Adaptive Control Using Collective Information Obtained from Multiple Models ⋆

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Abstract: A radically new approach to adaptive control using multiple models was recently proposed by the authors. Using parametric estimates and outputs generated by multiple adaptive identification models, an adaptive control procedure was proposed which resulted in significantly faster and more accurate response of the overall system, as compared to existing methods such as "switching" and "switching and tuning", which also use multiple models.
In this paper the ideas are extended to include fixed identification models. Further, many of the theoretical questions that have arisen concerning the stability of the overall system and the reasons for the improved performance are addressed in detail. Simulation results are included, towards the end of the paper, to indicate the effectiveness of the new methodology.

Keywords: Adaptive control, Adaptive systems, Linear systems, Convex hull, Second level adaptation.

1. INTRODUCTION

Adaptive control theory was first developed in the 1960s to deal with the control of linear time-invariant systems with unknown parameters. A large body of literature currently exists in this area, and it is generally accepted that the adaptive methods that have been developed are both stable and robust, when plant parametric uncertainty is small (Narendra and Annaswamy, 1989; Goodwin and Sin, 1984; Ioannou and Sun, 1996; Krstić et al., 1995; Tao, 2003; Middleton et al., 1988).

However, in numerous applications, it has been found that when the parameter errors are large, classical adaptive control methods result in large and oscillatory responses. Among the many efforts made to improve the response in such situations, one of the more successful approaches developed in the 1990s used multiple models to estimate the plant parameters. Both "switching", using fixed models (Morse, 1996, 1997), and "switching and tuning" utilizing fixed and adaptive models (Narendra and Balakrishnan, 1992, 1994, 1997) were proposed. In both cases, a large number of fixed models are distributed in the region of uncertainty, and based on the responses of the plant and the models, one of the models is chosen at every instant as the "best" fit for the unknown plant in some sense. The integral square error or a modification of that has been used by many authors as the index of performance. In "switching" the control input to the plant is based on the fixed model chosen at that instant. In "switching and tuning", an adaptive model is initialized from the location of the fixed model chosen, and the parameters of the best model determine the control to be used.

Switching is discontinuous, fast, but coarse, while tuning is continuous, slow, but accurate. Numerous past researches have shown that these methods offer satisfactory performance when no restrictions are put on the number of available models (Lainiotis, 1976; Lane and Maybeck, 1994; Yu et al., 1992; Bošković and Mehra, 1999; Narendra and Han, 2010; Kuipers and Ioannou, 2010).

From the preceding discussion it is seen that for "switching" to result in stability, a necessary condition is that there is at least one fixed model near the plant which when stabilized using a controller, will also stabilize the plant. Similarly, with "switching and tuning", adaptation from the model closest to the plant must result in improved performance. Both of the approaches call for a large number of models (Vinnicombe, 1993). The above number also increases exponentially with the dimension of the unknown parameter vector.

A second drawback of the above two methods is that very little information provided by all the models is actually used in the decision process. The control input at any instant is based entirely on one model which is closest to the plant according to some metric. The question naturally arises whether better decisions concerning the unknown plant parameter θ_p can be made by using the actual outputs of all the models i.e. whether the available resources can be utilized more efficiently to identify and control the plant.

In spite of the above shortcomings, the two methods discussed above have been successfully applied to control problems with large uncertainties.

The need for new tools for reacting effectively to large uncertainties is arising increasingly in widely differing fields including biology and medicine, economics and finance,

⋆ The research was supported by the National Science Foundation under Grant No. ECS-0824118.

and various engineering problems such as energy management and renewable power generation, aircraft and automotive control, and national security. These problems can be cast as adaptive decision making (or control) problems in which an unknown plant parameter vector may vary in a very large region. The objective in such cases is to estimate the unknown parameters rapidly and accurately in the presence of disturbances using very few models, and take appropriate control action. The adaptive methods that are currently available are generally inadequate to deal with such problems. Recently, a new framework based on multiple models was developed by the authors (Narendra and Han, 2010; Han and Narendra, 2010a,b) which can cope with the problems that arise in these contexts with very few models. In the new methods introduced, the identification of the unknown plant parameter vector is based on collective information provided by each of a small number of models during the entire process, so that resources are used efficiently.

2. THE GENERAL APPROACH

In this paper we first present the general framework used in (Han and Narendra, 2010b) for controlling an unknown plant using multiple adaptive models. As the adaptive models evolve, they provide multiple estimates of the plant parameter vector and the corresponding output identification errors. It is first shown that either the parameter estimates or the identification errors can be used collectively to determine the control input to the overall system. The proof of stability of the latter is first established before providing both qualitative and quantitative arguments to demonstrate the improvement in performance observed in simulations.

In the next stage, all the above results are extended to the case where the models used to estimate the plant parameters are linear and time-invariant. It is shown that the output errors of the different models can be used to adjust a parameter vector α in a stable fashion. The fact that both fixed and adaptive identification models can be used provides the foundation for a general methodology for adaptive control based on multiple models. In the final sections of the paper, many of the theoretical questions that arise in this context are discussed, and simulations are provided to indicate the substantial improvement in performance realized by the new approach.

