Consensus in symmetric multi-agent networks with sector nonlinear couplings.

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Abstract: We investigate consensus dynamics in multi-agent networked systems with time-varying undirected topology and uncertain scalar-valued nonlinear couplings, satisfying sector inequalities. Using the Kalman-Yakubovich-Popov lemma and techniques of the absolute stability theory, we obtain consensus criteria for the networks of the specified type, analogous to the celebrated circle stability criterion for Lurie systems.

Keywords: Multiagent networks, consensus, synchronization, absolute stability

1. INTRODUCTION.

Recent years the problems of cooperative control in networks of dynamic agents have attracted enormous attention of the research community, because of growing necessity to analyze and control complex and highly interconnected dynamical systems with limited communications between the agents. The problems of distributed consensus, referred also as agreement or averaging problem, constitute an important class of decentralized cooperative control problems.

The consensus problem consists in designing decentralized control policies or protocols, that make the agents' state vectors or some outputs, treated as individual "opinions", converge to some desired function ("consensus opinion").

The first known averaging algorithms seem to originate in Markov chains theory (see e.g. Seneta (1981)), decision making in expert communities (DeGroot (1974), Eisenberg and Gale (1959) and distributed computing (see D.Bertsekas and Tsitsiklis (1989)). More recent applications of the consensus dynamics include coordinating the groups of mobile autonomous vehicles and analysis of biological populations motion (Gazi and Fidan (2007), Ren et al. (2007), R. Sepulchre and Leonard (2008), Tanner et al. (2007).Olfati-Saber and Murray (2006) etc.), synchronization in various physical phenomena (Vicsek et al. (1995),Strogatz (2000)), data processing and fusion in sensor networks and others.

The most commonly used consensus protocols are linear ones, and a great deal of results for such protocols under different communication restrictions (fixed or time-varying communication topology, communication delays and asynchronous measurements, etc.) have been obtained, see Jadbabaie et al. (2003), Fax and Murray (2004),Olfati-Saber and Murray (2004).Moreau (2005),Hui and Haddad (2008),Ren (2008) or be passive (Chopra and Spong (2006),Arcak (2007)). Below we consider the case of identical linear agents with arbitrary state-space model. We assume only a scalar-valued output to be measured, and the couplings may be nonlinear and uncertain, but supposed to satisfy strict sector inequalities and some symmetry conditions. The undirected network topology may be fixed or switching. Using the absolute stability approach and the Kalman-Yakubovich-Popov lemma, we obtain frequency-domain consensus criteria, resembling the well-known circle criteria for Lurie systems.

2. SOME GRAPH-THEORETIC CONCEPTS.

Throughout the paper \( G_N \) stands for the set of all undirected graphs, having common set of vertices (nodes) \( V_N = \{1,2,\ldots,N\} \) and without loops (arcs with coincident ends). For any \( G \in G_N \) and \( 1 \leq j \leq N \) denote by \( N_j(G) \subset V_N \) the set of all neighbors (adjacent nodes) of the node \( j \) in the graph \( G \).

Let \( a_{ij}(G) = a_{ji}(G) \) (\( 1 \leq i,j \leq N, G \in G_N \)) be 1 if the nodes \( i,j \) if they are connected with an arc in \( G \) and 0 otherwise. The matrix \( (a_{ij}(G)) \) is called the adjacency matrix of the graph \( G \). The matrix associated with the quadratic form \( Q(z) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(G) (z_j - z_i)^2 \), that is
\[ L(G) = \begin{bmatrix} \sum_{j=1}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum_{j=1}^{N} a_{2j} & \cdots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \cdots & \sum_{j=1}^{N} a_{Nj} \end{bmatrix} \geq 0, \quad (1) \]

is called the Laplacian matrix of the graph. Its lowest eigenvalue \( \lambda_1(L(G)) \) is easily seen to be 0 (with eigenvector \( z = (1, 1, \ldots, 1)^T \)). The second eigenvalue \( \lambda_2(L(G)) \) is called the algebraic connectivity or Fiedler eigenvalue of the graph \( G \). It is easily shown Fiedler (1973) that

\[ \lambda_2(L(G)) = N \min_{z} \frac{Q_G(z)}{\sum_{i,j=1}^{N} (z_i - z_j)^2} \quad (2) \]

where minimum is taken over the set of all \( z \in \mathbb{R}^N \) with \( \sum_{i,j=1}^{N} (z_i - z_j)^2 > 0 \).

In particular, for the complete graph \( G \) with \( N \) vertices \((a_{ij} = 1 \text{ for any } i \neq j)\) one has \( \lambda_2(L(G)) = N \). Actually, the spectrum of \( L(G) \) in this case consists of two eigenvalues that are 0 and \( N \) (with multiplicity \( N - 1 \)).

