A design method of suboptimal LQ control law for mode transition systems

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Abstract: An LQ control problem for mode transition systems is considered and a design method of suboptimal control law, which guarantees the upper/lower bounds of the corresponding cost-functional, is derived. The control law is given by piecewise affine function of the state and each gain is easily obtained based on the solutions of finite-horizon LQ control problems. Furthermore, the achievable performance of the control law is characterized in terms of the initial state. The design method is applied to the start-up control of a hot strip mill tension/looper system and the feature of the control system is investigated.

Keywords: LQR control, mode transition system, constrained system, piecewise affine controller, tension/looper system.

1. INTRODUCTION

Optimal controller design for mode transition systems arises frequently in the practical system control and, for example in the steel process, various design methods are applied for improving the start-up performance of tension/looper systems which inherit the mode transition of dynamics (Imura et al. (2004); Asano et al. (2005); Masuda et al. (2008); Kojima et al. (2008)). The mode transition systems are addressed in a class of piecewise affine (PWA) systems and, based on the strategy of model predictive control, standard design procedures are established (e.g. Christophersen et al. (2007); Borrelli (2003)). However, the received design methods for the PWA systems require either online optimization or offline parameterization in the state space and the mimic application to the possibly high-order mode transition system is prohibitive even though the transition rule is quite simple.

In this paper, we focus on a simple but important class of mode transition systems, which covers typical start-up control problem (e.g. Imura et al. (2004); Asano et al. (2005)), and propose a design method of suboptimal LQ control law, which inherits the design method of LQ regulator for finite-horizon problems. The design method provides a PWA control law and each region is explicitly given by the difference of polyhedral regions. The feature of control law is further investigated and it is clarified that 1) the proposed control law is exactly optimal in the case the initial state lies in a certain set of state-space, 2) for any initial states, the upper and lower bounds of the optimal performance are given by the solutions of difference Riccati equation. The design method is applied to the tension/looper system reported by Kojima et al. (2008) and the feature of the control system is investigated.

2. BASIC PROBLEM

For the discrete-time mode transition system:

\[ x_0 \in U : \text{initial state} \]  
\[ \text{Mode I} \left\{ \begin{array}{l} x_k \in U, \quad U := \{ x_k | x_k \notin \mathcal{C} \} \\ x_{k+1} = Ax_k + Bu_k \end{array} \right\} \]  
\[ \text{Mode II} \left\{ \begin{array}{l} x_k \in \mathcal{C}, \\ x_{k+1} = \tilde{A}x_k + \tilde{B}u_k \end{array} \right\} \]

consider an LQ control problem defined as follows:

\[ \mathcal{P} : J^*(x_0) := \min_{N, U} J(x_0, U) \]  
\[ U := \{ u_0, u_1, \ldots \} \]

subj. to \( \left\{ \begin{array}{l} x_i \in U, \quad \forall i = 0, 1, \ldots, N_s - 1 \\ x_j \notin \mathcal{C}, \quad \forall j = N_s, N_s + 1, \ldots \end{array} \right\} \]

where \( x_k \in \mathbb{R}^m, u_k \in \mathbb{R}^m \) are the state and the control input and \( N_s \) is the transition time from Mode I to II. The cost-functional \( J(x_0, U) \) is defined by

\[ J(x_0, U) = J_I + J_{II} \]  
\[ J_I = \sum_{k=0}^{N_s-1} (x_k^T Q x_k + u_k^T R u_k), \quad Q > 0, R > 0 \]  
\[ J_{II} = \sum_{k=N_s}^{\infty} (x_k^T \tilde{Q} x_k + u_k^T \tilde{R} u_k), \quad \tilde{Q} \geq 0, \tilde{R} > 0 \]
where $>$, $\geq$ denote positive definiteness and positive semi-definiteness of the matrices respectively. For the problem $\mathcal{P}$, we make following assumptions (A1)-(A4) and propose a design method of sub-optimal control law which guarantees the upper and lower bounds of (3).

(A1) There exists a control $U$ which satisfies $J(x_0, U) < \infty$.

(A2) The region $C$ is convex and $0 \in \mathbb{R}^n$ is an interior point of $C$.

(A3) $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{Q}^{1/2}, \tilde{A})$ is detectable.

(A4) The region $C$ is positively invariant in the case the optimal control for (3c) is applied.

Under the conditions (A3), (A4), the optimal control in the Mode II and the resulting cost of (3c) are expressed by

$$ u_k = \tilde{K}_{LQ} x_k $$

$$ \tilde{K}_{LQ} = -(\tilde{R} + \tilde{B}^T \tilde{P} \tilde{B})^{-1} \tilde{B}^T \tilde{P} \tilde{A} $$

$$ J^*_k = x_k^T \tilde{P} x_k $$

where $\tilde{P} \geq 0$ is the solution of the following Riccati equation (e.g. Kwakernaak (1972)).

$$ \tilde{P} = \tilde{Q} + \tilde{A}^T \tilde{P} \tilde{A} - \tilde{A}^T \tilde{P} \tilde{B}(\tilde{R} + \tilde{B}^T \tilde{P} \tilde{B})^{-1} \tilde{B}^T \tilde{P} \tilde{A} $$

Hence in the simple case such that the transition time $N_s$ is a priori given, the control problem $\mathcal{P}$ is solved based on the auxiliary cost-functional

$$ J_{N_s}(x_0, U_{N_s}) = \sum_{k=0}^{N_s-1} (x_k^T Q x_k + u_k^T R u_k) + x_{N_s}^T \tilde{P} x_{N_s} $$

$$ U_{N_s} := \{ u_0, u_1, \ldots, u_{N_s-1} \} $$

with the system description (1b).

