Achieving consensus and synchronization
by adapting the network topology ⋆

Pietro DeLellis ∗† Mario diBernardo ∗‡ Maurizio Porfiri ∗∗

∗ Department of Systems and Computer Engineering, University of
Naples Federico II, 80125, Naples, Italy.
† Also with the Department of Engineering Mathematics, University of
Bristol, Bristol BS8 1TR, UK.
‡ Department of Mechanical and Aerospace Engineering, Polytechnic
Institute of New York University, Brooklyn, NY 11201, USA.

Abstract: In this paper, we present a novel control strategy for achieving consensus via network
evolution. Namely, we associate to each edge a multi-dimensional dynamical system forced
by the state mismatch of its nodes. We consider the case in which the network evolution is
driven by a multi-well potential, whose wells define the possible asymptotic levels of interactions
between each pair of nodes. We show that the proposed network evolution supports asymptotic
consensus while leading to an emerging steady-state weighted topology. Moreover, we illustrate
how a proper tuning of the potential parameters can be used to tailor the properties of this
emerging topology. Finally, numerical simulations confirm the theoretical predictions as well as
the viability of this approach for synchronizing nonlinear dynamical systems.

1. INTRODUCTION

Networked control systems have attracted a tremendous interest in the Applied Science and Engineering. Synchroni-
ization and consensus in complex networks of dynamical systems have been extensively studied (Bertsekas and Tsitsi-
[2006], Yu et al. [2009]), due to the broad range of possible applications in a diverse set of disciplines, see for example
Savkin [2004], Xiao and Boyd [2004], Cheah et al. [2009].

Let \( G = \{ N, E \} \) be a graph defined by the set of \( N \) nodes \( N \) and the set of \( M \) edges \( E \). A classical model that is
extensively used to describe a complex network of \( N \) nodes is

\[
\dot{x}_i = f(x_i, t) - c \sum_{j=1}^{N} \ell_{ij} \Gamma x_j, \quad i = 1, \ldots, N. \tag{1}
\]

Here, \( x_i \in \mathbb{R}^n \) is the state of node \( i \), \( f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \)
is the vector field describing the dynamics of an isolated node, \( c \in \mathbb{R} \) is the coupling strength, \( \Gamma \) is the inner coupling
matrix, and \( \ell_{ij} \) is the \( ij \)th element of the Laplacian matrix \( \mathcal{L} \) associated to the graph \( G \) describing the network
topology.

The classical consensus problem (Bertsekas and Tsitsiklis [1989], Olfati-Saber and Murray [2004], DeGroot [1974]) is defined by

\[
\dot{x}_i = u_i, \quad i = 1, \ldots, N, \tag{2}
\]

\[
u_i = -c \sum_{j=1}^{N} \ell_{ij} x_j, \quad i = 1, \ldots, N \tag{3}
\]

where \( u_i \) is the linear communication protocol, can be written in the format (1) by selecting \( f(x_i, t) = 0 \) and
\( \Gamma = 1 \).

In the above models, the network is considered static. While the state of the nodes evolve, both the topology of the
interconnections and the strength of the coupling remain fixed. In many practical applications, including wireless sensor networks (Cerpa and Estrin [2004], Xu et al. [2003]), this assumption is too restrictive as the
topology of interconnections can change in time to adapt to environmental changes or to minimize a cost function.

In the recent literature on synchronization over time-varying networks, topological changes are generally inde-
pendent of the states of the coupled dynamical systems, see for instance Porfiri and Fiorilli [2009], Porfiri et al.
[2008], Belykh et al. [2004]. In this paper, we tackle the problems of consensus and synchronization from a different
perspective. Following the pioneering work of Tseng and Siljak [1995], Šiljak [2008], we aim at defining the network
topology as a dynamical system itself. We model each edge in the network as a multi-dimensional dynamical system.

In particular, we consider the case in which the network evolution is driven by a multi-well potential, whose wells
define the possible asymptotic levels of interactions between each pair of network nodes. The self-organization
of the network topology is used as a control technique to synchronize/lead to consensus the network.

