Performance of Non-Homogeneous Multi-Agent Systems on a Graph

Stefania Tonetti * Richard M. Murray **

* Department of Aerospace Engineering, Politecnico di Milano, Milano, Italy (e-mail: tonetti@aero.polimi.it),
** Department of Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: murray@cds.caltech.edu)

Abstract: This paper considers distributed control of interconnected multi-agent systems. The dynamics of the individual agents are not required to be homogeneous and the interaction topology is described by an arbitrary directed graph. We derive the sensitivity transfer functions between every pair of agents and we analyze performance of non-homogeneous systems, showing that the low frequency behavior is influenced not only by topology, but also by static gain and poles of the agents.

1. INTRODUCTION

In numerous mission scenarios, the concept of a group of agents cooperating to achieve a determined goal is very attractive when compared with the solution of one single vehicle. In this class of systems, even if the agents are dynamically decoupled, they are coupled through the common task they have to achieve. When the number of agents grows, centralized control is no longer feasible and distributed control techniques become attractive. Applications of coordinated control of multiple vehicles can be found in many fields, including microsatellite clusters (Burns et al. [2000]), formation flying of unmanned aerial vehicles (Wolfe et al. [1996], Tonetti et al. [2011]), automated highway systems (Swaroop and Hedrick [1999]) and mobile robotics (Yamaguchi et al. [2001]).

The problem of distributed control has been widely studied with tools from graph theory (Corfmat and Morse [1976], Šiljak [1991], Mesbahi and Hadaegh [2001]). We consider agents with non-homogeneous linear dynamics and we model the interconnection topology as a graph in which the single agents are represented by a vertex, while the interaction links are the arcs.

The distributed control problem has been handled in different ways and with different tools: dissipativity theory and linear matrix inequalities in Langbort et al. [2004], edge agreement in Zelazo et al. [2008], linear quadratic regulator in Borrelli and Keviczky [2008], decomposition and linear matrix inequalities in Massioni and Verhaegen [2009]. In almost all the works mentioned above the control is applied to homogeneous agents interconnected by undirected graphs. If the graph is undirected the problem becomes easier because all the matrices associated with the graph, like the Laplacian, are symmetric.

One approach to distributed control is to use a leader-follower arrangement. This approach is well studied and representative papers exploring graph-theoretic ideas in the context of a leader-follower architecture include Mesbahi and Hadaegh [2001] and Jin [2007], where a double-graph control strategy was proposed. This topology represents a particular case, where the leader has a more important role than the other agents and this may not always be desirable.

The importance of cycles in distributed control has already been pointed out in several past works: Zelazo et al. [2008] investigated the role of cycles and trees in the edge Laplacian for the edge agreement problem, while Fax and Murray [2004] suggested a relation between the presence of cycles and the stability of formation. In Liu et al. [2009] gains over graph cycles are involved in stability conditions for nonlinear network models.

Limits on multi-agent systems performance have already been studied in Barooah and J. [2007] and Bamieh et al. [2009], showing that a global information, such as leader’s position or state, is needed to achieve reasonable performance.

In Tonetti and Murray [2010], we have considered only systems with the same identical dynamics $P(s)$ and local controller $C(s)$. However in the most of the distributed physical systems this is only an approximation. We can just think, for example, of formation of satellites in different orbits, robotics vehicles with different dynamics, internet routers and peer to peer systems. Even if there are agents with different dynamics, the analysis in Tonetti and Murray [2010] still holds, as long as the open loop transfer function $L(s)$ is the same for all the vehicles. This means to shape the controller in order to have $C_i(s) = L(s)/P_i(s)$ for every agent $i$. But this could not always be feasible. It becomes therefore important to develop results also for non-homogeneous systems. In literature we can find few papers dealing with distributed control of heterogeneous systems: in Dunbar and Murray [2006] a distributed receding horizon control for the stabilization of multi-vehicle formation is proposed, where the dynamics are not required to be homogeneous; in Motee and Jadabaie [2008] the structural properties of optimal control of spatially distributed systems is studied; in Rice and Verhaegen [2009] a distributed control for spatially heterogeneous linear sys-
tems is considered. Robustness analysis for interconnected heterogeneous dynamical systems controlled in feedback is studied for example in Lestas and Vinnicombe [2006, 2007], Jonsson et al. [2008] where scalable decentralized conditions are proposed.