It is known that \( \lambda_2(G) > 0 \) if and only if \( G \) is connected graph (i.e. any two modes may be connected by path). Moreover, as shown in Fiedler (1973), \( \lambda_2(G) \geq 2\eta(G)(1 - \cos \frac{\pi}{N}) \) for any connected graph \( G \in \mathbb{G}_N \), where \( \eta(G) \) is the minimum number of edges sufficient to delete in order to break graph's connectivity (the so called edge connectivity number).

Other properties of graph spectra, including upper and lower bounds on the algebraic connectivity, as well as proofs of the aforementioned facts can be found in Fiedler (1973), Cvetković et al. (1980), Morris (1994) and others.

3. PROBLEM STATEMENT.

We consider a group of SISO plants with identical dynamics indexed with \( j = 1, 2, \ldots, N \) called agents whose state-space models are

\[ \dot{x}_j(t) = Ax_j(t) + Bu_j(t), \quad y_j(t) = Cx_j(t). \quad (3) \]

Here \( t \geq 0, \ x_j(t) \in \mathbb{R}^d \) stands for the state vector of \( j \)-th agent, and \( u_j(t), y_j(t) \in \mathbb{R} \) are the input signal and the measured output signal respectively. Throughout the paper we assume that the state space model (3) is fully controllable and observable.

Below we investigate stability of distributed control policies (or protocols) of the following commonly used type. We assume that possible directions of the information exchange between the agents for any \( t \geq 0 \) are defined by a graph \( G(t) \), called the communication or underlying graph (or topology) of the network at time \( t \geq 0 \). This means that \( j \)-th agent obtains information about the value the output signal \( y_k(t) \) if and only if \( k \in N_j(G(t)) \). We assume that the graph-valued function \( G(\cdot) : [0; +\infty) \to \mathbb{G}_N \) is Lebesgue measurable.

The input function of any agent is determined by neighbors’ outputs as follows:

\[ u_j(t) = \sum_{k \in N_j(G(t))} \varphi_{jk}(t, y_k(t) - y_j(t)) \quad (4) \]

Here \( \varphi_{ij}(t, y_j) \) (with \( 1 \leq i, j \leq N, \ i \neq j \)) is a family of functions called couplings, describing the mutual interactions between the agents.

Definition 1. We say that the output consensus in the network (3), (4) is achieved, or that the protocol (4) provides the output consensus, if for any set of initial data \( (x_j(0))_{j=1}^{N} \) one has

\[ \lim_{t \to +\infty} |y_j(t) - y_i(t)| = 0, \quad 1 \leq i, j \leq N. \quad (5) \]

Analogously, the protocol (4) is said to provide the state consensus, if for any set of initial data \( (x_j(0))_{j=1}^{N} \) one has

\[ \lim_{t \to +\infty} |x_j(t) - x_i(t)| = 0, \quad 1 \leq i, j \leq N. \quad (6) \]

The couplings \( \varphi_{ij} \) are, in general, nonlinear functions and uncertain, we suppose only the following assumptions to be valid.

Assumption 2. (Symmetry in the network). Each coupling \( \varphi_{ij}(t, y_j) \) is odd with respect to \( y \): \( \varphi_{kj}(t, -y) = -\varphi_{kj}(t, y) \) and \( \varphi_{kj}(t, y) = \varphi_{jk}(t, y) \). In particular, \( \varphi_{kj}(t, 0) = 0 \) and \( \varphi_{kj}(y_k - y_j) = -\varphi_{kj}(y_j - y_k) \). Thus for any solution \( x_j \), \( u_j \) (with \( 1 \leq j \leq N \)) of the closed-loop system (3),(4) one has \( u_j(t) = 0 \) if \( y_1(t) = y_2(t) = \ldots = y_N(t) \).

Also one has

\[ \frac{\sum_{j=1}^{N} u_j(t) = 0}{1} \frac{\sum_{j=1}^{N} x_j(t) = e^{tA} \frac{1}{N} \sum_{j=1}^{N} x_j(0)}{1} \quad (7) \]

Assumption 3. (Sector inequality). There exist \( \alpha, \beta \) such that \( 0 \leq \alpha < \beta \leq +\infty \) and the inequalities hold:

\[ \alpha \leq \frac{\varphi_{jk}(t, \sigma)}{\sigma} \leq \beta, \quad \forall \sigma \neq 0, \forall j \neq k. \quad (8) \]

Equivalently, the graph \( \{(\sigma, \xi) : \xi = \varphi_{jk}(\sigma, \xi)\} \) of each function \( \varphi_{jk}(\cdot, \cdot) \) lies between the lines \( \xi = \sigma \alpha \) and \( \xi = \sigma \beta \) (degenerating into the vertical line \( \sigma = 0 \) for \( \beta = +\infty \)).

