Adaptive Tracking Control of Rigid-Link Flexible-Joint Robot Manipulator with Uncertainties

Chao Liu ∗ Xianbo Xiang ∗∗ Philippe Poignet ∗

Abstract: Joint flexibility is an important factor to consider in the robot control design if high performance is expected for the robot manipulators. In this paper, we propose an adaptive tracking control method which can deal with the kinematics uncertainty and uncertainties in both link and actuator dynamics of the rigid-link flexible-joint (RLFJ) robot system. Adaptive observers are designed to avoid acceleration measurements due to the fourth-order overall system dynamics. Convergence of both end-effector tracking errors and observing errors are proven and sufficient conditions are presented to guarantee system's asymptotic stability.

Keywords: Robot manipulator, flexible joint, adaptive control, observer

1. INTRODUCTION

For robot motion control, it has been a usual procedure to model the manipulator by chains of rigid links with rigid transmission joints to simplify dynamic analysis and the controller design. Many controllers for robots with rigid joints have been proposed and successfully implemented in industrial robot manipulator applications. However, joint flexibility due to gear elasticity, shaft wind-up and use of harmonic drives etc. has to be taken into account in the robot manipulator design if high performance is to be achieved. As the experimental investigations by Sweet and Good [1] show, the effects of joint flexibility can limit the robustness and performance of a given robot controller and can even lead to instability if neglected in the controller design. Moreover, the joint flexibility can serve as the first approximation of robot link flexibility [2] and hence the study of joint flexibility can provide another perspective into the study of flexible link which allows much lighter link weight and thus much faster motion of the robot.

As a benchmark for control designers, Spong [3] proposed a rigid-link flexible-joint (RLFJ) robot model which is globally feedback linearizable and reduces to the standard rigid-link robot model as the joint stiffness tends to infinity. As a result, this model has been widely used by many control researchers in the development of their controllers using robust, adaptive and other control techniques [4-6].

For all the aforementioned research works [4-6], the control tasks are defined in joint space or defined in task space then converted to joint space by using inverse kinematics. To eliminate the problem of solving inverse kinematics a few Cartesian-space controllers [7, 8] were proposed for RLFJ robot. One common problem with these control methods [4-8] is that they are no longer implementable if kinematics uncertainty exists in the robot system, since neither inverse kinematics nor robot Jacobian matrix calculation can be carried out in this case. Kinematics uncertainty has been recognized as one important factor affecting the performance of robots [9-11], especially in visual servoing applications or when the robot works in unknown environments. No result has been proposed for control of flexible-joint robots with kinematics uncertainty until the recent work by Liu et. al. [12]. However, this work focused on uncertain kinematics setpoint control of RLFJ robot. In some applications, it is necessary to specify the motion in much more details than simply setting the desired final position. Thus, a desired trajectory should be specified for the robot to follow with respect to time.

In this paper, we provide the first study on task-space tracking control problem of RLFJ robot with uncertainties in the system. A new adaptive control scheme is proposed to solve this problem. The proposed control scheme can deal with kinematic uncertainty as well as uncertainty in actuator dynamic parameters, such as stiffness constants, rotor inertia and actuator viscous friction coefficients. Despite the complex fourth-order overall system, acceleration is avoided in the control design by developing three adaptive observers. Stability of the whole system is proved rigorously through Lyapunov analysis and guaranteed by choosing the control parameters to satisfy certain sufficient conditions. Simulation study was conducted to verify the efficiency of proposed control scheme.

2. PROBLEM FORMULATION

For n-link RLFJ robot manipulator with all revolute joints and actuated directly by DC motors, the equations of motion can be described as [3]:
\begin{align*}
M(q)\ddot{q} + (B + C(q, \dot{q}))\dot{q} + g(q) &= K_m(q_m - q), \\
J_m\ddot{q}_m + B_m\dot{q}_m + K_m(q_m - q) &= u,
\end{align*}

where \( q \in \mathbb{R}^n \) denotes the vector of robot link angle, \( q_m \in \mathbb{R}^n \) denotes the vector of joint robot shaft position, \( M(q) \in \mathbb{R}^{n \times n} \) and \( J_m \in \mathbb{R}^{n \times n} \) denote the link and actuator inertia matrices respectively, \( B \in \mathbb{R}^{n \times n} \), \( B_m \in \mathbb{R}^{n \times n} \) are the damping matrices, \( C(q, \dot{q})\dot{q} \in \mathbb{R}^n \) is the Coriolis/centripetal vector and \( g(q) \in \mathbb{R}^n \) is the vector of gravity force. \( K_m \in \mathbb{R}^{n \times n} \) denotes the joint stiffness constant matrix which is diagonal and positive definite. \( u \in \mathbb{R}^n \) is the vector of actuator control torque input.

Some important properties of the dynamic equations (1), (2) that will be used in the controller design and stability analysis of this paper are stated as follows:

**Property 1** The inertia matrix \( M(q) \) is symmetric and uniformly positive definite for all \( q \in \mathbb{R}^n \) [13].