In the present paper we investigate how our previous results obtained for homogeneous agents can be extended to heterogeneous systems. The contribution of this work is to show a general method to derive the transfer functions between any pair of agents with different dynamics, where the interconnection topology is described by arbitrary directed graphs. We analyze mechanisms that rule the behavior of a non-homogeneous multi-agent system and we show intrinsic limits on the controller design due not only to the topology, but also to static gain and poles of the open loop transfer function of each agent.

The current paper is organized as follows. In section 2 we briefly review the principal concepts of graph theory, stability and performance of homogeneous systems. Section 3 presents the sensitivity transfer function for non-homogeneous systems, while in Section 4 performances are discussed. The conclusions of the paper are reported in Section 5.

2. PRELIMINARIES

In this section we summarize some of the key concepts from graph theory, stability and performance of homogeneous interconnected multi-agent systems that will be used in the paper.

2.1 Graph theory

A directed graph $G$ is a set of vertices or nodes $V$ and a set of arcs $A \subset V^2$ whose elements $a = (u, v) \in A$ characterize the relation between distinct pairs of vertices $u, v \in V$. For an arc $(u, v)$ we call $u$ the tail and $v$ the head. The in(out)degree of a vertex $v$ is the number of arcs with $v$ as its head (tail). A directed path in a graph is a sequence of vertices such that from each of its vertices there is an arc to the next vertex in the sequence. A directed path with no repeated vertices is called a simple directed path. A directed graph is called strongly connected if there is a directed path from each vertex in the graph to every other vertex. A directed graph is weakly connected if every vertex can be reached from every other but not necessarily following the directions of the arcs. A complete directed graph is a graph where each pair of vertices has an arc connecting them. A simple cycle is a closed path that is self-avoiding (does not revisit nodes, other than the first). A cyclic directed graph is a directed graph without cycles.

The structure of a graph can be described by appropriate matrices. The normalized Laplacian matrix $L$ of a directed graph $G$ is a square matrix of size $|V|$, defined by $L_{ij} = 1$ if $i = j$, $L_{ij} = 1/d_i$ if $(i, j) \in A$, where $d_i$ is the outdegree of the $i$th vertex, $L_{ij} = 0$ otherwise.

A more detailed presentation of graph theory can be found in Tutte [2005].

2.2 Stability of homogeneous systems

We consider a formation of $N$ agents with identical linear dynamics. The normalized Laplacian matrix $L$ of the graph is used to represent the interaction topology. Suppose each individual agent is a SISO system with local loop composed of a local controller $C(s)$ and a plant model $P(s)$. According to Fax and Murray [2004], the multi-agent system is stable if and only if the net encirclement of the critical points $-\lambda_i^{-1}(L)$ by the Nyquist plot of $P(s)C(s)/s$ is zero for all nonzero $\lambda_i(L)$, where $\lambda_i(L)$ are the eigenvalues of the normalized Laplacian matrix $L$ of the graph.

2.3 Performance of homogeneous systems

The Laplacian weight of a simple directed path of length $k$ from $i$ to $j$, where $i = i_0, i_1, \ldots, i_k = j$, is the product of the negative inverse of the outdegrees $d$ of all the nodes in the path besides the last one:

$$Lw_{i_0i_k}^k := \text{sgn}(k) \prod_{t=0}^{k-1} \left(-\frac{1}{d_{i_t}}\right),$$

where $\text{sgn}(k) = -1$ if $k$ is odd, $\text{sgn}(k) = +1$ if $k$ is even. A path is degenerate if it is a path of length zero between a node and itself and we define its Laplacian weight as one: $Lw_{ii}^0 = 1$. The Laplacian weight of a cycle of length $k$ is

$$Lw_{ii}^k := \text{sgn}(k-1) \prod_{t=0}^{k-1} \left(-\frac{1}{d_{i_t}}\right), \quad i_0 = i_k, \quad (2)$$

