKF (− − −) as a function of time for the DDE (15).

follows from Theorem 9 that (9) is an ISS-LKF for the DDE (15) with \( \rho_1(s) = \frac{s+1}{2} \) and \( \rho_2(s) = s \). Indeed, let \( \varepsilon = 0.04 \) and hence \( \kappa = 0.9984 \) and consider the function 
\[
V(x_{(k-1,k)}) = \max\left\{ \frac{s+1}{2} ||x(k-1)||_2, ||x(k)||_2 \right\}.
\]

\[
\dot{V}(x_{[k,k+1]}) = \max\left\{ \frac{\kappa + 1}{2} ||x(k)||_2, ||x(k+1)||_2 \right\}
\]

As \( \max\left\{ \frac{s+1}{2}, \frac{2\kappa}{s+1} \right\} \in \mathbb{R}_{(0,1)} \) it follows that \( \dot{V} \) is an ISS-LKF for the DDE (15).

Figure 1 shows the values of the ISS-LRF and the ISS-LKF as a function of time for a simulation of the DDE (15) from \( x(-1) = [0 0 0.7]' \) and \( x(0) = [-0.1 0 -0.9]' \) and with \( u(k) = 0 \) for all \( k \in \mathbb{Z}_+ \). Note that the ISS-LKF is strictly decreasing over time while the ISS-LRF is not.

5. CONCLUSION

ISS of DDEs subject to disturbances was studied. It was established that the Razumikhin method for DDEs is an exact application of the small-gain theorem for interconnected difference equations. Inspired by small-gain results for interconnected systems, an explicit construction of an ISS-LKF was provided. Furthermore, the use of the small-gain theorem naturally led to the derivation of a set of necessary conditions for the existence of an ISS-LRF. An example, which establishes that not every DDE that is ISS admits an ISS-LRF, indicates the significance of the developed necessary conditions. This example also indicates that further investigations are required before a converse ISS-Lyapunov theorem, such as Theorem 1 in (Jiang and Wang, 2001), can be proven for the ISS-LRF. Establishing such a converse Lyapunov theorem for the ISS-LRF makes the topic of future research.

REFERENCES


3377