**Property 2** \( J^T(q)M(q) - C(q, \dot{q}) \) is a positive definite matrix.

**Property 3** \( C(q, \dot{q}), g(q) \) is a linear function of \( q, \dot{q}, \theta \).

**Property 4** The left side of equation (1) is linear in \( \dot{q} \) and can be expressed as \[ J(q)\dot{q} + (B + C(q, \dot{q}))\dot{q} + g(q) = Y_d(q, \dot{q}, \dot{q}, q)\theta_d, \] (3) where \( Y_d(q, \dot{q}, \dot{q}, q) \in \mathbb{R}^{n \times p} \) is defined in (3):

\[
\dot{x} = J(q)\dot{q},
\] (4)

where \( J(q) \in \mathbb{R}^{n \times n} \) is the Jacobian matrix from joint space to task space. There stands a property regarding the Jacobian matrix as [15]:

**Property 5** \( J^T(q)y \) (for any \( y \in \mathbb{R}^n \)) is linear in a set of kinematic parameters \( \theta_k = (\theta_{k1}, \ldots, \theta_{kr})^T \):

\[
J^T(q)y = Y_k(q, \dot{q})\theta_k, \forall y \in \mathbb{R}^n
\] (5)

where \( Y_k(q, \dot{q}) \in \mathbb{R}^{n \times r} \) is the kinematic regressor matrix.

If the kinematic parameters are uncertain, then only estimated \( \hat{\theta}_k \) is available and hence the Jacobian matrix also becomes uncertain and only an approximate Jacobian matrix \( \hat{J}(q) \) can be obtained.

In this paper, we consider not only the link dynamics uncertainties which are considered in most of the recent robot control designs, but also the uncertainties in actuator dynamics and in robot kinematics. Hence in this study matrices \( K_m, J_m, B_m \) and physical parameters in \( M(q) \), \( C(q, \dot{q}) \), \( g(q) \) are assumed to be unknown.

### 3. ADAPTIVE TRACKING CONTROL OF RLFJ ROBOT WITH UNCERTAINTIES

Since the overall system is of fourth order. To synthesize the control torque input \( u \), the control design is separated into two steps, similar to backstepping procedure: first, a desired actuator position trajectory denoted by \( q_{nd} \) is formulated based on the link subsystem (1); then a control torque input \( u \) is designed to force the actuator shaft to follow the desired trajectory \( q_{nd} \).

#### 3.1 Desired Actuator Position Trajectory Design

With the desired actuator position trajectory \( q_{nd} \), the link subsystem dynamics (1) can be rewritten as:

\[
M(q)\ddot{q} + (B + C(q, \dot{q}))\dot{q} + g(q) = K_m(q_{nd} - q) + K_m\dot{q}_m, \] (6)

where \( q_m = q_m - q_{nd} \).

By defining \( q_{nd} \) using an auxiliary variable \( q'_m \) as \( q_{nd} = q'_m + q \), we can cancel off \( q \) from \( K_m(q_{nd} - q) \) and get

\[
K_m(q_{nd} - q) = K_mq'_m, \] (7)

Hence equation (6) can then be further written as

\[
M(q)\ddot{q} + (B + C(q, \dot{q}))\dot{q} + g(q) = K_mq'_m + K_m\dot{q}_m, \] (8)

In the presence of kinematics uncertainty, instead of the true Jacobian matrix \( J(q) \), only approximate Jacobian matrix \( \hat{J}(q, \hat{\theta}_k) \) is available and hence according to Property 5 we have

\[
\dot{x} = \hat{J}(q, \hat{\theta}_k)\dot{q} = Y_d(q, \dot{q}, \dot{q}, \hat{\theta}_k), \] (8)

where \( \dot{x} \) denotes the estimated task-space velocity; \( \hat{\theta}_k \in \mathbb{R}^p \) is the estimated kinematic parameters vector which will be updated through an adaptation law.

Let \( x_d(t) \in \mathbb{R}^n \) be the desired task space trajectory which is differentiable and bounded up to third order and \( K_m \in \mathbb{R}^{n \times n} \) be a fixed estimated joint stiffness matrix, the auxiliary signal \( y_{nd} \) can be proposed as:

\[
y_{nd} = K^{-1}_m[-J^T(q, \hat{\theta}_k)K_p\Delta x - K_d(\dot{\hat{q}}_o - \dot{q}_r)]
+ Y_d(q, \dot{q}_d, \dot{q}_r, \dot{\hat{\theta}}_k)\theta_d, \] (9)

where \( \hat{\theta}_d \in \mathbb{R}^p \) and \( \theta_d \in \mathbb{R}^n \) are vectors of estimated manipulator dynamic and actuator dynamic parameters which are updated through adaptation laws presented later. The regressor \( Y_d(q_{nd}, \theta_d) = diag(q'_{nd1}, \ldots, q'_{ndn}) \in \mathbb{R}^{n \times n} \) and \( q'_{ndi}(i = 1, \ldots, n) \) denotes the \( i \)-th element of the vector \( q'_{nd} \in \mathbb{R}^n \), which is defined as:

\[
y_{nd} = -J^T(q, \hat{\theta}_k)K_p\Delta x - K_d(\dot{\hat{q}}_o - \dot{q}_r) + Y_d(q, \dot{q}_d, \dot{q}_r, \dot{\hat{\theta}}_k)\theta_d, \] (10)

where \( \Delta x = x - x_d \) denotes position tracking error; \( K_p, K_d \in \mathbb{R}^{n \times n} \) are positive definite diagonal constant matrices; \( \dot{\hat{q}}_o \in \mathbb{R}^n \) is the observed signal of \( \dot{q} \) whose updating mechanism is defined in (17); \( \dot{\hat{q}}_r \in \mathbb{R}^n \) is the reference velocity signal and \( \tilde{\dot{q}}_r \) is the modified reference acceleration to be defined in (16); regressor \( Y_d(q, \dot{q}_d, \dot{q}_r, \dot{\hat{\theta}}_k) \in \mathbb{R}^{n \times p} \) possesses a similar form as the dynamic regressor \( Y_d(q, \dot{q}, \dot{q}, \dot{\theta}_k) \) defined in (3):

\[
Y_d(q, \dot{q}_d, \dot{q}_r, \dot{\hat{\theta}}_k)\theta_d = M(q)\dot{\hat{q}}_o + [\dot{B} + \dot{C}(q, \dot{q})]\dot{\hat{q}}_o + \dot{g}(q), \] (11)

where \( M(q), \dot{B}, \dot{C}(q, \dot{q}), \dot{g}(q) \) are the approximate manipulator dynamic matrices and vector respectively using the estimated dynamic parameters in \( \dot{\hat{\theta}}_k \).