Disjoint cycles in $G$ are a set of non-adjacent simple cycles, that is, two simple cycles that do not share any common nodes. The length of disjoint cycles is given by the sum of the lengths of the composing simple cycles, while the Laplacian weight of disjoint cycles is given by the product of the Laplacian weights of the composing simple cycles. The subgraph $G_{ij}^k$ is the subgraph of $G$ obtained from $G$ by removing all the nodes and all the arcs touching the simple directed path from node $i$ to node $j$ of length $k$. The subgraph $G_i$ is the subgraph of $G$ obtained from $G$ by removing node $i$.

According to Tonetti and Murray [2010], the transfer function between every pair of nodes $i$ and $j$ of a generic graph $G$ can be derived using a version of Mason’s Direct Rule (Mason [1953, 1956]). The low frequency behavior of the network sensitivity functions is studied, and it is proven that no matter how the controller is designed, there are fundamental limitations to performance. The analysis demonstrated that the presence of cycles in the interaction topology degenerates the system’s performance.

3. SENSITIVITY TRANSFER FUNCTION FOR NON-HOMOGENEOUS SYSTEMS

In this section we show how to derive the non-homogeneous networked sensitivity transfer functions between any pair of agents for a given topology, extending the results obtained in Tonetti and Murray [2010].

We consider a formation of $N$ agents. Each individual agent $i$ is a SISO system with local controller $C_i(s)$ and plant model $P_i(s)$. The normalized Laplacian matrix $L$ of the graph is used to represent the interaction topology. A
representation of the feedback control scheme is shown in Fig. 1, where \( r \in \mathbb{R}^N \) is the vector of the reference signals of all the agents, \( e \in \mathbb{R}^N \) are the errors between \( r \) and the process outputs \( y \in \mathbb{R}^N \), \( u \in \mathbb{R}^N \) is the control signal vector and \( d \in \mathbb{R}^N \) and \( n \in \mathbb{R}^N \) are the load disturbances and the measurement noises respectively. The open loop transfer function of each agent is \( L_i(s) = P_i(s)C_i(s) \). We define the networked sensitivity function matrix \( \tilde{S}(s) \) as
\[
\tilde{S}(s) = (I + \mathcal{L} \mathcal{T}(s))^{-1},
\]
where \( \mathcal{T}(s) = \text{diag}(L_1(s), L_2(s), \ldots, L_N(s)) \). From now on, without loss of generality, we will consider \( n = 1 \) so that each agent has a single output variable that is being controlled. In analogy with the single agent case, in order to guarantee stability, robustness and good performance, we want to have \( |\tilde{S}(j\omega)| \ll 1 \) for \( \omega \ll \omega_c \), and \( |\tilde{S}(j\omega)| \approx 1 \) for \( \omega \gg \omega_c \), where \( \omega_c \) is the cutoff frequency.

We define \( \mathcal{O}(a) \) the set of nodes belonging to the simple cycle \( a \), \( \mathcal{P}(p) \) the set of nodes belonging to the directed simple path \( p \) besides the starting node.

**Theorem 1.** The sensitivity transfer function between every pair of nodes \( i \) and \( j \) of a generic graph \( \mathcal{G} \) with arbitrary dynamics and local controller, can be still expressed through a version of Mason’s Direct Rule:
\[
\tilde{S}_{ij} = \frac{1}{\Delta} \sum_{\text{paths } p \in \mathcal{G}} T_p \Delta_p,
\]
where now the determinant of \((I + \mathcal{L} \mathcal{T}(s))\) is
\[
\Delta = \prod_{f=1}^{N} (1 + L_f) + \sum_{\text{cycles } a \in \mathcal{G}} \left( \mathcal{L} w_o \prod_{z \in \mathcal{O}(a)} (L_z) \prod_{m \notin \mathcal{O}(a)} (1 + L_m) \right),
\]
the ‘gain’ of the \( p \)-th simple directed path from node \( i \) to node \( j \) of length \( k \) is
\[
T_p = \mathcal{L} w_{ij} \prod_{z \in \mathcal{P}(p)} (L_z),
\]
and the value of \( \Delta \) for the subgraph \( \mathcal{G}^i_j \) not touching the \( p \)-th simple directed path from node \( i \) to node \( j \) of length \( k \) is
\[
\Delta_p = \prod_{f \in \mathcal{G}^i_j} (1 + L_f) + \sum_{\text{cycles } a \in \mathcal{G}^i_j} \left( \mathcal{L} w_o \prod_{z \in \mathcal{O}(a)} (L_z) \prod_{m \notin \mathcal{O}(a)} (1 + L_m) \right),
\]
where \( k \) represents the length of the cycles in \( \mathcal{G}^i_j \).