If \( \beta = +\infty \), we additionally assume that \( \varphi_{jk}(t, \sigma) \) is uniformly bounded whenever \( \sigma \) runs over bounded set: for any compact \( K \subset \mathbb{R} \) one has

\[ \sup\{|\varphi_{jk}(t, \sigma)| : t \geq 0, \sigma \in K\} < +\infty. \]

Assumption 4. (Strictness of sector inequality). If \( \sigma \) remains bounded and separated from 0, then \( \varphi_{jk}(t, \sigma) \) is separated from \( \alpha, \beta \) uniformly for \( t \geq 0 \). More precisely, for any compact set \( K \subset \mathbb{R} \setminus \{0\} \) there exist \( \alpha' = \alpha'(K) > \alpha, \beta' = \beta'(K) < \beta \) such that

\[ \alpha' \leq \frac{\varphi_{jk}(t, \sigma)}{\sigma} \leq \beta', \quad \forall t \geq 0, \sigma \in K. \]

Remark 5. Note that due to (7) the state consensus condition (6) implies that

\[ \lim_{t \to +\infty} |x_j(t) - e^{tA} \tilde{x}| = 0, \quad \tilde{x} = \frac{1}{N} \sum_{j=1}^{N} x_j(0), \]

and the output consensus condition (5) implies that

\[ \lim_{t \to +\infty} |y_j(t) - C e^{tA} \tilde{x}| = 0 \]

for any \( j = 1, 2, \ldots, N \).

Lemma 6. Let the couplings \( \varphi_{ij}(t, \sigma) \) be uniformly continuous at \( \sigma = 0 \) (with respect to \( t \geq 0 \)), i.e. \( \sup_{\sigma > 0} |\varphi_{ij}(t, \sigma)| \to 0 \) as \( \sigma \to 0 \) (e.g. \( \beta < +\infty \)). Then the output consensus (5) implies the state consensus (6).

Indeed, the synchronization of outputs (5) implies that \( u_j(t) \to 0 \) as \( t \to +\infty \) for any \( j \). Since the functions \( X_{ij} = x_j - x_i, U_{ij} = u_j - u_i \) and \( Y_{ij} = y_j - y_i \) are evident to obey the controllable and observable state-space equations

\[ \dot{X}_{ij} = AX_{ij} + BU_{ij}, Y_{ij} = CX_{ij}, \]

one has \( X_{ij}(t) \to 0 \) as \( t \to +\infty \).
The aim of the present paper is to give effective sufficient conditions for achievement of state or output consensus whenever the network (3), (4) satisfies the Assumptions 2-4. Such consensus criteria, involving the agent’s transfer functions and sector bounds only, are presented in the next section. Their proofs can be found in the Appendix.

4. MAIN RESULTS.

For the sake of brevity we introduce some auxiliary notations. Introduce constants $\gamma, \delta$ as follows
\[
\gamma = \frac{1}{\beta + \alpha}, \quad \delta = \frac{\alpha \beta}{\beta + \alpha}
\] (9)
(by definition we take $\delta = \alpha$ for $\beta = +\infty$). Here $\alpha, \beta$ stand for the sector bounds from the Assumption 3. Assumptions 3 and 4 can now be reformulated as follows.

Lemma 7. Suppose the couplings $\varphi_{ik}(t, \sigma)$ satisfy Assumption 3. Then
\[
\eta_{ij}(t, \sigma) := \varphi_{ij}(t, \sigma)\sigma - \delta \sigma^2 - \gamma \varphi_{ij}(t, \sigma)^2 \geq 0.
\] (10)
Assumption 4 being valid, $\eta_{ij}(t, \sigma)$ is separated from 0 (uniformly for $t \geq 0$) if $\sigma$ remains bounded and separated from 0. More precisely, for any compact set $K \subset \mathbb{R} \setminus \{0\}$ we have $\inf\{\eta_{ij}(t, \sigma) : \sigma \geq 0, \sigma \in K\} > 0$.

The proof of Lemma 7 is obvious and omitted.

Let $W_x, W_y$ stand for the transfer functions of the plant (3) from $u$ to $x$ and $y$ respectively
\[
W_x(\lambda) = (\lambda I_d - A)^{-1} B, \quad W_y(\lambda) = CW_x(\lambda).
\] (11)
Define a function $R_N(\omega, \theta)$ as follows:
\[
R_N(\omega, \theta) = Re W_y(\omega) + \delta |\omega| W_y(\omega)|^2 + \frac{\gamma}{2(N-1)},
\] (12)
where $\theta \in \mathbb{R}$ and $\omega \in \mathbb{R}$, $det(\omega I_d - A) \neq 0$.