The reference velocity \( \dot{\hat{q}}_r \) is defined as:

\[
\dot{\hat{q}}_r = \hat{J}^{-1}(q, \hat{\theta}_k)[\dot{x}_d - \alpha(\dot{x}_o - x_d)], \] (12)

where \( \alpha \) is a positive control constant, \( \dot{x}_o \in \mathbb{R}^n \) is the the observed signal of \( x \) updated through the following observer updating law:

\[
\dot{x}_o = \hat{J}(q, \hat{\theta}_k)\dot{\hat{q}}_o + \Lambda(x - \dot{x}_o), \] (13)
where $\Lambda \in \mathbb{R}^{n \times n}$ is a positive definite diagonal constant matrix.

From equation (12), the time derivative of $\dot{q}_r$ (reference acceleration $\ddot{q}_r$) can be obtained as

$$\ddot{q}_r = -\mathbf{J}^{-1}(q, \dot{\theta}_k)Z_r(q, \dot{\theta}_k, \Delta x, \bar{x}) \ddot{q} + \mathbf{J}^{-1}(q, \dot{\theta}_k)[\ddot{x}_a - \alpha(\ddot{x}_a - \ddot{x}_d)] \quad (14)$$

where $\mathbf{J}^{-1}(q, \dot{\theta}_k)$ has been used, matrix $Z_r(q, \dot{\theta}_k, \Delta x, \bar{x}) \in \mathbb{R}^{n \times n}$ is constructed as

$$Z_r(q, \dot{\theta}_k, \Delta x, \bar{x}) \ddot{q} = \dot{J}(q, \dot{\theta}_k)\dot{q}, \quad (15)$$

$\dot{J}(q, \dot{\theta}_k)$ is a function of $\dot{q}$ and $\dot{\theta}_k$; $\ddot{q}_r$ is a function of $\dot{\theta}_k$, $\ddot{x}_a$, $x_d$ and $\dot{x}_d$.

Using the observed velocity $\ddot{q}_o$ instead of $\ddot{q}$, the modified reference acceleration $\ddot{q}_o$ for $\ddot{q}_r$ can be defined as

$$\ddot{q}_o = \ddot{q}_r + \mathbf{J}^{-1}(q, \dot{\theta}_k)Z_r(q, \dot{\theta}_k, \Delta x, \bar{x}) \ddot{q}_o \quad (16)$$

where $\ddot{q}_o = \ddot{q}_r - \ddot{q}_o$.

So far, we have been using the observed signal $\ddot{q}_o$, whose role is clearly to replace actual velocity $\dot{q}$, and its updating law is now given as

$$\ddot{q}_o = M^{-1}(q)\{K_m(q_m - q) - [B + \dot{C}(q, \dot{q})]\ddot{q}_o - \ddot{g}(q) + [K_d + Z^T_r(q, \dot{\theta}_k, \Delta x, \bar{x})J^{-1}(q, \dot{\theta}_k)M(q) - C^T(q, \dot{q})]s + J^T(q, \dot{\theta}_k)K_d\dot{x} + K_dY_a(q_m - q)\dot{\theta}_a\} \quad (17)$$

where $\ddot{q}_o$, $\ddot{q}_r$, $\dot{q}_o$, $\dot{\theta}_o$, and $\dot{\theta}_r$ are same as used in (11); $Y_a(q_m - q)$ is defined in the same way as $Y_a(q_m - q)$ but with $(q_m - q)$ as the argument, $\ddot{q}_o \in \mathbb{R}^n$ is an adaptive parameter vector updated through an adaptation law defined later; $s$ is an adaptive sliding vector defined using $\ddot{q}_o$ as

$$s = \ddot{q} - \ddot{q}_r, \quad (18)$$

such that

$$\dot{J}(q, \dot{\theta}_k)s = \dot{J}(q, \dot{\theta}_k)\ddot{q} - \dot{x}_d + \alpha(\ddot{x}_a - x_d) = \Delta \ddot{x} + \alpha \Delta x - Y_k(q, \dot{q})\Delta \ddot{q}_r - \alpha \ddot{x} \quad (19)$$

where $\Delta \ddot{q}_r = \ddot{q}_r - \ddot{q}_o$ and equations (5), (8) have been used.

Substituting equation (9) into (6), we can get the closed-loop link subsystem dynamic equation

$$M(q)\ddot{q} + [B + C(q, \dot{q})]\dot{q} + g(q) + J^T(q, \dot{\theta}_k)K_p\Delta x = K_d(\ddot{q}_o - \ddot{q}_r) - Y_a(q_o, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)(I - K_mK_m^{-1})q_{mdo} - K_mK_m^{-1}Y_a(q_{mdo})\dot{\theta}_a = K_m\ddot{q}_o \quad (20)$$

where $I$ denotes identity matrix.