**Proof.** This follows from proof of Theorem 1 in Tonetti and Murray [2010] where now the weight of each arc is equal to
\[
w_{ij} = \frac{1}{d_{oi}} L_{ij}, \quad \forall (i, j) \in \mathcal{G},
\]
and self-loops in each node has weight
\[
w_{ii} = -L_{ii}, \quad \forall i \in \mathcal{G}.
\]
Of course we can observe that if \( L_i = L_j \) for all \( i, j \), equations (4)-(6) became exactly like equations for homogeneous multi-agent systems in Tonetti and Murray [2010]. Even if polynomials of the network sensitivity functions include different plant models and local controllers, paths and cycles structures influence the performances in the same way as homogeneous systems.

4. PERFORMANCE OF NON-HOMOGENEOUS SYSTEMS

We are going now to analyze the system low frequency behavior, in order to see where considerations done for homogeneous agents can be extended to non-homogeneous systems.

**Theorem 2.** In a non-homogeneous interconnected multi-agent system all the asymptotic values of \( \tilde{S}_{ii} \), for graph with \( d_o > 0 \), \( \forall i \in \mathcal{G} \), sum up to the unity:
\[
\lim_{|L_i| \to \infty} \sum_{i=1}^{N} \frac{1}{|L_i|} \tilde{S}_{ii} = 1.
\]

**Proof.** Even if now polynomials of \( \tilde{S}_{ii} \) have more than one variable, if \( d_o > 0 \) the coefficient of \( \prod_{f=1}^{N} L_f \) at the denominator is always zero, as proved in Theorem 3 (Tonetti and Murray [2010]). So we will consider only terms given by \( \mathcal{P}(p) \subset \mathcal{G} \) and \( \Delta_p \). Starting from \( \Delta_p/\Delta \), with \( \Delta_p \) expressed in (6) and \( \Delta \) in (4), with some algebraic considerations, as \( |L_i| \to \infty \) the asymptotic value for a diagonal interconnected sensitivity function is
\[
\lim_{|L_i| \to \infty} \tilde{S}_{ii} = \frac{\left(1 + \sum_{f \in \mathcal{G}^i_j} \mathcal{L} w_o \right) \prod_{f \in \mathcal{G}^i_j} L_f}{\left(1 + \sum_{f \in \mathcal{G}^i_j} \mathcal{L} w_o \right) \prod_{f \in \mathcal{G}^i_j} L_f},
\]
and it is clear that the sum over all the nodes is equal to one, despite the expression of \( L_i \).

From Theorem 2 we can see that even in non-homogeneous multi-agent systems there are fundamental limitations to what can be achieved by control, and control design is a redistribution of disturbance rejection at low frequencies among agents.

**4.1 Example**

We are now going to consider the strongly connected graph of Fig. 2. Agents 1 and 2 have the same stable open
loop transfer function equal to \( L_1 = (800s + 2000)/(s^3 + 41s^2 + 44s) \), while agents 3, 4 and 5 have \( L_2 = (50s + 100)/(0.0475s^3 + 2.375s^2 + s) \). The network sensitivity functions are shown in Fig. 3.

Suppose now we want to improve low frequency behavior of agents 1 and 2 choosing a higher gain for \( L_1 \). As we can see in Fig. 4, a lower asymptotic value for agents 1 and 2 implies a higher value for the other agents, as predicted in Theorem 2.