4.1 Fixed topology case.

In this paragraph we deal with the case of fixed network topology $G(t) = G_0$ with the graph $G_0$ connected (for the disconnected network the protocol, evidently, can not guarantee synchronization between different connectivity domains).

The class of all nonlinear protocols (4) satisfying Assumptions 2-4 contains linear protocols with the couplings $\varphi_{ik}(t, \sigma) = \mu \sigma$ ($\mu \in (\alpha; \beta)$). Therefore one can expect that any of those nonlinear protocols guarantee consensus only if the consensus is achieved in a linear network as follows:
\[
\dot{x}_j = Ax_j + \mu BC \sum_{k \in N_j(G_0)} (x_k - x_j), 1 \leq j \leq N.
\] (13)
Introducing the joint vector $\tilde{x}(t) = col(x_1(t), \ldots, x_N(t)) \in \mathbb{R}^{dN}$, the system (13) may be rewritten as follows:
\[
\dot{\tilde{x}}_j = [I_N \otimes A - \mu L(G_0) \otimes BC] \tilde{x}_j
\] (14)
Note that accordingly to Lemma 6, the output and state consensus for the networked system (13) are equivalent. The achievement of state consensus for the linear network (13) is known to be easily verifiable, given the graph Laplacian eigenvalues (Fax and Murray (2004), Seo et al. (2009)).

Lemma 8. Consider a linear networked system (13). The state consensus condition (6) is fulfilled for any initial data if and only if for all nonzero eigenvalues $\lambda$ of $L(G_0)$ the matrix $A - \mu \lambda BC$ is Hurwitz.

The following result gives frequency-domain sufficient condition for the state consensus with exponential convergence rate in (6). Note that it does not require the couplings to satisfy Assumption 4.

Theorem 9. Suppose that $G_0$ is connected graph and for some $\mu \in (\alpha; \beta)$ and any eigenvalue $\lambda > 0$ of $L(G_0)$ the matrix $A - \mu \lambda BC$ is Hurwitz. Suppose that a constant $\varepsilon > 0$ exists such that for any $\omega \in \mathbb{R}$ one has
\[
R_N(\omega, \lambda_2(L(G_0))) \geq \varepsilon |W_x(\omega)|^2.
\] (15)
Then any protocol (4) with $G(t) = G_0$ and couplings, satisfying Assumptions 2,3, provides the state consensus (6). Furthermore, a constant $\nu > 0$ exists such that
\[
\sum_{i,j=1}^{N} |x_i(t) - x_j(t)|^2 \leq Me^{-\nu t} \sum_{i,j=1}^{N} |x_i(0) - x_j(0)|^2.
\] (16)

Remark 10. Suppose the matrix $A$ to have no eigenvalues on the imaginary axis. Then (15) may be rewritten as $R_N(\omega, \lambda_2(L(G_0))) > 0$ for any $\omega \in [-\infty; +\infty]$. Geometrically the latter inequality may be treated as follows: if $\alpha > 0$ and thus $\delta > 0$, the Nyquist curve of the plant (3), i.e. the locus of points $W_y(\omega)$ on the complex plane, lies outside the circle centered at $z_0$ with radius $R_0$, where
\[
z_0 = -\frac{1}{2\delta\lambda_2(L(G_0))}, \quad R_0 = \frac{1}{2\delta\lambda_2(L(G_0))} \sqrt{1 - 2\gamma\lambda_2(L(G_0))}.
\]
For $2\gamma\lambda_2(L(G_0)) > N - 1$ the circle vanishes, and the frequency domain inequality is always fulfilled. In the case of $\alpha = 0$, the frequency domain inequality has the form $Re W_y + \frac{\gamma}{2(N-1)} \geq 0$, i.e. the Nyquist plot should lie on the right from the vertical line $\left\{ -\frac{\gamma}{2(N-1)} + i\theta, \theta \in \mathbb{R} \right\}$.

The result of Theorem 9 appears to be a generalization of the celebrated circle criterion in the absolute stability theory (Khalil (1996), Gelig et al. (2004)). Assume for that $det(\omega I_d - A) \neq 0$ for simplicity and consider a group of $N = 2$ agents (3), coupled with a function $\varphi_{12} = \varphi_{21}$ satisfying the Assumptions 2,3. Evidently $\lambda_2(L(G_0)) = 2$. Let $X(t) = x_2(t) - x_1(t)$, $Y(t) = y_2(t) - y_1(t)$ $= CX(t)$, $U(t) = u_2(t) - u_1(t)$ and $\Phi(t, Y) = -2\varphi_{12}(t, Y)$ so $\Phi(t, Y) \in [-2\beta; -2\alpha]$. The achievement of exponential state consensus means the exponential stability of equilibrium point $X = 0$ of the nonlinear system as follows:
\[
\dot{X}(t) = AX(t) + BU(t), U(t) = \Phi(t, Y(t))
\] (17)
The circle criterion provides the global stability under assumption that the Nyquist curve lies strictly outside the circle with the diameter $[-(2\alpha)^{-1}, -(2\beta)^{-1}]$. A straightforward computation shows this circle to coincides with one from the Remark 10.