In the sequel, we introduce auxiliary problems $\mathcal{P}_N$, $\mathcal{P}_N^U$:

$$ \mathcal{P}_N : J_N^*(x_0) := \min_{U_N} J_N(x_0, U_N) $$

$$ U_N := \{ u_0, u_1, \ldots, u_{N-1} \} $$

$$ \text{subj. to } \begin{cases} x_k \in U, & \forall k = 0, 1, \ldots, N-1 \\ x \in C \end{cases} $$

$$ \mathcal{P}_N^U : J_N^{U_s}(x_0) := \min_{U_N} J_N(x_0, U_{N_s}) $$

$$ U_{N_s} := \{ u_0, u_1, \ldots, u_{N_s-1} \} $$

and derive a sub-optimal control law for the problem $\mathcal{P}$. The problem $\mathcal{P}_N$ is a constrained finite-horizon LQ control problem which imposes the mode transition time at $N$. While $\mathcal{P}_N^U$ is a standard (unconstrained) finite-horizon LQ control problem. Employing the solutions $\{ P_0, P_1, \ldots \}$ of the difference Riccati equation:

$$ P_{k+1} = Q + A^T P_k A - A^T P_k B (R + B^T P_k B)^{-1} B^T P_k A $$

$$ P_0 := \tilde{P} \quad \text{(defined by (7))}, $$

the optimal control law for $\mathcal{P}_N^U$ and the resulting cost are given as follows (e.g. Kwakernaak (1972)).

$$ U_{N_s} := \{ u_0, u_1, \ldots, u_{N_s-1} \} $$

$$ u_k = K_{N_s - \ell} x_{\ell} \quad (\ell = 0, 1, \ldots, N-1) $$

$$ K_k = -(R + B^T P_{k-1} B)^{-1} B^T P_{k-1} A \quad (k = 1, 2, \ldots, N) $$

$$ J_N^{U_s}(x_0) := J_N(x_0, U_{N_s}) = x_{N_s}^T P_N x_0 $$

The problems $\mathcal{P}_N$, $\mathcal{P}_N^U$ enable to obtain the inequalities:

$$ J_N^{U_s}(x_0) \leq J^*(x_0) \leq J_N^*(x_0), \quad \forall N > 1 $$

3. CONTROL LAW

Based on $\mathcal{P}_N$, $\mathcal{P}_N^U$, we will derive a sub-optimal control law for the problem $\mathcal{P}$. The problems $\mathcal{P}$, $\mathcal{P}_N$ preserve the equality

$$ J^*(x_0) = \min_{U_N} J_N^*(x_0) $$

and, in principle, the control law for $\mathcal{P}$ is introduced by employing the solutions of $\mathcal{P}_N$ which horizon is appropriately defined for each initial state of (1). Furthermore as described below (Theorem 1), the problem $\mathcal{P}_N$ coincides with $\mathcal{P}_N^U$ in the case the initial state of (1) lies in a certain region of $U$.

Employing the relation between the problems $\mathcal{P}$ and $\mathcal{P}_N$, $\mathcal{P}_N$, we propose the following design method of sub-optimal control law and clarify the performance in terms of the cost-functional (3).
Algorithm [A]

1) Define a control law (4) in the state region \( \mathcal{C} \).
\[
    u_k = \tilde{K}_L Q x_k, \quad x_k \in \mathcal{C}
\] (19)

2) \( j = 1, U_0 = U_e =: \mathcal{C} \).

3) Solve the problem \( P^u_j \) (with the control horizon \( j \)) and define the control law \( u_k = K_j x_k \) and state regions \( U_j, U_j^* \) (Fig. 1).
\[
    u_k = K_j x_k, \quad x_k \in U_j
\] (20)
\[
    U_j := \{x|(A + BK_j)x \in U_{j-1}, x \notin U_{j-1}'\}
\] (21)
\[
    U_j^* := U_j \cup U_{j-1}'
\] (22)

4) If \( x_0 \notin U_j, j := j + 1 \) and go to Step 3).

5) Halt: \( N := j \), the sub-optimal control law for \( \mathcal{P} \) is given as follows.
\[
    U^* : u_k = \begin{cases} 
        K_j x_k, & x_k \in U_j \\
        \tilde{K}_L Q x_k, & x_k \in \mathcal{C}
    \end{cases}
\] (23)

The algorithm [A] yields a piecewise linear feedback control law for the basic problem \( \mathcal{P} \) and each state region \( U_j \) is easily obtained by the difference of the polyhedron \( \{U^*_j\} \).