**Corollary 3.** In a non-homogeneous multi-agent system, interconnected by an arbitrary directed graph with \( d_o > 0 \) \( \forall i \in \mathcal{G} \), the low frequency asymptotic value of the diagonal sensitivity function is

\[
\lim_{{|L_i| \to \infty}} S_{ii} = \lim_{{s \to 0}} \sum_{{i = 1}}^{N} \frac{\prod_{{f \in G_i}} \mu_f}{\prod_{{f \in G_i}} \gamma_f}, \quad (9)
\]

where \( \mu_f \) is the zero frequency gain, which will be called static gain, and \( g_f \) is the number of poles in the origin of the open loop transfer function of agent \( f \neq i \).

**Proof.** A generic open loop transfer function \( L(s) \) can be expressed as

\[
L(s) = \frac{\mu_0 \prod_{{m = 1}}^{n}(1 + s\tau_m)}{\prod_{{n = 1}}^{m}(1 + sT_n)},
\]

where \( \mu_0 \) is the static gain, \( \gamma \geq 0 \) is the number of poles in the origin, \( \tau_m \) and \( T_n \) are zeros and poles time constants, respectively.

Equation (9) comes from equation (8), where each \( L_j \) has been substituted by \( \lim_{{s \to 0}} L(s) = \mu_0/s^\gamma \) of equation (10).

In the following some special cases of Corollary 3 will be considered, showing examples.

### 4.2 Regular graph and same low frequency behavior

We start by first considering directed regular graphs, where there is a full symmetry and in a homogeneous case the asymptotic value of the sensitivity functions would be the same for all the nodes and equal to 1/\( N \). We moreover consider open loop transfer functions with the same number of poles in the origin, \( g_i = g_j \) \( \forall i, j \in \mathcal{G} \). This means they approach \( s = 0 \) with the same speed and therefore they have the same low frequency behavior.

Starting from equation (9) and following our hypothesis: if the graph is regular, \( 1 + \sum_{{f \in G_i}} (L_{i\rightarrow f}) \) is the same for all the nodes and it can first be taken out of the sum at the denominator and then simplified with the one at the numerator; if all the open loop functions have the same number of poles we can simplify all the \( g_f \). So the asymptotic value becomes

\[
\lim_{{s \to 0}} S_{ii} = \frac{\prod_{{f \in G_i}} \mu_f}{\sum_{{i = 1}}^{N} \prod_{{f \in G_i}} \mu_f}, \quad (11)
\]

and it is evident that, since \( \mu_i \) does not appear in the numerator of \( S_{ii} \), if agent \( i \) has the highest \( \mu_i \) with respect to the other agents, the numerator of \( S_{ii} \) will have the lowest low frequency magnitude. An agent with the lowest \( \mu_i \), will have the highest \( S_{ii} \) low frequency magnitude and therefore the poorest disturbance rejection behavior. Same considerations can be done for all the other agents.

In a non-homogeneous interconnected multi-agent system, connected by a directed regular graph and with \( L_i(s) \) with the same number of poles in the origin for all the nodes, the highest is the static gain \( \mu_i \) of an agent with respect to the others, the better disturbance rejection properties has that agent in the formation. This tell us that even if the interconnection topology is symmetric, the agents behavior is different and it is ruled by the static gain of the open loop function.

Consider a formation of 3 agents connected by a complete graph and with the following stable open loop transfer functions with one pole in the origin: \( L_1 = (80s + 200)/(s^3 + 41s^2 + 44s) \), \( L_2 = 18/(s^2 + 5s^2 + 17s) \) and \( L_3 = (5s + 10)/(0.0475s^3 + 2.375s^2 + s) \). Static gains are: \( \mu_1 = 4.54, \mu_2 = 1.06, \mu_3 = 10. \) The magnitude of \( L_i(s) \) is depicted in Fig. 5(a) and it is clear that all the functions approach \( s = 0 \) with the same speed, but \( |L_2(s \to 0)| < |L_1(s \to 0)| < |L_3(s \to 0)| \). We expect \( S_{33} \) to have the lowest value at low frequencies, while agent
2 to have the worst disturbance rejection behavior, as shown in Fig. 5(b). We can also verify that the sum of the asymptotic values is equal to one: \( |\bar{S}_{11}(s \to 0)| = 0.174 \), \( |\bar{S}_{22}(s \to 0)| = 0.747 \), \( |\bar{S}_{33}(s \to 0)| = 0.079 \).