3. ADAPTIVE CONTROL BASED ON MULTIPLE ADAPTIVE MODELS

A linear time-invariant plant Σ_p is described by the equation

$$\Sigma_p : \quad \dot{x}_p(t) = A_p x_p(t) + bu(t) \tag{1}$$

which is in companion form, with $x_p(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$. The last row of the matrix A_p is denoted as $\theta_p^T = [a_{p(1)}, a_{p(2)}, \ldots, a_{p(n)}]$ and represents the unknown parameter vector. (In the discussions, we shall refer to θ_p) as the unknown parameter vector.) A stable reference model Σ_m is described by the differential equation

$$\Sigma_m : \quad \dot{x}_m(t) = A_m x_m(t) + br(t) \tag{2}$$

where (A_m, b) is also in companion form and the last row of A_m is denoted by the known parameter vector θ_m where $\theta_m^T = [a_{m(1)}, a_{m(2)}, \ldots, a_{m(n)}]$. $r(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is a known bounded piecewise continuous reference signal.

Assuming $\theta_p \in \mathcal{S}_\theta$, where $\mathcal{S}_\theta$ is a compact set in parameter space, the objective is to estimate θ_p and determine the control input $u(\cdot)$ to the plant so that all signals in the system are bounded and $\lim_{t\to\infty}[x_p(t) - x_m(t)] = 0$.

3.1 Identification Models

It is well known in classical adaptive control theory that θ_p can be estimated using an identification model Σ_I described by

$$\Sigma_I : \quad \dot{\hat{x}}(t) = A_m \hat{x}(t) + [A(t) - A_m]x_p(t) + bu(t) \tag{3}$$

so that the error $e_I(t)$ defined as $e_I(t) = \hat{x}(t) - x_p(t)$ satisfies the differential equation

$$\begin{aligned} \dot{e}_I(t) &= A_m e_I(t) + b[\hat{\theta}(t) - \theta_p]^T x_p(t) \\ &= A_m e_I(t) + b\tilde{\theta}(t)^T x_p(t). \end{aligned} \tag{4}$$

The corresponding adaptive law for adjusting $\hat{\theta}(t)$ is

$$\dot{\hat{\theta}}(t) = \dot{\tilde{\theta}}(t) = -e_I(t)^T P b x_p(t) \tag{5}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta_p$ and the stability of the adaptive system described by the error equations (4) and (5) is assured by the Lyapunov function $V(e_I, \tilde{\theta}) = e_I^T P e_I + \tilde{\theta}^T \tilde{\theta}$ whose time-derivative is $\dot{V}(e_I, \tilde{\theta}) = -e_I^T Q e_I$ where $A_m^T P + P A_m = -Q \le 0$.

To control the plant, state feedback of the form $u(t) = k(t)^T x_p(t) + r(t)$ is used where $k(t)$ is computed algebraically from the equation $k(t) = \theta_m - \hat{\theta}(t)$. The corresponding error differential equation describing the output state error vector is $\dot{e}_c(t) = A_m e_c(t) + b\tilde{k}(t)^T x_p(t)$, $\dot{\tilde{k}}(t) = -e_c(t)^T P b x_p(t)$, which is shown to be stable using a quadratic Lyapunov function $V(e_c, \tilde{k}) = e_c^T P e_c + \tilde{k}^T \tilde{k}$.

When multiple identification models are used to estimate the unknown vector θ_p, they are described by the same equations but with different initial conditions. In such a case the models are denoted as Σ_i $(i = 1, 2, \ldots, N)$, the parameter estimates as θ_i and the corresponding identification errors as $e_i(t)$. The error equations describing the N models can be stated as

$$\begin{aligned} \dot{e}_i(t) &= A_m e_i(t) + b\tilde{\theta}_i(t)^T x_p(t) \\ \dot{\tilde{\theta}}_i(t) &= -e_i(t)^T P b x_p(t). \end{aligned} \tag{6}$$

Comment: When a single model is used to identify the plant, $\hat{\theta}(t)$ provided by that model is used to determine the control law. In contrast to this, when multiple models are used and provide the estimates $\theta_i(t)$ as determined by (6) the following questions arise and are addressed throughout the paper:

(i) how can the overall system be stabilized by a controller using the N parameter estimates $\theta_i(t)$ and/or the output identification errors $e_i(t)$?

(ii) to what extent does the proof of stability differ from that used in the single model case, which is based on an explicit Lyapunov function?
(iii) assuming that stability can be demonstrated, what arguments can be provided to demonstrate the markedly improved performance observed in simulation studies?

From the above questions it is clear that both stability and performance are major issues. As we shall describe later in this paper, methods that assure stability in a straight forward fashion may not result in improved performance, while those that improve performance may not follow the conventional approach used in adaptive control to prove stability.