Substituting equation (16) into (11) and using Property 3, we have

$$Y_d(q, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)\dot{\theta}_d = Y_d(q, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)\dot{\theta}_d + M(q)\ddot{q}_o + [B + C(q, \dot{q})]\dot{q}_r + \ddot{g}(q) \quad (21)$$

where according to Property 5 it is defined that

$$Y_d(q, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)\dot{\theta}_d = M(q)\ddot{q}_o + [B + \dot{C}(q, \dot{q})]\dot{q}_r + \ddot{g}(q). \quad (22)$$

From equation (18), it has

$$\ddot{s} = \ddot{q} - \ddot{q}_r, \quad (23)$$

Using equation (21) and substituting (18) and (23) into (20), we get the closed-loop manipulator dynamic equation in the following form:

$$M(q)\ddot{s} = [B + C(q, \dot{q})]s + Y_d(q, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)\Delta \ddot{q}_d - M(q)J^{-1}(q, \dot{\theta}_k)Z_r(q, \dot{\theta}_k, \Delta x, \bar{x})\ddot{q}_o - \dot{C}(q, \dot{q})\ddot{q}_o$$

+ $J^T(q, \dot{\theta}_k)K_p\Delta x + K_d(\ddot{q}_o - \ddot{q}_r) + Y_a(q_{mdo})(\theta_a - K_mK_m^{-1}\ddot{\theta}_a) = K_m\ddot{q}_m \quad (24)$$

where Property 4 has been used; $\Delta \ddot{q}_d = \ddot{q}_d - \ddot{q}_a; \theta_a = [1 - \frac{k_i}{k_m} \cdots 1 - \frac{k_i}{k_m}]^T$ with $k_m$ and $k_i$ being the $i$th diagonal element of $K_m$ and $K_m$, respectively.

Next, the closed-loop observer dynamics for $\ddot{q}$ is to be developed. From equation (17), multiplying both sides by $M(q)$ and using the link subsystem dynamic equation (1), we can get the closed-loop observer dynamic equation as

$$M(q)\ddot{q} + [B + K_q + C(q, \dot{q})]\dot{q} + Y_d(q, \dot{q}_o, \dot{\theta}_o, \dot{\theta}_a)\Delta \ddot{q}_d + [K_d + Z^T_r(q, \dot{\theta}_k, \Delta x, \bar{x})J^{-1}(q, \dot{\theta}_k)M(q) - C^T(q, \dot{q})]s$$

+ $J^T(q, \dot{\theta}_k)K_p\Delta x + K_d(\ddot{q}_o - \ddot{q}_r) + Y_a(q_{mdo})(\theta_a - \ddot{\theta}_a) = 0 \quad (25)$$

where $\ddot{q} = \ddot{q} - \ddot{q}_o$, and similar as the definition of $Y_d(q, \ddot{q}_o, \dot{\theta}_a)$ in equation (22), regressor $Y_d(q, \ddot{q}_o, \dot{\theta}_a, \dot{\theta}_a)$ is defined as

$$Y_d(q, \ddot{q}_o, \dot{\theta}_a, \dot{\theta}_a)\dot{\theta}_d = M(q)\ddot{q}_o + [B + \dot{C}(q, \dot{q})]\ddot{q}_o + \ddot{g}(q), \quad (26)$$

Now that the closed-loop dynamics of the link and observer have been obtained, we can proceed to analyze the stability of closed-loop link subsystem with the desired actuator trajectory proposed. A Lyapunov function candidate $V_1$ is proposed as

$$V_1 = \frac{1}{2}s^TM(q)s + \frac{1}{2}\Delta x^T K_p\Delta x + \frac{1}{2}J^T M(q)\ddot{q}$$

$$+ \frac{1}{2}\ddot{x}^T K_s\ddot{x} + \frac{1}{2}\Delta \ddot{\theta}_d^T L_{\theta d}^{-1}\Delta \ddot{\theta}_d + \frac{1}{2}\Delta \ddot{\theta}_a^T L_{\theta a}^{-1}\Delta \ddot{\theta}_a$$

$$+ \frac{1}{2}(K_mK_m^{-1}\theta_a - \ddot{\theta}_a)^T L_{\theta a}^{-1}(K_mK_m^{-1}\theta_a - \ddot{\theta}_a) + \frac{1}{2}(\theta_a - \ddot{\theta}_a)^T L_{\theta a}^{-1}(\theta_a - \ddot{\theta}_a) \quad (27)$$

where $K_s$, $L_{\theta d}$, $L_{\theta a}$, $L_{\theta a}$ are positive definite constant diagonal matrices.

It’s obvious that $V_1$ is positive definite in all the signals contained. Differentiating $V_1$ with respect to time, using equations (19), (24) and (25) and noting that from equations (8) and (13)

$$\dot{J}(q, \dot{\theta}_k)\ddot{q} = \ddot{x} + \Lambda \ddot{x} - Y_k(q, \dot{q})\Delta \ddot{q}_k, \quad (28)$$

we can get
\[ V_1 = -s^T (B + K_d)s - \tilde{q}^T (B + K_q)\tilde{q} - \alpha \Delta x^T K_p \Delta x + \alpha \tilde{x}^T K_x \tilde{x} + \Delta \theta^T (Y_a(q, \dot{q}, q_r, \hat{\theta}_n)) \tilde{q} + L_d^{-1} \theta_d \]

\[ -x^T \Lambda K_x \tilde{x} - (\theta_a - K_m \theta_a) \tilde{q} + Y_a(q, \dot{q}, \hat{\theta}_m) s + L_d^{-1} \theta_a \]