We next investigate the performance of the proposed control law \( U^* \) and clarify that \( U^* \) is exactly optimal in the case the initial state \( (1a) \) is in the certain regions of \( \mathcal{U} \). Introducing auxiliary partitions such that \( U_j := V_j \cup W_j \) (\( j = 1, 2, \ldots \), \( N \)) holds (Fig. 2):
\[
    V_0 := U_0
\] (24a)
\[
    V_j := \{x|(A + BK_j)x \in U_{j-1}, x \notin U_{j-1}'\} \quad (j = 1, 2, \ldots , N)
\] (24b)
\[
    W_1 := \phi
\] (24c)
\[
    W_j := U_j \setminus V_j \quad (j = 2, 3, \ldots , N)
\] (24d)
the performance of the control law \( U^* \) is characterized as follows.

**Theorem 1.** The control law \( U^* \) guarantees the following performance in terms of the initial state of (1):

(a) \( \forall x_0 \in V_j \subset U \) (\( j = 1, 2, \ldots \)):
\[
    x_0^T P_{j-1} x_0 \leq J^*(x_0) = J(x_0, U^*) = x_0^T P_j x_0 \quad (j = 1, 2, \ldots)
\] (25)

(b) \( \forall x_0 \in W_j \subset U \) (\( j = 1, 2, \ldots \)):
\[
    x_0^T P_{j-1} x_0 \leq J^*(x_0) \leq J(x_0, U^*) \leq x_0^T P_j x_0 \quad (j = 1, 2, \ldots)
\] (26)

where \( J^*(x_0) \) is the optimal cost for the problem \( \mathcal{P} \) and \( J(x_0, U^*) \) denotes the resulting cost in the case the control law \( U^* \) is applied.

For the proof of Theorem 1, we first provide some preliminaries on the inequalities (25)-(26).

**Lemma 2.** For the problems \( \mathcal{P}_j \) (\( j = 1, 2, \ldots \)) defined by (9), the following inequality holds:
\[
    x_0^T P_j x_0 \leq J^*_j(x_0), \quad x_0 \in \mathcal{U}
\] (27)

where \( P_j \) are the solutions of (11).

**Proof** By Chmielewski et al. (1996) Lemma 3, the functional
\[
    J^*_N(x_0) - x_0^T P_N x_0 \geq 0
\] (28)

is convex in any convex set of \( \mathcal{U} \). Furthermore, the inequality
\[
    J^*_N(x_0) - x_0^T P_N x_0 = J^*_N(x_0) - J^*_N(x_0) \geq 0
\] (29)
holds since the problem \( \mathcal{P}_N \) is obtained by imposing constraints on \( \mathcal{P}_N \). Thus (27) follows from (28), (29).

**Lemma 3.** For the feedback gains \( K_i \) defined by (23), the state transitions are characterized as follows.
\[
    \forall x \in V_\ell, (A + BK_\ell)x \in V_{\ell-1} \quad (\ell = 1, 2, \ldots)
\] (30)
\[
    \forall x \notin U_{\ell}, (A + BK_\ell)x \notin U_{\ell-1} \quad (\ell = 1, 2, \ldots)
\] (31)
\[
    \forall x \in W_\ell, (A + BK_\ell)x \in U_{\ell-1}' \setminus V_{\ell-1} \quad (\ell = 2, 3, \ldots)
\] (32)

**Proof** (30): The inclusion (30) is directly derived from (24a),(24b).

(31): By definition (21), \( x \notin U_\ell \) holds if \( (A + BK_\ell)x \notin U_{\ell-1}' \) or \( x \notin U_{\ell-1}' \). By (22), \( x \notin U_{\ell-1}' \) does not hold for \( x \notin U_\ell' \) and the inclusion (31) is derived.

(32): Since \( x \in W_\ell \subset U_\ell \) holds, \( (A + BK_\ell)x \in U_{\ell-1}' = V_{\ell-1}' \cup W_{\ell-1}' \cup U_{\ell-1}' \) follows from (21),(22),(24d). If \( (A + BK_\ell)x \in V_{\ell-1}' \) holds, (24b),(24d) yield \( x \in V_\ell \) which contradicts the assumption \( x \in W_\ell \). Thus \( (A + BK_\ell)x \in U_{\ell-1}' \cup U_{\ell-1}' = V_{\ell-1}' \setminus V_{\ell-1} \) holds and (32) is derived.

Based on Lemma 2,3, Theorem 1 is proved as follows.

**Proof of Theorem 1** (a): \( J^*(x_0) \leq J(x_0, U^*) \) is directly obtained by definition. We will derive the following relations based on Lemma 2,3.
\[
    x_0^T P_{\ell-1} x_0 \leq J^*(x_0), \quad x_0 \in V_\ell
\] (33)
\[
    J^*(x_0) = J(x_0, U^*) = x_0^T P_{\ell} x_0, \quad x_0 \in V_\ell
\] (34)

(33): For \( \ell = 1 \), verify that the relation
\[
    \forall x_0 \in V_1, \ x_0^T P_0 x_0 \leq J^*(x_0)
\] (35)
holds. Suppose the state transition occurs at the time \( s \) in the case the optimal control for \( \mathcal{P} \) is applied to (1):
\[
    x_0, x_0, \ldots , x_{s-1} \notin C, \quad x_s \in C
\] (36)
then the optimal values of the cost-functional for the problems \( \mathcal{P}_s \) ((2) and (9)) satisfies the following inequality (Lemma 2).
\[
    J^*_s(x_0) = J^*_s(x_0) \geq x_0^T P_s x_0, \quad s \geq 1
\] (37)
Thus, under the assumption (A5), (35) is obtained.