The idea of using network topology as a control input can also be found in Liu et al. [2009]. Here, the controlled
network is modeled as

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Fig. 1. Consensus via a bistable potential. Evolution of the states (top) and coupling gains (bottom).

\[ \dot{x}_i = f(x_i, t) + c \sum_{j=1}^{N} a_{ij} \Gamma (x_j - x_i), \quad i = 1, \ldots, N, \]  
\[ u_i = -c \sum_{j=1}^{N} a_{ij} \Gamma (x_j) + \sum_{j=1}^{N} b_{ij} \Gamma (x_j), \quad i = 1, \ldots, N, \]

where \( A = (a_{ij}) \) corresponds to \(-L\) in (1), \( A^c = (a_{ij}^c) \) represents the edges to be removed, and \( B = (b_{ij}) \) the control edges added to the network. In Liu et al. [2009], \( a_{ij} \) is chosen solving an off-line optimization problem. In our work, instead, we propose to control the network topology on-line, without the need of solving off-line an optimization problem. The motivation is overcoming the main limitation of the edge snapping mechanism proposed in DeLellis et al. [2010b]. In this strategy, the only possible levels of interactions at steady-state are zero (the connection is absent) or one (the connection is activated).

The focus of this paper is on consensus problems. We show that consensus can be achieved as a steady-state weighted topology asymptotically emerges. Moreover, we illustrate how a proper tuning of the potential can tailor the properties of this emerging topology. Finally, the viability of this approach is numerically assessed against a synchronization problem.

2. EDGE SNAPPING CONTROL

In our case, the controlled network can be modeled as follows

\[ \dot{x}_i = f(x_i, t) + u_i, \quad i = 1, \ldots, N, \]  
\[ u_i = c \sum_{j=1}^{N} \sigma_{ij}(t) \Gamma (x_j - x_i), \quad i = 1, \ldots, N, \]  
\[ \sigma_{ij}(0) = 0, \quad i, j = 1, \ldots, N, i \neq j, \]

where \( c \) represents the maximum steady state control gain, and the final control topology is determined by the evolution of the \( \sigma_{ij} \)-s.

We consider an adaptive law for \( \sigma_{ij} \) such that a steady-state control topology is achieved while guaranteeing the network synchronization. To this aim, a state \( s_{ij} \in \mathbb{R}^m \) is associated to each possible edge in the network

\[ \dot{s}_{ij} = g(s_{ij}, x_j - x_i, t), \]  
\[ \sigma_{ij} = h^T s_{ij}, \]

where \( g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^m \) is the vector field defining the evolution of the edges, and \( h \) is the \( m \)-dimensional output vector. Thus, the control gain \( \sigma_{ij} \in \mathbb{R} \) associated to each edge is considered as a linear combination of the edge states.

A simple network evolution model is presented in DeLellis et al. [2009], where \( m = 1 \) and equations (9) and (10) reduce to

\[ \dot{s}_{ij}^{(1)} = s_{ij}^{(2)}, \]  
\[ \dot{s}_{ij}^{(2)} = -\zeta s_{ij}^{(2)} - \frac{d}{ds_{ij}^{(1)}} V(s_{ij}^{(1)}) + \kappa(||x_j - x_i||), \]

\[ \sigma_{ij} = [1 \quad 0] s, \]

where \( \zeta \) is a damping parameter.

In what follows, we focus on the selection of the potential \( V \) to control the properties of the steady-state emerging topology.

2.1 Bistable potential

If the coupling strength is assumed to be equal throughout the activated network links, then the most natural selection for \( V \) is a classical double-well potential. This selection is studied in DeLellis et al. [2010b] and reads as:

\[ V(z) = bz^2(z - 1)^2. \]

This selection leads to the emergence of an unweighted final topology, whose properties are strongly related to the initial conditions on the nodes’ states. In Fig. 1, we show a simulation of a network of \( N = 30 \) integrators. According to the network evolution, only a subset of the potential edges are activated at steady-state. Edge activation is controlled in a fully decentralized way by the individual nodes.