Unfortunately conditions of the Theorem 9 are often violated, e.g. the frequency domain inequality (15) implies that $\gamma > 0$, excluding the case of infinite sector $(\beta = +\infty)$. Also it is usually violated for $\alpha = 0$ (consider for example single integrator dynamics with $A = 0$ and $B = C = 1$).

Under Assumption 4 it appears possible to weaken the frequency domain inequality, replacing the exponential state consensus condition with the output one.

Theorem 11. Suppose that $G_0$ is connected graph such that for some $\mu \in (\alpha; \beta)$ and any eigenvalue $\lambda > 0$ of $L(G_0)$ the matrices $A - \mu \lambda BC$ is Hurwitz. Let a frequency domain inequality hold as follows:
\[
R_N(\omega, \lambda_2(L(G_0))) \geq 0, \forall \omega \in \mathbb{R}, det(\omega I_d - A) \neq 0.
\] (18)
Then any protocol (4) with \( G(t) = G_0 \), satisfying Assumptions 2-4, provides that

(1) deviations between the state vectors \( |x_j(t) - x_i(t)| \) remain bounded: for some constant \( M > 0 \) one has

\[
\sum_{i,j=1}^{N} |x_j(t) - x_i(t)|^2 \leq M \sum_{i,j=1}^{N} |x_j(0) - x_i(0)|^2 \tag{19}
\]

(2) the output consensus (5) is achieved;

(3) if \( \sup_{t \geq 0} |\varphi_{ij}(t, \sigma)| \to 0 \) as \( \sigma \to 0 \) for all \( i, j \) (e.g. \( \beta < +\infty \)) the state consensus (6) is also achieved.

**Remark 12.** The frequency domain inequality (18) means that the points of the Nyquist curve should not go inside the circle (or half-plane) from Remark 10.

### 4.2 Time-varying network graph.

This paragraph is devoted to more complicated situation when the network topology is time-varying and may be unknown.

Denote by \( G_N(\theta) \) the set of all graph-valued Lebesgue measurable functions \( G : [0; +\infty) \to G_N \) such that \( \lambda_2(G(t)) \geq \theta \) for almost all \( t \geq 0 \). Evidently \( G_N(\theta) \) consists of all measurable graph-valued functions \( G : [0; +\infty) \to G_N \) for \( \theta = 0 \), and is empty set for \( \theta > N \).

Main results of this paragraph are Theorems 13 and 15 that are counterparts of Theorems 9 and 11 but concern time-varying topology function \( G(\cdot) \). The latter function may be uncertain, but belongs to \( G_N(\theta) \) for some \( \theta \geq 0 \). The less \( \theta \) one takes, the more restrictive constraints on the agents’ dynamics and the sector \( [\alpha; \beta] \) one has to pose in order to guarantee the consensus. For the case \( \theta = 0 \) (the topology is completely unknown) we also need some connectivity assumptions.

For any \( 0 \leq \theta < N \) the set \( G_N(\theta) \) contains the graph-valued function \( G(t) \equiv C_N \), where \( C_N \) is a complete graph. Therefore if any protocol (4) with \( G(\cdot) \in G_N(\theta) \) and satisfying Assumptions 2-4 the consensus, the consensus is achieved for \( G(t) \equiv C_N \) and \( \varphi_{ij}(t, \sigma) = \mu \sigma, \mu \in (\alpha; \beta) \). Accordingly to the Lemma 8, the matrix \( A - \mu NBC \) should be Hurwitz.

**Theorem 13.** Let \( G(\cdot) \in G_N(\theta) \) for some \( \theta > 0 \). Assume that for some \( \mu \in (\alpha; \beta) \) the matrix \( A - \mu NBC \) is Hurwitz. Suppose that a constant \( \varepsilon > 0 \) exists such that for any \( \omega \in \mathbb{R} \), \( \det(\omega I - A) \neq 0 \) one has

\[
R_N(\omega, \theta) \geq \varepsilon |W_x(\omega)|^2. \tag{20}
\]

Then any protocol (4) with given graph \( G(\cdot) \) and couplings, satisfying Assumptions 2,3, provides the state consensus with exponential convergence rate (16).