Next, focus on the cases \( \ell = 2, 3, \ldots \) and let \( x_0 \in V_\ell \) be the initial state of (1). If the control sequence \( U_{\ell-1}' \) defined by (12) is applied to (1), the inclusions
\[
    x_k \notin U_{\ell-1}' \quad (k = 0, 1, \ldots, \ell - 2), \quad x_{\ell-1} \notin C
\] (38)
are obtained by (31). In other words, the control sequence \( U_{\ell-1}' \) does not yield mode transition in the interval \( [0, \ell - 1] \) and \( J^*_s(x_0) = x_0^T P_{\ell-1} x_0 \) holds. Thus, under the assumption (A5), the lower bound of the optimal cost \( J^*(x_0) \) satisfies (33).

(34): Let \( x_0 \in V_\ell \) be the initial state of (1) and apply the control law \( U^* \). By (30), the inclusions \( x_\ell \in V_{\ell-1} \) (\( k = 1, 2, \ldots, \ell - 1 \)), \( x_\ell \in C \) hold and \( U^* \) yields the optimal control sequence \( U^*_\ell \) for the problem \( \mathcal{P}_\ell \). Since the constraints in the problems \( \mathcal{P} \) and \( \mathcal{P}_s \) are not active in the case \( U^*_\ell \) is applied, (34) is derived with (15).

(b): \( J^*(x_0) \leq J(x_0, U^*) \) is obtained by definition. We will derive the following inequalities based on Lemma 2,3.
\[ x_0^T P_{\ell-1} x_0 \leq J^*(x_0), \quad x_0 \in W_\ell \quad (39) \]
\[ J(x_0, U^\#) \leq x_0^T P_{\ell} x_0, \quad x_0 \in W_\ell \quad (40) \]

(39): Similarly derived to (33).

(40): By induction, we prove that (a) holds for \( \ell = 2, 3, \ldots \). For \( \ell = 2 \), \( x_0 \in W_2 \) and \( x_1 = (A + BK_2) x_0 \in C \) follows from (32). In this case, the control law \( U^\# \) yields
\[ u_k = \begin{cases} K_2 x_k & k = 0 \\ K_{1Qx_k} & k = 1, 2, \ldots \end{cases} \quad (41) \]
for the system (1) with \( x_0 \in W_2 \) and we have
\[ x_0^T Q x_0 + u_0^T R u_0 = J_2^{U^*}(x_0) - J_1^{U^*}(x_1) = x_0^T P_{\ell} x_0 - x_0^T P_{\ell} x_1. \quad (42) \]
Hence, by (17),(42) and the inequality:
\[ J(x_0, U^\#) = x_0^T Q x_0 + u_0^T R u_0 + x_1^T P_{\ell} x_1 \]
\[ = x_0^T P_{\ell} x_0 - x_0^T P_{\ell} x_1 + x_1^T P_{\ell} x_1 \]
\[ = x_0^T P_{\ell} x_0 - x_0^T (P_1 - P_0)x_1 \]
\[ \leq x_0^T P_{\ell} x_0. \quad (43) \]

(40) is derived for \( \ell = 2 \).

Assume (40) holds for \( \ell = 2, \ldots, k \), we will show that (40) holds for \( \ell = k + 1 \) as well. For the system (1) with the initial state \( x_0 \in W_{k+1} \), the control law \( U^\# \) yields \( u_0 = K_{k+1} x_0 \) and the inclusion
\[ x_1 \in U_k \setminus V_k = C \cup \left( \bigcup_{l=1}^{k-1} V_l \right) \cup \left( \bigcup_{l=1}^{k-1} W_l \right) \quad (44) \]
is derived from (32). Since one of the evaluations:
\[ J(x_1, U^\#) = x_1^T P_{\ell} x_1, \quad x_1 \in C \quad (45) \]
\[ J(x_1, U^\#) = x_1^T P_{\ell} x_1, \quad x_1 \in V_l \quad (\ell = 1, 2, \ldots, k - 1) \quad (46) \]
\[ J(x_1, U^\#) \leq x_1^T P_{\ell} x_1, \quad x_1 \in W_k \quad (\ell = 2, \ldots, k - 1) \quad (47) \]
holds, we have the following inequality with (A4).
\[ J(x_1, U^\#) \leq x_1^T P_{\ell} x_1 \quad (48) \]
Since
\[ J(x_0, U^\#) = x_0^T Q x_0 + u_0^T R u_0 + J(x_1, U^\#) \]
\[ = x_0^T P_{\ell+1} x_0 - x_0^T P_{\ell} x_1 + J(x_1, U^\#) \]
\[ \leq x_0^T P_{\ell} x_0 \quad (49) \]
holds, (40) is derived for \( \ell = k + 1 \). \( \square \)

Remark 4. By Theorem 1, the proposed control \( U^\# \) turns optimal in the case the initial state (a1) lies in the regions \( V_i \) \((i = 1, 2, \ldots)\). While in the case the initial state is in \( W_i \), any optimal control for the problems \( P_N^i \) does not meet the transition time of \( P_N \) (Lemma 3). Thus \( U^\# \) is sub-optimal for \( x_0 \in W_i \) and only upper/lower bounds are guaranteed. \( \square \)

Remark 5. The assumption (A5) is employed for the evaluation of the lower bounds of \( J^*(x_0), J(x_0, U^\#) \). The upper bounds of (25),(26) are valid even if (A5) is relaxed. \( \square \)

4. GENERALIZED MODE FOR AFFINE SYSTEMS

Based on the results stated in Section 3, we clarify a sub-optimal control law for a class of affine systems. The problem focused here enables to deal with an affine system such that the dimension of the state-space changes over the mode transition.