The main drawback of this potential is the absence of weights in the final topology. While active connections are adaptively selected in a decentralized way, the intensities of active links are identical at steady-state. In this paper, we propose a multi-well potential to guide the evolution of the network. The wells are positioned in a discrete set of points representing the set of admissible of interactions between nodes.

\[ F : I \rightarrow \mathbb{R} \text{ is positive definite if } F(z) > 0, \forall z \in I, z \neq 0 \text{ and } F(0) = 0. \text{ A function } f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \text{ is of class } k \text{ if it is continuous, positive definite and strictly increasing. A unbounded function of class } k \text{ belongs to class } k_{\infty}. \]
\[ V(z) = \prod_{i=0}^{n_w-1} [(z - w_i)^2 - \alpha_i], \]  

where \( n_w \geq 2 \) is the number of wells, \( w_i > 0 \) describes their position (defining the set of possible asymptotic levels of interactions), and \( \alpha_i \) is a parameter used to tune the wells’ depth. Without loss of generality, we assume \( w_{n_w-1} > \ldots > w_0 \). Moreover, we set \( w_0 = 0 \) to account for the case in which an edge remains deactivated.

Under these assumptions, \( w_{n_w-1} \) represents the maximum level of interactions between each pair of nodes. It is worth remarking that the choice of the parameters \( w_i \) and \( \alpha_i \) is subject to the constraint that each \( w_i \) is a minimum of the potential, as further displayed in Fig. 2.

### 3. EDGE SNAPPING CONSENSUS

In the presence of the edge snapping mechanism described above, and assuming, without loss of generality, \( c = 1 \), we select the following linear communication protocol to achieve consensus:

\[ u_i = \sum_{j=1}^{N} \sigma_{ij}^2 (x_j - x_i), \]

Equations (12)-(14) describing the evolution of each \( \sigma_{ij} \) can be written as:

\[ \dot{\sigma}_{ij} = -\zeta \sigma_{ij} - \frac{d}{d\sigma_{ij}} V(\sigma_{ij}) + (x_j - x_i)^2. \]  

In this case, we have \( x_i \in \mathbb{R} \) and we consistently replace the squared error norm with the square of the state differences in (18). Notice that we consider positive coupling gains in (17). In this way, none of the possible emerging topologies is unstable, see for example Porfiri and Stilwell [2007].

In order to assess the consentability of this protocol, we recast the equation set in the following handleable form:

\[ \dot{\delta} = -\mathcal{L}_\sigma \delta, \]  

\[ \dot{\sigma}_{ij} = -\zeta \sigma_{ij} - \frac{d}{d\sigma_{ij}} V(\sigma_{ij}) + (\delta_i - \delta_j)^2, \]

where, consistently with Olfati-Saber and Murray [2004], \( \delta \) is the disagreement vector \( \delta = x - \alpha_1 N \). \( 1_N \) is the \( N \)-dimensional vector composed of all ones, and \( \alpha \) is the time-constant mean state \( \sum_{i=1}^{N} x_i(t)/N \). Moreover, \( \mathcal{L}_\sigma = (l_{ij}) \) is the time-varying Laplacian matrix of the network, that is defined as:

\[ l_{ij} = \begin{cases} \sum_{k=1}^{N} \sigma_{ik}^2 & \text{if } i = j, \\ -\sigma_{ij}^2 & \text{otherwise}. \end{cases} \]

From equation (19), the evolution of the total disagreement, defined as \( \delta_{tot} = \delta^T \delta \), is governed by:

\[ \delta_{tot} = -2\delta^T L_\sigma \delta = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij}^2 (\delta_i - \delta_j)^2 \leq 0. \]

Clearly, the total disagreement asymptotically approaches a constant value \( 0 \leq \delta_{tot} \leq \delta_{tot}(0) \). At steady-state, we have \( \delta_{tot} = 0 \), which implies that \( \sigma_{ij} = 0, \forall (i,j) \) and/or \( (\delta_i - \delta_j) = 0, \forall (i,j) \), where \( \delta_i \)'s are the components of the disagreement vector. Nonetheless, if for some \( (i,j) \), \( (\delta_i - \delta_j) \neq 0 \), then equation (20) would imply that the corresponding \( \sigma_{ij} \) is different than 0 and thus \( \delta_{tot} < 0 \). Hence, we would get a contradiction. Therefore, \( \delta_{tot} = 0 \) and the network of integrators asymptotically reaches consensus on \( \alpha \).