**Remark 14.** Note that for \( \theta = 0 \) we do not need any connectivity assumptions on the graph \( G(\cdot) \). In particular, for the totally disconnected graph and \( u_j(t) \equiv 0 \) one has (6) as well. Therefore the assumptions of Theorem 13 for \( \theta = 0 \) imply stability of agent’s dynamics: \( A \) is Hurwitz matrix. In most applications this is not the case so the proposition of Theorem 13 is of main interest for \( \theta > 0 \).

The next result is analogous to Theorem 11 and deals with the case of non-strict frequency domain inequality.

**Theorem 15.** Let \( G(\cdot) \in G_N(\theta) \) for some \( \theta > 0 \). Assume that for some \( \mu \in (\alpha; \beta) \) the matrix \( A - \mu NBC \) is Hurwitz. Let a frequency domain inequality hold as follows:

\[
R_N(\omega, \theta) \geq \varepsilon |W_x(\omega)|^2. \tag{21}
\]

Then for any protocol (4) with given graph \( G(\cdot) \) and couplings, satisfying Assumptions 2-4, the following statements are true:

(1) deviations between the state vectors \( |x_j(t) - x_i(t)| \) remain bounded: for some constant \( M > 0 \) (19) holds;

(2) if \( G(t) \) is connected for almost all \( t \geq 0 \), the output consensus is achieved;

(3) if \( G(t) \) is connected for almost all \( t \geq 0 \) and \( \sup_{t \geq 0} |\varphi_{ij}(t, \sigma)| \to 0 \) as \( \sigma \to 0 \) (e.g. \( \beta < +\infty \)) the state consensus (6) is achieved;

### 5. EXAMPLES

#### 5.1 Single integrators and the Kuramoto model.

Consider the first order agents’ dynamics given by

\[
\dot{x}_j = u_j = \sum_{k \in N_j(G(t))} \varphi(x_k - x_j), j = 1, \ldots, N \tag{22}
\]

where the function \( \varphi(\sigma) \) is odd, continuous and satisfying \( \varphi(\sigma) > \alpha_0|\sigma|^2 \) for \( \sigma \neq 0 \), where \( \alpha \geq 0 \) is some constant. It is evident that the protocol in (22) satisfies Assumptions 2-4 for \( \alpha = \alpha_0, \beta = +\infty \) corresponding to \( \delta = \alpha_0, \gamma = 0 \).

Since \( W_y = W_{y2} = \frac{1}{\sqrt{N}} \), one easily sees that the strict frequency-domain inequalities (15), (20) hold if and only if \( \theta, \alpha_0 > 0 \), but (21) is fulfilled whenever \( \theta, \alpha_0 \geq 0 \). Note also that since \( \alpha = 0, B = C = 1 \) the system \( \dot{x} = (A - \lambda NBC)x \) is exponentially stable for any \( \lambda > 0 \).

Combining the results of Theorems 9,11 to the networked system, one obtains the following convergence condition (proved by other methods in Olfati-Saber and Murray (2003), Chopra and Spong (2006)):

**Theorem 16.** Consider a network (22) with coupling function \( \varphi(\cdot) \) odd, continuous and satisfying \( \varphi(\sigma)\sigma > \alpha_0|\sigma|^2 \) for \( \sigma \neq 0 \). Then

(1) the boundedness condition (19) holds;

(2) if \( G(t) \) is connected almost everywhere, then the state consensus (6) is achieved, the convergence rate in (6) is exponential for \( \alpha_0 > 0 \);

Among practical examples of the networked systems having dynamics (22) we mention the famous Kuramoto model of coupled oscillators dynamics. A population of \( N \) oscillators with phase angles \( \theta_j \) is assumed to be governed by the following equations:

\[
\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{i=1}^{N} \sin(\theta_i - \theta_j)
\]

Here \( K > 0 \) is a coefficient describing the coupling strength. A natural generalization is the system of oscillators with time-varying interconnectedness topology \( G(\cdot) : [0; +\infty) \to G_N \) with pairs of oscillators not coupled:

\[
\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{i \in N_j(G(t))} \sin(\theta_i - \theta_j)
\]
Assume that the oscillators are identical: \( \omega_j = \omega \) for any \( 1 \leq j \leq N \). Without loss of generality, we may assume that \( \omega_j = 0 \) (otherwise one can replace \( \theta_j \) with \( \theta_j - \omega \)). Suppose also that initial data satisfy inequalities \( |\phi_j(0)| < \frac{\alpha}{2} - \varepsilon \) for some \( \varepsilon > 0 \). It can be easily shown that \( |\phi_j(t)| < \frac{\alpha}{2} - \varepsilon \) for any \( t \geq 0 \), therefore
\[
\alpha \leq \sin(\phi_j - \phi_i) \leq 1
\]
for some \( \alpha = \alpha(\varepsilon) > 0 \). From Theorem 16 the following result is immediate:

**Lemma 17.** Suppose that \( \omega_j = 0 \) for any \( j \). If \( |\phi_j(0)| < \frac{\pi}{2} \) and \( G(t) \) is connected for any \( t \geq 0 \), the phase synchronization is achieved: \( \lim_{t \to +\infty} (\phi_j(t) - \phi_i(t)) = 0 \), and the convergence is exponential.