Define a generalized mode transition system by
\[ I_0 = 0, x_0 \in U : \text{initial state, } \mathcal{U} := \{ x_k \mid x_k \not\in C \} \quad (50a) \]

Mode I
\[
\begin{align*}
I_{k+1} &= I_k \\
x_{k+1} &= A x_k + B u_k + a \\
I_k &= 0, x_k \in C
\end{align*}
\]

Mode II
\[
\begin{align*}
I_{k+1} &= 1 \\
x_{k+1} &= E_0 x_k + e_0 \\
I_k &= 1, x_k \in \tilde{C}, \tilde{C} := \{ \tilde{x}_k \mid \exists x_k \in C : \tilde{x}_k = E_0 x_k + e_0 \}
\end{align*}
\]

and consider the following LQ control problem
\[ \bar{P} : J^*(x_0) := \min_{N, \bar{U}} \bar{J}(x_0, \bar{U}) \quad (51) \]
\[ \bar{U} := \{ u_0, u_1, \ldots, u_{N-1}, \bar{u}_{N_s}, \bar{u}_{N_s+1}, \ldots \} \]
subject to
\[ \begin{align*}
x_j &\in U, \forall i = 0, 1, \ldots, N_s - 1 \\
\bar{x}_j &\in \tilde{C}, \forall j = N_s, N_s + 1, \ldots
\end{align*} \]
where \( I_k \in \{ 0, 1 \} \) is mode value, \( x_j \in \mathbb{R}^n, u_k \in \mathbb{R}^m \) and \( \tilde{x}_k \in \tilde{C}, \bar{x}_k \in \tilde{C} \) are the state, the control input of the Mode I, respectively. The Mode II is newly introduced for embedding the internal state and \( N_s \) denotes the transition time. For the problem \( \bar{P} \), the cost-functional \( J(x_0, \bar{U}) \) is defined by
\[ \bar{J}(x_0, \bar{U}) = \tilde{J}_I + \tilde{J}_II \quad (52a) \]
\[ \bar{U} := \{ u_0, u_1, \ldots, u_{N_s-1}, \bar{u}_{N_s}, \bar{u}_{N_s+1}, \ldots \} \]
\[ \tilde{J}_I = \sum_{k=0}^{N_s-1} (x_k^T Q x_k + u_k^T R u_k), \quad Q > 0, R > 0 \quad (52b) \]
\[ \tilde{J}_II = \sum_{k=N_s}^{\infty} (x_k^T Q x_k + u_k^T R u_k), \quad \tilde{Q} \geq 0, \tilde{R} > 0. \quad (52c) \]

Under the assumptions (B1)-(B4), we will generalize the design method of Section 3 and clarify the corresponding performance.

(B1) There exists a control \( \bar{U} \) which satisfies \( \bar{J}(x_0, U) < \infty \).

(B2) The region \( \tilde{C} \) is convex and \( 0 \in \tilde{C} \) is an interior point of \( \tilde{C} \).

(B3) \( (\bar{A}, \bar{B}) \) is stabilizable and \( (\tilde{Q}^{1/2}, \tilde{A}) \) is detectable.

(B4) The region \( \tilde{C} \) is positively invariant in the case the optimal control for (52c) is applied.

Since (B3),(B4) hold in the Mode II, the optimal control for (52c) and the resulting cost are given as follows.
\[ \bar{u}_k = \tilde{K}_{1Qx} \bar{x}_k \quad (53) \]
\[ \tilde{K}_{1Qx} = -(\bar{R} + \bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A} \quad (54) \]
\[ \bar{J}_{\text{opt}} = \bar{x}_N^T \bar{P} \bar{E}_N \quad (55) \]

where \( \bar{P} \geq 0 \) is a solution of the Riccati equation
\[ \bar{P} = \bar{Q} + \tilde{A}^T \bar{P} \tilde{A} - \tilde{A}^T \bar{P} \bar{B} (\bar{R} + \bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \tilde{A}. \quad (56) \]

While, the affine system (50b) is further transformed to the linear system:
and the corresponding state region $\mathcal{U}$ and the cost-functional (52b) are expressed as follows.

$$
\mathcal{U} = \{ \bar{x}_k | \bar{x}_k = \begin{bmatrix} x_k \\ 1 \end{bmatrix}, x_k \in \mathcal{U} \} 
$$

$$
J_1 = \sum_{k=0}^{N-1} (\bar{x}_k^T Q \bar{x}_k + u_k^T R u_k), \quad Q = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}
$$

Hence, in like manner of Section 3, a sub-optimal control for $\bar{\mathcal{P}}^U_N$ is given based on the auxiliary control problem $\bar{\mathcal{P}}^U_N$.