From equation (22), no claim can be made on the evolution of \( \sigma_{ij} \). For the sake of clarity, we refer to a possible class of potentials \( V(z) \) of the form (16). In particular, we assume that there are only three wells \( (n_w = 3) \). Moreover, we fix \( \alpha_0 = \alpha_2 = 0 \) and we consider only equally spaced wells, that is, \( w_2 = 2w_1 \). In this case, the potential (16) reduces to

\[ V(z) = z^2[(z - w_1)^2 + \alpha_1(z - 2w_1)^2 \]  

Now, considering that equation (20) is sufficiently smooth and that \( (\delta_i - \delta_j)^2 \) is bounded and asymptotically vanishing, we have that \( \dot{\sigma}_{ij} \to -\sigma_{ij}^2 \) as \( \dot{\sigma}_{ij} \to +\infty \). Therefore, we conclude that each \( \sigma_{ij} \) is upper-bounded. Similarly, it is possible to show that it is also lower-bounded. As a consequence, each \( \sigma_{ij} \) converge to one of its equilibria.
The equilibria of (20) correspond to the extrema of (23). As illustrated in Fig. 2, this potential has five extrema \( z_1 = 0, z_2 = w_1, z_3 = w_2, \) and

\[
z_{4,5} = w_3 + \sqrt{-2a_1 + w_1^2 \over 3}.
\]

In order to ensure that \( z_{1,2} \) are local minima and \( z_{4,5} \) local maxima, we must enforce that \( a_1 \) and \( w_1 \) satisfy

\[
-w_1 < a_1 < w_1^2 \over 2.
\]  \hspace{1cm} (24)

We first analyze the local stability of \( \sigma_{ij} = z_1 \). From (22), we have that \( |\delta_i| \leq \sqrt{\delta_{tot}}(0) \). Thus, we note that \( (\delta_i - \delta_j)^2 \leq 4\delta_{tot}(0) = \gamma < +\infty \) and that the initial condition for the total disagreement can be selected to have \( \gamma \) arbitrary small. For local perturbations of \( \sigma_{ij} \) and \( \delta_{ij} \) from the origin and for sufficiently small \( \gamma \), equation (20) can be approximated by:

\[
\dot{\sigma}_{ij} + \zeta \dot{\sigma}_{ij} + 8w_1^2(a_1 + w_1^2)\sigma_{ij} = \varepsilon,
\] \hspace{1cm} (25)

where \( \varepsilon \) is bounded by \( \gamma \) and asymptotically vanishes. Since \( w_1 > 0 \) and (24) holds, we have that the coefficient \( 8w_1^2(a_1 + w_1^2) \) is positive. Moreover, the damping parameter \( \zeta \) is positive and, therefore, the eigenvalues of the linear second order system in (25) have negative real part. Thus, for sufficiently small perturbations, \( \sigma_{ij} \) remains bounded in the neighborhood of 0 and asymptotically goes to zero as \( \varepsilon \to 0 \). The same results can be obtained by linearizing equation (18) in the neighborhood of the equilibrium \( \sigma_{ij} = z_1 \). In the neighborhood of \( \sigma_{ij} = z_2 = w_1 \), equation (20) is approximated by:

\[
\dot{\sigma}_{ij} + \zeta \dot{\sigma}_{ij} + 2w_1^2(w_1^2 - 2a_1)(\sigma_{ij} - w_1) = \varepsilon.
\] \hspace{1cm} (26)

Again, from (24) we have that \( (w_1^2 - 2a_1) > 0 \) and therefore (26) has negative eigenvalues.