Here we propose the adaptation laws for \( \hat{\theta}_k, \hat{\theta}_d, \hat{\theta}_a, \hat{\theta}_m \) as follows:

\[ \dot{\hat{\theta}}_k = L_k Y_k^T(q, \dot{q})(K_x \tilde{x} + K_D \tilde{q}), \quad (30) \]

\[ \dot{\hat{\theta}}_d = -L_d Y_d^T(q, \dot{q}, q_r, \dot{q}_r) s + Y_d(q, \dot{q}, \hat{\theta}_m, \dot{q}_m), \quad (31) \]

\[ \dot{\hat{\theta}}_a = -L_a Y_a(q_m, q - q, \tilde{q}), \quad (32) \]

\[ \dot{\hat{\theta}}_m = -L_m Y_m(q_m, q, \tilde{q}), \quad (33) \]

Bring the adaptation laws (30)-(33) into (29), we can get \( V_1 \) as

\[ V_1 \leq -\lambda_{\min}[B + K_d] ||s||^2 - \lambda_{\min}[B + K_q] ||\tilde{q}||^2 - \gamma ||\Delta x|| ||\tilde{x}|| ||\bar{P}(\Delta x)|| + s^T K_m \tilde{q}_m \]

where \( \lambda_{\min}[\cdot] \) and \( \lambda_{\max}[\cdot] \) mean the minimum and maximum eigenvalue of a matrix respectively; \( P \) is a matrix of the form

\[ P = \begin{bmatrix} \alpha \lambda_{\min}[K_p] & -\frac{1}{2} \alpha \lambda_{\max}[K_p] \\ -\frac{1}{2} \lambda_{\max}[K_p] & \lambda_{\min}[K_p] \end{bmatrix} \quad (35) \]

Note that the matrix \( P \) is positive definite if the following condition is satisfied:

\[ 4 \lambda_{\min}[K_p] \lambda_{\min}[\Lambda K_x] > \alpha^2 \lambda_{\max}[K_p] \quad (36) \]

then

\[ V_1 \leq -\lambda_{\min}[B + K_d] ||s||^2 - \lambda_{\min}[B + K_q] ||\tilde{q}||^2 - \lambda_{\min}[\bar{P}] ||\Delta x||^2 - \lambda_{\min}[\bar{P}] ||\tilde{x}||^2 + s^T K_m \tilde{q}_m \]

Hence \( V_1 \) is negative definite in \( s, \tilde{q}, \Delta x \) and \( \tilde{x} \) when the actuator position perturbation \( \tilde{q}_m = 0 \).

### Lemma

In the absence of actuator position perturbation, i.e., \( \tilde{q}_m = 0 \) and with the condition (36) satisfied, the desired actuator position trajectory (3.1) (9), the parameter adaptation laws (30), (31), (32), (33) together with the observers (13) and (17) guarantee the asymptotic convergence of \( s \rightarrow s_o \) and \( \tilde{x} \rightarrow \tilde{x}_d \) to 0 as \( t \rightarrow \infty \). Moreover, \( \dot{s} \rightarrow \dot{s}_o \) and \( \dot{\tilde{x}} \rightarrow \dot{\tilde{x}}_d \) to 0 as \( t \rightarrow \infty \).

### Proof

From equation (27) it can be seen that \( V_1 \) is positive definite in \( s, q, \Delta x, \tilde{x}, \Delta \theta, \Delta \theta^T \bar{K}_m \bar{K}^{-1}_m \tilde{\theta}_a - \tilde{\theta}_a \) and \( \theta_a - \hat{\theta}_a \). Since \( V_1 \) is negative definite in \( s, q, \Delta x, \tilde{x}, \hat{\theta}_a \) and \( V_1 \) is bounded as well the variables included in \( V_1 \). This implies that \( \hat{\theta}_k, \hat{\theta}_d, \hat{\theta}_a, \hat{\theta}_m \) are bounded, \( \tilde{x} \) and \( \tilde{x}_d \) are also bounded if \( \tilde{x}_d \) is bounded. Hence \( \dot{q} \) in equation (12) is bounded if the approximate Jacobian matrix is non-singular and \( \tilde{x}_d \) is bounded. From the boundedness of \( s \), it can be seen from equation (18) that \( \dot{\tilde{q}} \) is also bounded which means \( \dot{\hat{q}} = J(q) \hat{q} \) is bounded. Since \( \Delta \dot{x} = \Delta \tilde{x}_d, \Delta \tilde{x} \) is bounded if \( \tilde{x}_d \) is bounded. Then boundedness of \( \dot{\tilde{q}} \) can be concluded because \( \hat{q} = \tilde{q} - \hat{q}_o \) is bounded. From equation (13) it can be seen now that \( \tilde{x}_o \) is bounded and hence \( \tilde{x} = \tilde{x} - \tilde{x}_o \) is also bounded. From equation (30), \( \hat{\theta}_m \) is bounded since \( \Delta \theta, \Delta \theta^T \bar{K}_m \tilde{\theta}_a - \tilde{\theta}_a \) is bounded. Then from equation (14), \( \dot{q} \) is bounded. Hence it can be seen from equation (9) that \( \dot{q}_m \) is bounded since all signals contained in it are bounded. The boundedness of \( \dot{q}_m \) is not required since it is the result of equation (3). In the absence of actuator position perturbation, \( \tilde{q}_m = 0 \) or \( (q_m - q) = \dot{q}_m \) and hence from equation (44) it has \( \tilde{q}_m \) is bounded. The boundedness of \( \tilde{q}_m \) is not required since it is the result of equation (3). In the absence of actuator position perturbation, \( \tilde{q}_m = 0 \) or \( \tilde{q}_m = \dot{q}_m \) and hence from equation (44) it has \( \tilde{q}_m \) is bounded. However, if \( \tilde{q}_m \) is bounded, then \( \dot{\tilde{q}} \) is bounded if all signals contained in it are bounded. In this case, \( \dot{q} \) is bounded since all signals contained in it are bounded. The boundedness of \( \dot{q} \) is not required since it is the result of equation (9). In the absence of actuator position perturbation, \( \tilde{q}_m = 0 \) or \( \tilde{q}_m = \dot{q}_m \) and hence from equation (44) it has \( \dot{q}_m \) is bounded. However, if \( \tilde{q}_m \) is bounded, then \( \dot{\tilde{q}} \) is bounded if all signals contained in it are bounded. In this case, \( \dot{q} \) is bounded since all signals contained in it are bounded. The boundedness of \( \dot{q} \) is not required since it is the result of equation (9).
\[
\begin{align*}
\dot{\hat{q}}_{md} &= \eta + \Lambda \dot{\eta} + (u - \hat{K}_{mo}(q_m + \hat{q}_{md} - q - q_{md}) \\
\dot{\eta} &= \hat{J}^{-1}_{mo}(u - \hat{K}_{mo}(q_m + \hat{q}_{md} - q - q_{md})),
\end{align*}
\]