$$
\bar{\mathcal{P}}^U_N : \quad \bar{J}_N(x_0) := \min_{\bar{U}_N} \bar{J}_N(x_0, \bar{U}_N)
$$

$$
\bar{U}_N := \{ u_0, u_1, \ldots, u_{N-1} \}
$$

$$
\bar{J}_N(x_0, U_N) = \sum_{k=0}^{N-1} (\bar{x}_k^T Q \bar{x}_k + u_k^T R u_k) + \bar{x}_0^T \bar{P}_0 \bar{x}_N
$$

$$
\bar{P}_0 := \text{defined by (62)}
$$

Introducing a Riccati difference equation:

$$
\bar{P}_{k+1} = Q + A^T \bar{P}_k A - A^T \bar{P}_k B (R + \bar{B}^T \bar{P}_k B)^{-1} B^T \bar{P}_k A
$$

$$
\bar{P}_0 := \text{defined by (62)}
$$

with a following assumption:

**B5** The solutions of (63) are monotonically non-decreasing:

$$
\bar{P}_{k+1} \geq \bar{P}_k, \quad (k = 0, 1, 2, \ldots),
$$

a design method of sub-optimal control law for $\bar{\mathcal{P}}$ is summarized as follows.

**Algorithm [B]**

1) Define a control law (53) in the state region $\mathcal{C}$.

$$
\bar{u}_k = \bar{K}_{LQ} \bar{x}_k, \quad \bar{x}_k \in \mathcal{C}
$$

2) $j = 1, \bar{U}_0 = \bar{U}_0^j := \mathcal{C}$.  
3) Solve the problem $\bar{\mathcal{P}}^U_{j}$ (with the control horizon $j$) and define the following control law $\bar{u}_k = \bar{K}_j \bar{x}_k$ and state regions $\bar{U}_j, \bar{U}_j^c$.

$$
\bar{u}_k = \bar{K}_j \bar{x}_k, \quad \bar{x}_k \in \bar{U}_j
$$

$$
\bar{K}_j := -(R + \bar{B}^T \bar{P}_{k-1} \bar{B})^{-1} \bar{B}^T \bar{P}_{k-1} A
$$

$$
\bar{U}_j := \{ \bar{x} | (A + \bar{B} \bar{K}_j) \bar{x} \in \bar{U}_{j-1}, \bar{x} \notin \bar{U}_{j-1}^c \}
$$

$$
\bar{U}_j^c := \bar{U}_j \cup \bar{U}_{j-1}^c
$$

4) If $\bar{x}_0 \notin \bar{U}_j, j := j + 1$ and go to Step 3.  
5) Halt: $N := j$, the sub-optimal control law for $\bar{\mathcal{P}}$ is given as follows.

$$
\bar{U}^\# := \bar{u}_k = \begin{cases} 
\bar{K}_j \bar{x}_k, & \bar{x}_k \in \bar{U}_j, \\
\bar{K}_{LQ} \bar{x}_k, & \bar{x}_k \in \mathcal{C}
\end{cases} \quad (j = 1, 2, \ldots, N)
$$

For the control law $\bar{U}^\#$ obtained by the algorithm [B], the following properties are obtained along the derivation of Theorem 1.

**Corollary 6.** Introduce auxiliary partitions such that $\bar{U}_k = \bar{V}_k \cup \bar{W}_k (k = 1, 2, \ldots, N)$ holds:

$$
\bar{V}_0 := \bar{U}_0
$$

$$
\bar{V}_k := \{ \bar{x} | (A + \bar{B} \bar{K}_j) \bar{x} \in \bar{V}_{k-1}, \bar{x} \in \bar{U}_k \}
$$

$$
\bar{W}_k := \bar{U}_k \setminus \bar{V}_k (k = 2, 3, \ldots, N).
$$

The performance attained by the control law $\bar{U}^\#$ is characterized as follows:

(a) $\forall \bar{x}_0 \in \bar{V}_j \subset \bar{U} (j = 1, 2, \ldots)$:

$$
\bar{x}_0^T \bar{P}_{j-1} \bar{x}_0 \leq \bar{J}^* (x_0) = \bar{J}(x_0, \bar{U}^\#) = \bar{x}_0^T \bar{P}_j \bar{x}_0
$$

(b) $\forall \bar{x}_0 \in \bar{W}_j \subset \bar{U} (j = 1, 2, \ldots)$:

$$
\bar{x}_0^T \bar{P}_{j-1} \bar{x}_0 \leq \bar{J}^* (x_0) \leq \bar{J}(x_0, \bar{U}^\#) \leq \bar{x}_0^T \bar{P}_j \bar{x}_0
$$

where $\bar{J}^* (x_0)$ is the optimal cost for the problem $\bar{\mathcal{P}}$ and $\bar{J}(x_0, \bar{U}^\#)$ denotes the resulting cost in the case the proposed control law $\bar{U}^\#$ is applied.

By Corollary 6, it is shown that the control law $\bar{U}^\#$ guarantees the upper bound of the performance and it is exactly optimal in the case $\bar{x}_0 \in \bar{V}_j (j = 1, 2, \ldots)$. Similarly to Remark 5, the assumption (B5) is relaxed if the lower bound of (73),(74) is not needed.