The case of \( \sigma_{ij} = z_{4,5} \) is different as expected from the depiction in Fig. 2. In fact, in the neighborhood of \( \sigma_{ij} = z_{4,5} \) and \( \delta_{ij} = 0 \), system (20) can be approximated by:

\[
\dot{\sigma}_{ij} + \zeta \dot{\sigma}_{ij} - 8(w_1^2 - 2a_1)(a_1 + w_1^2)(\sigma_{ij} - z_{4,5}) = \varepsilon.
\] \hspace{1cm} (27)

From (24), system (27) has one positive real eigenvalue that leads to an unstable equilibrium point. We conclude that almost all the solutions converge to \( \delta = 0 \) and \( \sigma_{ij} = 0 \), or \( \sigma_{ij} = w_1 \), in other words, if we exclude the trivial case in which \( \sigma_{ij}(0) = z_{4,5} \) for all node pairs and \( \delta(0) = 0 \), the network evolves to a final topology, whose structure depends on the initial conditions, and on the damping \( \zeta \).

We note that the above stability consideration can be easily extended to any smooth multi-well potential which is radially unbounded with power growth greater than 2.

### 4. PARAMETER TUNING

As illustrated in Section 3, the edge snapping strategy can be successfully used to enforce consensus in a network of integrators. Moreover, a weighted network topology emerges at steady-state. The possible levels of interactions are given by the stable equilibria of the potential. Nonetheless, the performance of the proposed scheme is strongly influenced by the potential selection. Therefore, depending on the control objectives, a proper tuning of the potential is required. For the sake of clarity, we refer to the potential (23) and we fix the damping \( \zeta \) in (13) to one. Moreover, we assume that the two possible levels of interactions are 1 and 2, that is, we set \( w_1 = 1 \). If a coupling gain asymptotically converges to \( w_1 \), or \( w_2 \), we call the corresponding edge inactive, weak, or strong link, respectively.

In this case, the only tuning parameter in the potential is \( \alpha_1 \), which regulates the depth of the wells. According to Fig. 2, we denote with \( a \) the depth of the first and third well and with \( b - a \) the depth of the second well. Thus, we have

\[
a = \frac{4}{27}(a_1 + w_1^2)^3,
\]

\[
b - a = \frac{4}{27}(a_1 + w_1^2)^3 - \alpha_1 w_1^4,
\]

where \( b \) is the maximum corresponding to \( z_{4,5} \). By increasing the parameter \( \alpha_1 \), the depth \( a \) increases while \( b - a \) reduces. These variations are expected to affect the number of weak and strong links that appear at steady-state.

To illustrate the effects of the choice of \( \alpha_1 \), we consider a network of \( N = 30 \) integrators and fix the initial conditions on both the nodes and edge dynamics. In particular, we take each \( x_i \) randomly from a normal distribution and we set \( s_i^{(1)}(0) = s_i^{(2)}(0) = 0 \). Simulations are performed for different values of the parameter \( \alpha_1 \), ranging from \(-1 \) to \( 0.5 \) as from inequality (24). According to the stability analysis performed in Section 3, consensus is always achieved as further displayed in Fig. 3, with the coexistence of weak and strong links at steady-state, see Fig. 4.

#### 4.1 Emerging topologies

Now, we analyze the properties of the emerging topologies and how these influence the time needed to reach consensus. This is measured as the first time instant \( t_c \) in which \( r(t) > 0.9999 \), where \( r(t) \) is the order parameter of the network (Kuramoto [1984]). This positive parameter is defined as:

![Fig. 4. Network of 30 integrators. Three-well potential (23) with \( \alpha_1 = 0 \). Plot of the emerging topology.](image)
\( r(t) = \frac{1}{N} \sum_{j=1}^{N} |Ie_j| \),

where \( I \) is the imaginary unit. To characterize the network topology, we consider the average node degree \( k_{av} \) and a measure of its heterogeneity, namely its standard deviation \( k_{std} \). Two possible measures of the network cohesiveness are the average path length \( apl \) and the average clustering coefficient \( cc_{av} = \frac{1}{N} \sum_{i=1}^{N} c_i / N \). Here, the clustering coefficient \( c_i \) of a node \( i \) is the fraction of connected neighbors of \( i \), see Boccaletti et al. [2006] for further details. The centrality of a node in the network is also very relevant, especially in traffic flow problem, see for example Wang and Zhou [2007]. A possible measure of centrality is the betweenness centrality \( bc \) (Anthonisse [1971], Freeman [1977]), defined as:

\[
bc(v) = \sum_{i \neq v \neq j} \frac{sp_{ij}(v)}{sp_{ij}}
\]