5.2 Second order dynamics.

Suppose that agent model has a form
\[
\dot{z}_j = u_j, \quad y_j = p z_j + q z_j
\]
with \( p, q > 0 \) and consider a protocol as follows:
\[
u_j = \sum_{k \in N_j} \varphi(y_k - y_j), \quad j = 1, \ldots, N
\]
with \( \varphi(\sigma) \) odd, continuous and satisfying \( \varphi(\sigma)\sigma \geq a|\sigma|^2 \). Thus we consider a sector \([\alpha; \infty)\) for which \( \delta = \alpha, \gamma = 0 \).

Transforming the system (23) into the state-space form by taking \( x = (z, \dot{z})^T, A = (0, 1)^T, C = (q, p) \) and \( B = (0, 1)^T \), it is immediate that \( A - \lambda BC \) is Hurwitz whenever \( \lambda > 0 \).

Note that \( W_p(\omega) = (p \omega + q)/(\omega^2) \). The function \( R \) from (12) is as follows:
\[
R_N(\omega, \theta) = -\frac{q}{\omega^2} + \theta \left( \frac{\delta p^2 \omega^2 + q^2}{\omega^4} \right) = \left( \frac{\theta \alpha p^2 - q}{\omega^2} \right)^2 + \frac{\theta \alpha q^2}{\omega^2}
\]
In particular, for \( \theta > q/(\alpha p^2) \) the strict frequency domain inequality (20) is fulfilled. Thus we obtain the convergence result as follows.

**Theorem 18.** Suppose that \( \lambda_2(L(G(t))) > q/(\alpha p^2) \) for almost all \( t \geq 0 \) (this is the case, e.g. if \( G(t) \) is connected and \( q < 2\alpha p^2(1 - \cos \frac{\pi}{N}) \)). Then the state consensus with exponential convergence rate (16) is achieved.

Appendix A. PROOFS OF THE MAIN RESULTS.

A.1 Technical lemmas.

Denote by \( F_2(x, u) \) a quadratic form of variables \( x \in \mathbb{R}^d \), \( u \in \mathbb{R} \) as follows:
\[
F_2(x, u) = -u C x - \delta [C x]^2 - \frac{\delta}{2(N-1)^2} u^2.
\]
Here \( \delta, \gamma \) are defined by (9) and \( C \) is from (3).

**Lemma 19.** Suppose that functions \( \varphi_{ij} \) satisfy Assumptions 2.3 and \( u_i(t) \) is given by (4) for any \( i = 1, \ldots, N \). Suppose also that \( \lambda_2(L(G(t))) \geq \theta \). Then any solution of the system (3), (4) satisfies a quadratic constraint
\[
\sum_{i,j=1}^N F_2(x_j(t) - x_i(t), u_j(t) - u_i(t)) \geq 0.
\]

To prove the Lemma 19 introduce functions \( \xi_{ij}(t) = a_{ij}(G(t))\varphi_{ij}(t, y_j(t) - y_i(t)), \eta_{ij} = a_{ij}(G(t))\varphi_{ij}(y_j(t) - y_i(t)), \) \( i \neq j \), where \( a_{ij}(G) \) stands for the adjacency matrix of the graph \( G \). In particular, \( a_{ij}^2 = a_{ij} \). Due to (4) one has \( u_i(t) = \sum_{k=1}^N \xi_{ik}(t) \) and therefore \( |u_i(t)|^2 \leq N - 1 \sum_{k=1}^N |\xi_{ik}(t)|^2 \). Applying the inequalities (10) to \( \sigma = y_i(t) - y_j(t) \), one obtains
\[
|\xi_{ij}(y_i(t) - y_j(t)) - \delta \alpha_{ij} G_0| |y_i(t) - y_j(t)|^2 - \gamma |\xi_{ij}(t)|^2 \geq 0.
\]
By summation due to Assumption 2 \( (\xi_{ij} = -\xi_{ji}) \) and the definition of algebraic connectivity (2) one obtains that
\[
-2 \sum_{j=1}^N |y_j(t) - y_i(t)|^2 - \gamma \sum_{j=1}^N |u_j(t)|^2 \geq 0.
\]
Multiplying the latter inequality by \( N \) and using the identity \( \sum u_j = 0 \), one easily derives (A.1).