5. DESIGN EXAMPLE

In this section, the design method is applied to the case study of tension/looper system in the finishing mill (Asano et al. (2005); Kojima et al. (2008)). The feature of the control law is discussed in terms of the improvement of the transient. The formulation of the tension/looper system is summarized along Asano et al. (2005); Kojima et al. (2008).

The tension and looper systems are separately modeled and, with account of interactions which depends on the looper angle, a unified model is obtained along the following procedure.
Table 1: Notation and parameters

<table>
<thead>
<tr>
<th>Sign</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>-</td>
<td>[rad]</td>
<td>Looper angle</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>-</td>
<td>[MPa]</td>
<td>Interstand tension</td>
</tr>
<tr>
<td>$V_R$</td>
<td>-</td>
<td>[m/s]</td>
<td>Roll velocity</td>
</tr>
<tr>
<td>$T_{Lref}$</td>
<td>-</td>
<td>[rad]</td>
<td>Looper torque</td>
</tr>
<tr>
<td>$J$</td>
<td>2.16 $\times$ 10$^3$</td>
<td>[Nm$^2$]</td>
<td>Looper inertia</td>
</tr>
<tr>
<td>$H$</td>
<td>3.1 $\times$ 10$^{-3}$</td>
<td>[m]</td>
<td>Strip thickness</td>
</tr>
<tr>
<td>$b$</td>
<td>1.0</td>
<td>[m]</td>
<td>Strip width</td>
</tr>
<tr>
<td>$p$</td>
<td>7.85 $\times$ 10$^3$</td>
<td>[kg/m$^3$]</td>
<td>Strip density</td>
</tr>
<tr>
<td>$r$</td>
<td>0.6</td>
<td>[m]</td>
<td>Looper arm length</td>
</tr>
<tr>
<td>$W_L$</td>
<td>2.5 $\times$ 10$^3$</td>
<td>[kg]</td>
<td>Looper weight</td>
</tr>
<tr>
<td>$l$</td>
<td>2.8</td>
<td>[m]</td>
<td>Half of length between stands</td>
</tr>
<tr>
<td>$r_L$</td>
<td>0.125</td>
<td>[m]</td>
<td>Distance between looper axis and center of gravity of looper</td>
</tr>
<tr>
<td>$D$</td>
<td>49.0</td>
<td>[Nm]</td>
<td>Looper damping factor</td>
</tr>
<tr>
<td>$\theta_L$</td>
<td>5.0</td>
<td>[deg]</td>
<td>Offset angle between looper angle and center of gravity of looper</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.16</td>
<td>[deg]</td>
<td>Strip angle with passline</td>
</tr>
<tr>
<td>$f(\sigma)$</td>
<td>9.8</td>
<td>[m/s$^2$]</td>
<td>Acceleration of gravity</td>
</tr>
<tr>
<td>$E$</td>
<td>1.96</td>
<td>[GPa]</td>
<td>Young’s modulus of strip</td>
</tr>
<tr>
<td>$L_{ASR}$</td>
<td>23.48</td>
<td>[m]</td>
<td>Interstand strip length</td>
</tr>
<tr>
<td>$\theta_{min}$</td>
<td>10$\pi$/180</td>
<td>[rad]</td>
<td>Looper angle when the looper is risen to the passline</td>
</tr>
</tbody>
</table>

Table 2: Operating points at modes I and II

<table>
<thead>
<tr>
<th>Sign</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_n$</td>
<td>2027/180</td>
<td>[rad]</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>9.8</td>
<td>[MPa]</td>
</tr>
<tr>
<td>$T_{Lrefc}$</td>
<td>4.2 $\times$ 10$^3$</td>
<td>[Nm]</td>
</tr>
<tr>
<td>$V_{Rc}$</td>
<td>9.29</td>
<td>[m/s]</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>10$^7$/180</td>
<td>[rad]</td>
</tr>
</tbody>
</table>

5.1 Dynamic equations

Based on Fig.3 and Table 1, the dynamic equation of the looper system is described as follows:

$$\dot{\theta} = T_{Lref} - \delta[K_1(\theta)\sigma + K_2(\theta)] - K_3(\theta) - D \dot{\theta}$$

$$K_1(\theta) = 2Hbr \cos \theta \sin \beta$$

$$K_2(\theta) = 2pHbr L \cos \theta$$

$$K_3(\theta) = W_L g r_L \cos(\theta + \theta_L)$$

where $K_1$, $K_2$, $K_3$ denote the looper load torque by the tension, the strip weight and the looper weight, respectively. A binary variable $\delta \in \{0, 1\}$ in (75) describes both contact mode (Mode II) and non-contact mode (Mode I) (Fig.3). The case $\delta = 0$ stands for the Mode I as the interaction with the tension system is neglected. While $\delta = 1$ stands for the Mode II which considers the interaction with the tension system. The mode transition rule is given by

$$\delta = \begin{cases} 
0 & \text{if } \theta < \theta_{min} \\
1 & \text{if } \theta \geq \theta_{min} 
\end{cases}$$

where $\theta_{min}$ is the minimal looper angle when it reaches the pass line (Table 2).