where

\[
z = \hat{q}_{md} - q = \Delta \dot{q}_{md} + \Lambda \eta
\]

and \(z\) is an auxiliary variable, \(\Lambda\) is a positive definite diagonal constant matrix, \(\eta = \Delta \dot{q}_{md} - \eta_{md}\). \(J_{mo}\), \(B_{mo}\), \(K_{mo}\) are the approximate models used in the observer dynamics and \(\theta_{mo}, \theta_{Kmo}, \theta_{Kmo}\) are vectors of the estimated model parameters updated through adaptation laws defined in the following:

\[
\begin{align*}
\dot{\hat{J}}_{mo} &= -L_{Jmo}N_{Jmo}(\hat{q})z, \\
\dot{\hat{B}}_{mo} &= -L_{Bmo}N_{Bmo}(\eta)z, \\
\dot{\theta}_{Kmo} &= -L_{Kmo}N_{Kmo}(q_m + \hat{q}_{md} - q - q_{md})z,
\end{align*}
\]

where \(L_{Jmo}, L_{Bmo}, L_{Kmo} \in \mathbb{R}^{2n_m}\) are positive definite constant matrices, \(N_{Jmo}(\hat{q}) = diag(\hat{q})\), \(N_{Bmo}(\eta) = diag(\eta)\), \(N_{Kmo}(q_m + \hat{q}_{md} - q - q_{md}) = diag(q_m + \hat{q}_{md} - q - q_{md})\).

Using equations (2) and (43), we can get the closed-loop observer dynamics as

\[
\begin{align*}
J_{mo}\hat{z} + (B_{mo} + K_{s})z + K_{m}e_m &= -J_{mo}(\hat{q})\Delta \theta_{Jmo} - N_{Bmo}(\eta)\Delta \theta_{Bmo} \\
&- N_{Kmo}(q_m + \hat{q}_{md} - q - q_{md})\Delta \theta_{Kmo},
\end{align*}
\]

where \(\Delta \theta_{Jmo} = \theta_{Jmo} - \dot{\theta}_{Jmo}, \Delta \theta_{Bmo} = \theta_{Bmo} - \dot{\theta}_{Bmo}, \Delta \theta_{Kmo} = \theta_{Kmo} - \dot{\theta}_{Kmo}\).

Then a Lyapunov function candidate \(V_2\) can be proposed for the actuator-observer integrated closed-loop system as

\[
V_2 = \frac{1}{2}(\Delta \hat{q}_m + \beta \Delta \hat{q}_m)^T J_m(\Delta \hat{q}_m + \beta \Delta \hat{q}_m)
\]

\[
+ \frac{1}{2} \Delta \hat{q}_m^T (\beta B_{mo} + \beta K_{mv} + K_{mp} - \beta^2 J_m) \Delta \hat{q}_m
\]

\[
+ \frac{1}{2} \Delta \hat{q}_m^T \hat{J}_{mo} L_{Jmo}^T \Delta \theta_{Jmo} + \frac{1}{2} \Delta \hat{q}_m^T \hat{B}_{mo} L_{Bmo}^T \Delta \theta_{Bmo}
\]

\[
+ \frac{1}{2} \Delta \hat{q}_{md}^T L_{Kmo}^T \Delta \theta_{Kmo} + \frac{1}{2} \Delta \hat{q}_{md}^T \hat{B}_{mo} L_{Bmo}^T \Delta \theta_{Bmo}
\]

\[
+ \frac{1}{2} \Delta \hat{q}_{md}^T L_{Kmo}^T \Delta \theta_{Kmo} \quad (49)
\]

where the control gain \(K_{mv}\) should be chosen large enough or \(\beta\) small enough such that

\[
\lambda_{min} B_{mo} + K_{mv} > \beta \lambda_{max} J_m
\]

and hence \(V_2\) is positive definite.