Analyzing the proof of the previous lemma, the following result is easily shown:

**Lemma 20.** Suppose that the protocol (4) to satisfy Assumptions 2-4 and let \( S(t) = \sum_{i,j=1}^N |y_i(t) - y_j(t)| \). If \( G(t) \) is connected and \( S(t) > 0 \) for some \( t \geq 0 \), then the inequality in (A.1) is strict. Moreover, if \( G(t) \) is connected almost everywhere, \( S(t) \) is bounded and the set \( \{ t : S(t) > \varepsilon \} \) has infinite Lebesgue measure for some \( \varepsilon > 0 \), then
\[
\sum_{i,j=1}^N \int_0^\infty F_2(x_j(t) - x_i(t), u_j(t) - u_i(t)) dt = +\infty.
\]

The next lemma is obvious consequence of the Kalman-Yakubovich-Popov lemma for the quadratic form \( F_2 \):

**Lemma 21.** Suppose that the frequency domain inequality (21) or, respectively, (20) is fulfilled for some \( \theta \). Then there exists a \( d \times d \)-matrix \( H_\theta = H_\theta^T \) such that
\[
2x^T H_\theta (Ax + Bu) + F_2(x, u) \leq 0,
\]
or, respectively,
\[
2x^T H_\theta (Ax + Bu) + F_2(x, u) \leq -\varepsilon |x|^2.
\]

A.2 Proofs of Theorems 9,11,13,15.

A scheme of the proof of all the four theorems may be outlined as follows. Suppose the frequency domain inequality (21) (respectively (20)) is fulfilled for some \( \theta \), which equals to \( \lambda_2(G_0) \) in (18), (15).

We introduce a Lyapunov function
\[
V(\bar{x}) = \sum_{i,j=1}^N (x_j - x_i)^T H_\theta (x_j - x_i), \quad \bar{x} = \text{col}(x_1, \ldots, x_N).
\]
Here \( H_\theta \) is a matrix from Lemma 21. From (A.2) or, respectively from (A.3), one easily obtains that
\[
\dot{V} + \sum_{i,j=1}^N F_2(x_j - x_i, u_j - u_i) \leq 0
\]
or, respectively,
\[
\dot{V} + \sum_{i,j=1}^N F_2(x_j - x_i, u_j - u_i) \leq -\varepsilon \sum_{i,j=1}^N |x_j - x_i|^2.
\]

Assumptions of all Theorems 9,11,13,15 imply that the state consensus is provided by some (4) with \( \varphi_{ij}(t, \sigma) = \mu \sigma \) \( (\mu \in (\alpha; \beta)) \) and \( G(\cdot) \in G_N(\theta) \) being constant \( (G_0 \) in Theorems 9,11 or complete graph in Theorems 13,15).
This protocol satisfies Assumptions 2-4. Substituting a corresponding solution of (3), (4) into (A.4) and integrating, one obtains due to (6) and $V(\bar{x}(t)) \rightarrow \infty$ as $t \rightarrow +\infty$ that

$$V(\bar{x}(0)) \geq \int_0^{+\infty} \sum_{i,j=1}^N F(x_i(t) - x_j(t), u_i(t) - u_j(t)) dt \quad (A.6)$$

Due to Lemma 20, we have $V(\bar{x}(0)) > 0$ whenever $\sum_{i,j=1}^N |y_i(t) - y_j(t)|^2 > 0$ for some $t \geq 0$. Thus $V(\bar{x}(0)) \leq 0$ is possible only for $y_i - y_j \equiv 0$ and thus $u_i \equiv 0$, which implies $x_i(0) = x_j(0) = 0$ for any $i, j$. Thus $H_0 > 0$.

The remainder of proofs of Theorems 9, 13 are trivial: due to (A.5) one has that $\frac{dV(\bar{x}(t))}{dt} \leq -\nu V(\bar{x}(t))$ for some $\nu > 0$, thus $V(\bar{x}(t)) \leq V(\bar{x}(0))e^{-\nu t}$, which is followed by the exponential rate state consensus (16).

Inequality (19) is immediate from Lemma 19, (A.4) and $H_0 > 0$. To prove the output consensus, assume on the contrary that a sequence $t_n \rightarrow +\infty$ and $\epsilon > 0$ exist such that $S(t) = \sum_{i,j=1}^N |y_i(t) - y_j(t)| > 2\epsilon$ for $t = t_k$, $k \geq 1$. Due to (19) and Assumption 3 one obtains that both $y_j - y_i$ and $y_j - y_i$ are globally bounded functions. Thus for some $r > 0$ one has $|S(t)| > \epsilon$ for $t \in (t_n - r; t_n + r)$. But this contradicts Lemma 20 since (A.6) should be fulfilled. The contradiction shows that the output consensus is achieved.

The last proposition follows from the Lemma 6.

REFERENCES