The tension system is described by

$$\dot{\theta} = E \frac{1}{2} \left(- (1 + f(\sigma))V_R + \frac{\partial L}{\partial \theta} \right)$$

$$\dot{V_R} = - \frac{1}{T_{ASR}} (V_R - V_{Rref})$$

and the state transition to the contact mode (Mode II) is assumed by

$$\dot{\theta}(t) = \epsilon_1 \theta(t-$$)

$$\sigma(t) = \sigma(t-) + \epsilon_2 \dot{\theta}(t-)$$

where $\epsilon_1$, $\epsilon_2$ are appropriately estimated constants (Asano et al. (2005); Kojima et al. (2008)).

5.2 Model Description

Linearizing around each operating point (Table 2), then discretizing the dynamic equations, the mode transition system (57) is derived in the discrete-time setting.

We first formulate Mode II and II (contact mode), which cover whole dynamics of (75)-(78), (80)-(81), then formulate Mode I (non-contact mode) by neglecting redundant dynamics.

**Mode II, II (contact mode):** Let $(\theta_c, \sigma_c, V_{Rc}, T_{Lrefc}, V_{Rvec})$ be the operating points such that

$$T_{Lrefc} = K_1(\theta_c)\sigma_c + K_2(\theta_c) + K_3(\theta_c)$$

$$V_{Rvec} = V_{Rc}$$

hold (Table 2). Denoting the variation by $(\bar{\theta}, \bar{\sigma}, \bar{V}_R, \bar{T}_{Lref}, \bar{V}_{Rvec}) = (\theta - \theta_c, \sigma - \sigma_c, V_R - V_{Rc}, T_{Lref} - T_{Lrefc}, V_{Rvec} - V_{Rvec})$, then linearizing and discretizing the equations (75)-(78), (80), (81) with $\delta = 1$, the description for the Mode II is obtained.

$$\bar{x}_{k+1} = A_c^x \bar{x}_k + B_c^x \bar{u}_k$$

$$\bar{x}_k = \left[ \bar{\theta}_k \bar{\theta}_k \bar{V}_{Rk} \right]^T, \quad \bar{u} = [T_{Lrefk} \bar{T}_{Lrefk}]^T$$

The description of Mode IIo is also obtained by imposing (82), (83).

**Mode I (non-contact mode):** Let $(\theta_n, T_{Lrefn})$ be the operating point which preserves

$$T_{Lrefn} = K_3(\theta_n)$$

$$V_{Rn} = V_{Rvec}$$

Denoting the variation by $(\bar{\theta}, \bar{T}_{Lref}) = (\theta - \theta_n, T_{Lref} - T_{Lrefn})$, then linearizing and discretizing the equations (75)-(78) with $\delta = 0$, a linear system description is obtained in the coordinate of Mode I. Finally, substituting the coordinate for Mode II: $\bar{\theta} + (\theta_n - \theta_n), \bar{T}_{Lref} = \bar{T}_{Lref} + (T_{Lref} - T_{Lrefn})$, the following affine system is obtained.

$$\bar{x}(t) = A_n^x \bar{x}(t) + B_n^x \bar{u}(t) + a_n$$

$$x = \left[ \bar{\theta} \bar{\theta} \bar{T}_{Lref} \right]^T, \quad \bar{u} = \bar{T}_{Lref}$$

A mode transition model is obtained by combining the descriptions Mode I, IIo and II with the transition rule (79).

5.3 Controller Design and Simulation

Apply the design method (Algorithm [B]) to the tension/looper system with the sample period $h = 0.02$ [s], we will investigate the performance of the resulting control system.

For the cost-functional (52a) with the following matrices:

$$Q = \text{diag}(6 \times 10^6, 6 \times 10^6, 8.5 \times 10^5), R = 1, \quad (89)$$

$$Q = \text{diag}(1, 1, 1), \quad \bar{R} = \text{diag}(1, 1), \quad (90)$$

design the sub-optimal LQ control law based on algorithm [B].

In (Step 1), the LQ control in the Mode II is obtained by

$$\bar{u}_k = \begin{bmatrix} 3110 & 3602 & -0.003 & 0.0002 \\ -4.511 & 1.855 & 0.966 & 0.422 \end{bmatrix} \bar{x}_k, \bar{x}_k \in \tilde{C}$$

(91)
is depicted by Fig. 4 (a),(b) and it is observed that the control is switched to the standard LQ control (71) at 42 unit-time.

The system responses and the control input are summarized by Fig. 5. In this case, $\tilde{U}^\#$ yields a control strategy such that the magnitude of control input is attenuated before the mode transition occurs.

As described by Corollary 6, the performance of the proposed control law $\tilde{U}^\#$ depends on the regions where the initial state lies. In this example, $\mathcal{U}_i = \mathcal{V}_i$ $(i = 1, 2, \ldots, 42)$ holds and the sub-optimal regions $\{W_i\}$ shrink. In other words, $\tilde{U}^\#$ provides optimal control for the system (84)-(88).

6. CONCLUSION

For a class of mode transition systems, a design method of sub-optimal LQ control law is derived based on auxiliary finite-horizon LQ control problems. The control law is given as a piecewise affine function of the state and the each region is fairly characterized by the difference of polyhedral regions. The feature of the resulting control law is further investigated and it is shown that the proposed control law turns optimal in the case the initial state lies in certain regions of the state space.

Generalization to the sequential mode transition systems and the performance analysis of the $H^\infty$-type disturbance attenuation are the direction of future research.

REFERENCES


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