Here, \( sp_{ij} \) is the number of shortest paths connecting node \( i \) to \( j \) and \( sp_{ij}(v) \) is the number of shortest paths from \( i \) to \( j \) where vertex \( v \) lies. In particular, a key parameter is the maximal betweenness centrality \( bc_{M} \). In fact, a high value of \( bc_{M} \) hinders synchronization in networks of nonlinear oscillators and represents a bottleneck for the throughput in traffic flow problems (Wang and Zhou [2007]). The second smallest eigenvalue \( \lambda_2 \) of the Laplacian of the emerging topology is important in estimating a lower bound for the exponential rate of convergence in linear consensus problems over static network topologies Olfati-Saber and Murray [2004]. Finally, to have a further hint on the characteristics of the final topology, we report the percentage of inactive, weak, and strong links, denoted with \%\(w_0\), \%\(w_1\), and \%\(w_2\), respectively.

Numerical results are summarized in Table 1. In the first and last row, the two degenerate cases are analyzed. When \( \alpha_1 = -1 \), the potential (23) has only three extrema: a minimum \( w_1 \) and two saddle points \( 0 \) and \( 2 \). Therefore, all the edges converge to their unique equilibrium point and the emerging topology is all-to-all and characterized by the 100% of weak links. In contrast, when \( \alpha_1 = 0.5 \), the network is very sparse and disconnected. In this case, all the existing links are strong since the potential has two minima in \( 0 \) and \( 2 \) and a saddle in \( 1 \). Between these two extremal cases, it is possible to tune the potential to balance appropriately the percentage of strong and weak links. In addition, other topological features can be tuned appropriately. For instance, a connected network with nonnull clustering coefficient and an average degree lower than 10 can be constructed by setting \( \alpha_1 = 0 \), according to Table 1. If we want to enhance the synchronization properties, negative \( \alpha_1 \) values are required, while if synchronization is undesired, positive \( \alpha_1 \) are preferred, since a high maximal betweenness centrality is induced. Applications in which synchronization should be avoided include Parkinson disease (Hammond et al. [2007]). As for the speed of convergence, we notice that its reduction as \( \alpha_1 \) increases is very slow when compared to the decrease of the average degree. Therefore, consensus can be rapidly achieved with sparse networks as well.

5. CONCLUSIONS

The proposed scheme for network evolution was shown to be effective in enforcing consensus in networks of integrators. Nonetheless, numerical evidence suggests that it is also effective in synchronization problems. As an example, we show a simulation of a network of 30 Lorenz oscillators, described by equations (6)-(8) and (12)-(14), with \( f(x) = (10(q−p),−pr+28z−q,pq−\frac{r}{3^2})^T \), where \( x = [p\ q\ r]^T \). The initial conditions on the nodes are taken from a uniform distribution between 0 and 15, while all the links are initially inactive. As depicted in Fig. 5, synchronization is asymptotically achieved, while the network reaches a weighted emerging topology, see Fig. 6.

Moreover, it was shown in Section 4 that the choice of the multistable potential can be used to increase the convergence speed to consensus or to tailor the properties of the emerging topology. Ongoing work focuses on the development of adaptive potentials, capable of self-tuning according to the prescribed control objectives for the final topology.
Table 1. Properties of the emerging topology versus time to consensus.

<table>
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<tr>
<th>$\alpha_1$</th>
<th>$k_{av}$</th>
<th>$k_{std}$</th>
<th>$bc_{av}$</th>
<th>$bc_{std}$</th>
<th>$\lambda_2$</th>
<th>$apl$</th>
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REFERENCES