Differentiating \(V_2\) with respect to time and using equations (39)-(42) and (45)-(48) one has

\[
\dot{V}_2 \leq -(\lambda_{min} B_{mo} + K_{mv}) \Delta \hat{q}_m^2 - (\lambda_{min} J_m) \Delta \hat{q}_m^2 - \lambda_{min} e_m^2 - \lambda_{max} J_m \Delta \hat{q}_m^2 (51)
\]

Now an overall Lyapunov function candidate \(V\) can be proposed for the whole closed-loop robot system as

\[
V = V_1 + V_2,
\]

where \(V_1\) is defined in equation (27) and it’s easy to see that \(V\) is positive definite with condition (50) satisfied.

Then the time derivative of \(V\) can be obtained according to inequalities (37) and (51) and using \(\Delta q_m = (q_m - \hat{q}_{md}) - (q_{md} - q_{md}) = \Delta q_m - \Delta \hat{q}_m \in C^0\),

\[
\dot{V} \leq -[\|s\|^2 + \|\Delta \hat{q}_m\|^2 + \|\Delta q_m\|^2 + \|e_m\|^2]
\]

\[
A \cdot [\|s\| + \|\Delta \hat{q}_m\| + \|\Delta q_m\| + \|\Delta r\| + \|e_m\| + \|\Delta \eta\|]
\]

\[
\leq -\lambda_{min} [B_{mo} + K_{s}] [\|s\|^2 + \|\Delta \hat{q}_m\|^2 + \|\Delta q_m\|^2 + \|e_m\|^2]
\]

where

\[
A = \begin{bmatrix}
\frac{1}{2} \lambda_1 & \frac{1}{2} \lambda_1 & 0 & 0 & \frac{1}{2} \lambda_9 \\
\frac{1}{2} \lambda_1 & \lambda_7 & 0 & 0 & \frac{1}{2} \lambda_9 \\
0 & 0 & \lambda_7 & \lambda_7 & \lambda_9 \\
0 & 0 & \frac{1}{2} \lambda_9 & \frac{1}{2} \lambda_9 & \lambda_9 \\
\frac{1}{2} \lambda_9 & \frac{1}{2} \lambda_9 & \lambda_9 & \lambda_9 & \lambda_9
\end{bmatrix}
\]

and

\[
\lambda_{1} = \lambda_{min} [B_{mo} + K_{s}], \lambda_{7} = \lambda_{max} [K_{m}], \lambda_{9} = \lambda_{min} [B_{mo} + K_{mv} - \lambda_{max} [J_m]]
\]

Then if matrix \(A\) is positive definite, inequality (53) can be written as

\[
\dot{V} \leq -\lambda_{min} [B_{mo} + K_{s}] [\|s\|^2 + \|\Delta \hat{q}_m\|^2 + \|\Delta q_m\|^2 + \|e_m\|^2]
\]

and \(V\) is negative definite in \(\hat{q}, z, \Delta x, \hat{x}, s, \Delta q_m, \Delta q_m, e_m\). According to Sylvester’s Theorem [17], the symmetric matrix \(A\) is positive definite iff all its principal minors \(A_i (i = 1, 2, 3, 4)\) be strictly positive, where principle minor is defined as \(A_1 = a_{11}, A_2 = a_{12}a_{22} - a_{11}a_{21}, \ldots, A_4 = det(A)\).

Since \(\lambda_1\) is positive, it’s easy to see that \(A_1\) is positive. \(A_2\) can be calculated as

\[
A_2 = \lambda_1 \lambda_7 - \frac{1}{4} \lambda_9^2
\]

and it is positive by choosing a large enough \(K_{s}\). From the structure of \(A\), it can be seen that \(A_3\) is simply \(A_3 = A_2 \lambda_6\) and \(\lambda_6\) is positive if \(K_{mv}\) is chosen large enough or \(\beta\) is chosen small enough so that

\[
\lambda_{min} [B_{mo} + K_{mv}] - \beta \lambda_{max} [J_m] > 0
\]

The determinant \(det(A)\) or \(A_1\) can be calculated as

\[
A_4 = A_2 + \frac{1}{4} \lambda_9^2 - \frac{1}{4} \lambda_9^2 - \lambda_7 - \lambda_9
\]

which requires a small enough \(K_{mp}\) or \(A_2\). Hence if the control parameters \(K_{s}, K_{mv}, K_{mp}\) and \(A_2\) are chosen to satisfy conditions (56), (57) and (58), then matrix \(A\) is positive definite and \(V\) is negative definite in all signals contained.

We can now state the following Theorem:

**Theorem** The rigid-link flexible-joint robot system described by equations (20)-(22) together with the observers (13)-(17) (43)-(45) and the parameter adaptation laws (30)-(33), (39)-(41) and (45)-(47) gives rise to the convergence of tracking error \(\Delta x = x - x_d \to 0\) as \(t \to \infty\) if the control parameters \(\alpha, \beta, K_{s}, K_{mp}, K_{mv}\) and \(A_2\) are chosen to satisfy conditions (36), (50), (56), (57), (58), moreover the observed signals \(\hat{x}, \hat{q}_d\) and \(\hat{q}_{md}\) converge to their corresponding true variables \(x, \dot{q}\) and \(q_{md}\) as \(t \to \infty\).
Proof: Since $V$ is positive definite in $s$, $\Delta x$, $\tilde{q}$, $\tilde{x}$, $\Delta \theta_d$, $\Delta \theta_k$, $K_m K_m^{-1} \theta_a - \tilde{\theta}_a$, $\theta_a - \tilde{\theta}_a$, $\Delta \dot{\theta}_m + \beta \Delta \dot{\theta}_m$, $\Delta \theta_m$, $\Delta \theta_{\bar{m}}$, $\Delta \dot{\theta}_m$, $\Delta \theta_{\bar{m}}$ with condition (50) satisfied, and $V$ is negative definite in $\tilde{q}$, $\tilde{z}$, $\tilde{x}$, $s$, $\Delta \dot{q}_m$, $\Delta \dot{\theta}_m$, $\epsilon_m$ with conditions (36), (56), (57), (58) satisfied, the variables included in $V$ are all bounded. This implies that $\hat{\theta}_d$, $\hat{\theta}_a$, $\hat{\theta}_a$ are bounded, $x$ and $\hat{x}$ are also bounded if $x_d$ is bounded. Hence $\hat{q}_d$ in equation (12) is bounded if the approximate Jacobian matrix is non-singular and $\hat{x}_d$ is bounded. From the boundedness of $s$, it can be seen from equation (18) that $\hat{q}_d$ is also bounded which means $x = J(q) \hat{q}_d$ is bounded. Since $\Delta \dot{x} = \dot{x} - \dot{x}_d$, $\Delta \dot{x}$ is bounded if $\dot{x}_d$ is bounded. Since $V$ is bounded and $\dot{V}$ is negative definite in $\Delta x$, it’s obvious that $\Delta x \in L_2(0, \infty)$. Then it follows from [13, 16] that $\Delta x \to 0$ as $t \to \infty$, i.e., $x \to x_d$ as $t \to \infty$. Similarly, since $\Delta \dot{q}_m$, $\tilde{z}$, $\tilde{x}$, $\tilde{\dot{q}}$ is bounded in $L_2(0, \infty)$ and are bounded it can be concluded that $\Delta \dot{q}_m$, $\tilde{z}$, $\tilde{x}$ is bounded as $\tilde{\dot{q}} = \tilde{\dot{q}} - \dot{q}_d \to 0$ as $t \to \infty$. From (44) it has $\epsilon_m = \Lambda_2 (z - \Delta \dot{q}_m)$ and from the convergences of $z$ and $\Delta \dot{q}_m$ it can be concluded $\epsilon_m = q_{md} - \dot{q}_m \to 0$ as $t \to \infty$. The proof is complete.

4. SIMULATION STUDY

In the simulation study, the RLFJ robot is requested to follow a desired circle trajectory despite unknown joint flexibility and uncertainties in both system kinematics and dynamics. The simulation is based on a two link robot holding an uncertain object as shown in Figure 1.

Fig. 1. 2-Link Robot Holding Uncertain Object

The robot arm’s kinematics and dynamics parameters are set to imitate PUMA 560 manipulator (2nd and 3rd link) in order to make the simulation study as realistic as possible. The true robot arm kinematics parameters are set to $l_1 = 0.4318m$, $l_2 = 0.2m$, $q_0 = 30^\circ$ and main dynamics parameters are set to $m_1 = 17.4kg$, $m_2 = 4.8kg$, $m_o = 2kg$. The true actuator dynamics parameters are set to $J_m = 5I$, $B_m = 5I$ and in this study both robot joint stiffness parameters are set to $K_m = 600 Nm/rad$. To introduce uncertainties in robot system kinematics and dynamics, the estimated initial parameters used in the controller are set with errors to $l_1 = l_2 = 0.45m$, $\hat{l}_o = 0.18m$, $\hat{q}_0 = 45^\circ$, $\hat{m}_1 = 16kg$, $\hat{m}_2 = 5.5kg$, $\hat{m}_o = 1.5kg$ and $\hat{J}_m = 4I$, $\hat{B}_m = 4I$, $\hat{K}_m = 650I$.

The trajectory to be tracked in this study is a circle defined in Cartesian space with radius of 0.2m as: $(x_d = 0.65 + 0.2sin(t + \pi/2), y_d = 0.5 + 0.2cos(t + \pi/2))$. The control parameters are set as $K_p = 550I$, $K_d = 100I$, $\alpha = 1.2$, $\beta = 1$, $K_{mp} = 600I$, $K_{mo} = 450I$, $L_k = 0.001I$, $L_d = 0.1I$, $L_a = L_{\bar{a}} = 0.002I$, $L_{Jm} = L_{\bar{Jm}} = 0.1I$, $L_{Bm} = L_{\bar{Bm}} = 0.1I$, $L_{Km} = L_{\bar{Km}} = 0.1I$. The parameters used in the observers are set as $\Lambda = 100I$, $K_g = 100I$, $K_q = 450I$, $\Lambda_2 = 250I$ and $K_s = 50I$.

Figure 2 and 3 illustrate the performance of proposed controller and observers respectively. It’s seen that the actuator shaft angles $\dot{q}_m$ converge to desired angles $\dot{q}_{md}$ and both the manipulator position and velocity tracking errors converge fast despite all the uncertainties existed, and the designed adaptive observers function well in the overall nonlinear adaptive tracking controller proposed.

5. CONCLUSION

In this paper, the trajectory tracking control problem of rigid-link flexible-joint robot with uncertainties is studied. An observer-based adaptive control method is proposed as the first result in literature to address this problem. It’s shown that the closed-loop system is asymptotically stable.
with certain sufficient conditions satisfied. Exact knowledge of the robot link and actuator dynamics parameters as well as kinematics parameters is not required in control design.

REFERENCES