4.3 Arbitrary graph and same low frequency behavior

What does it happen if the graph is not regular but the number of poles is still the same for all the \( L_i(s) \)? Starting from equation (9) we will have that \( 1 + \sum_{\mathcal{G}_i} (L_{w_0}) \) will be different from agent to agent, while all the \( g_f \) can be simplified and the asymptotic value will be

\[
\lim_{s \to 0} \bar{S}_{ii} = \frac{\left( 1 + \sum_{\mathcal{G}_i} (L_{w_0}) \right) \prod_{f \in \mathcal{G}_i} \mu_f}{\sum_{i=1}^N \left( 1 + \sum_{\mathcal{G}_i} (L_{w_0}) \right) \prod_{f \in \mathcal{G}_i} \mu_f}.
\]

(12)

Here the product of the static gains of all the other agents has to be weighted by a value depending on the cycles not passing through that agent. The fewer cycles pass through a node, the lower is \( 1 + \sum_{\mathcal{G}_i} (L_{w_0}) \) and the better is the agent low frequency behavior.

Consider the formation of Example 4.1 but with arc 35 added. The open loop transfer functions have the same number of poles in the origin. With this topology we have added a cycle passing through nodes 3 and 5 (1-2-3-5-1) with respect to topology of Example 4.1. Therefore, even if agents 3, 4, and 5 have the same open loop transfer function, we can see in Fig. 6 that agent 4 has a better low frequency behavior with respect to 3 and 5, while in Fig. 3 the behavior was the same.

4.4 Regular graph and different low frequency behavior

Consider now a formation connected by a regular graph but with at least one vehicle with different number of poles with respect to the others \( g_i \neq g_j \). The asymptotic value is

\[
\lim_{s \to 0} \bar{S}_{ii} = \frac{\prod_{f \in \mathcal{G}_i} \mu_f}{\sum_{i=1}^N \prod_{f \in \mathcal{G}_i} \mu_f}.
\]

(13)

The agent with the highest \( g_i \), which will be called \( g_{\text{max}} \), since it will appear only at the denominator, will lead its asymptotic value to zero. From Theorem 2 we know that the asymptotic values sum up to the unity, so the redistribution of disturbance rejection will affect only the agents with \( g < g_{\text{max}} \). For example if there is only one agent with \( g < g_{\text{max}} \) the disturbance entering in it will not be attenuated at all because its asymptotic value will be equal to one.

Consider the same formation of the example in Section 4.2, but with \( L_2 \) with two poles in the origin, instead of one. In Section 4.2 the low frequency behavior of agent 2 was the poorest, but in Fig. 7, because of the pole added, the disturbance rejection is very good. The asymptotic values of \( |\bar{S}_{11}| = 0.687 \) and \( |\bar{S}_{33}| = 0.312 \) get higher if compared to the example in Section 4.2 because they have to sum up to the unity.
5. CONCLUSION
In this paper we have explored stability and performance of non-homogeneous systems, extending results obtained by Fax and Murray [2004] and Tonetti and Murray [2010] for homogeneous systems.

We have shown that cycles and paths are still involved in the network sensitivity functions, but if the agent dynamics is different, topology is not the only player in determining system’s performance. The low frequency behavior is also influenced by static gain and poles of the open loop function of each agent. If every single agent has the same number of poles, the larger the low frequency gain, the better the formation disturbance rejection. If an agent has a higher number of poles with respect to the others, it will behave like a single agent in the formation.

We can conclude that there are fundamental limitations to what can be achieved by distributed control of non-homogeneous systems. If the behavior of one agent improves, the behavior of the others get worse. Control design is a redistribution of disturbances at low frequency among agents.

REFERENCES