The principal ideas for achieving the latter, when all models are adaptive were first presented in (Han and Narendra, 2010b), and are stated compactly in what follows. Also the facts derived in (Han and Narendra, 2010b) provide the rationale for the adaptive control law chosen, and are summarized below.

(i) If θ_p lies in the convex hull $\mathcal{K}(t_0)$ of $\theta_i(t_0)$ $(i \in \Omega = \{1, 2, \ldots, N\})$, it can be expressed as

$$\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t_0) \qquad \sum_{i=1}^{N} \alpha_i = 1 \qquad 0 \leq \alpha_i \leq 1 \quad (7)$$

(ii) θ_p lies in $\mathcal{K}(t)$, the convex hull of $\theta_i(t)$ for all $t \geq t_0$ and $\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t)$.
(iii) In view of the above facts, estimation of $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ is equivalent to the estimation of θ_p. Since $\theta_i(t)$ are continually evolving α can be estimated on-line and used to determine the desired feedback controller.

In (Han and Narendra, 2010b) a proof of stability of the overall system was also presented. In what follows we merely derive the adaptive laws proposed in (Han and Narendra, 2010b). Following this we demonstrate that a similar procedure can also be used with fixed (stable) identification models, i.e. it is shown that if a set of suitably chosen (stable) fixed identification models are used with the same input as the plant, the identification errors can be used collectively to locate θ_p, the unknown parameter vector of the plant. In Section 5, a detailed investigation of the stability problem is undertaken, and it is shown that different adaptive laws can be chosen to assure stability. However, only some of these assure significant improvement in performance, and these are described in Section 6.

3.2 Adaptive Laws for Identification

It was stated earlier that determining the parameter vector α in equation (7) is equivalent to determining θ_p. In (Han and Narendra, 2010b) it was shown that the constant vector α can be estimated as $\hat{\alpha}(t)$ at every instant, based on the identification errors $e_i(t)$ resulting from the $(n+1)$ identification models described in equation (6). Representing $e_i(t) - e_{n+1}(t)$ $(i = 1, 2, \ldots, n)$ as columns of a matrix $\bar{E}(t)$, and $-e_{n+1}(t) = \ell(t)$ it was shown that α satisfies the algebraic equation

$$\bar{E}(t)\bar{\alpha} = \ell(t) \qquad (8)$$

where $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ whose 'n' elements are the first n components of the vector $\alpha \in \mathbb{R}^{n+1}$ (and $\alpha_{n+1} = 1 - \sum_{i=1}^{n} \alpha_i$). Estimating $\bar{\alpha}$ as $\hat{\bar{\alpha}}(t)$ is achieved on-line by using a model described by the equation $\bar{E}(t)\hat{\bar{\alpha}}(t) = \hat{\ell}(t)$ where $\hat{\ell}(t)$ is the output of the model. The adaptive law for adjusting $\hat{\bar{\alpha}}(t)$ can be readily derived as

$$\dot{\hat{\bar{\alpha}}}(t) = -\bar{E}^T(t)\tilde{\ell}(t) \qquad (9)$$

where $\tilde{\ell}(t) = \hat{\ell}(t) - \ell(t)$.

Since $\bar{E}(t)$ and $\tilde{\ell}(t)$ are accessible, the adaptive law (9) can be implemented in practice. From $\hat{\alpha}(t)$, the corresponding estimate $\hat{\theta}_p(t)$ and the feedback controller $\hat{k}(t)$ used to stabilize the system can be determined. Extensive simulation results have shown that the response is considerably faster than conventional adaptive methods. It is the proof of stability of the above scheme (as well as other related schemes) and qualitative and quantitative reasons for the improved performance observed that will be discussed in Sections 5 and 6 respectively.

4. ADAPTIVE CONTROL USING FIXED MODELS

Let Σ_{Fi} be stable fixed models describe by the differential equations

$$\Sigma_{Fi} : \dot{x}_{Fi}(t) = A_m x_{Fi}(t) + b[\theta_{Fi} - \theta_m]x_p(t) + bu(t) \qquad (10)$$

where θ_{Fi} is a constant parameter vector. The question to be addressed is whether a control input can be generated based on the identification errors $e_{Fi}(t)$ $(i \in \Omega)$ where $e_{Fi}(t) = x_{Fi}(t) - x_p(t)$. Since it is assumed that the unknown parameter vector θ_p lies in the convex hull of θ_{Fi} $(i \in \Omega)$, it can be expressed as

$$\theta_p = \sum_{i=1}^{N} \beta_i \theta_{Fi} \qquad \sum_{i=1}^{N} \beta_i = 1 \qquad 0 \leq \beta_i \leq 1 \qquad (11)$$

it follows from the linearity of the models that

$$[e_{F1}(t), e_{F2}(t), \ldots, e_{FN}(t)]\beta = 0 \quad \text{or} \quad E(t)\beta = 0 \qquad (12)$$

where $E(t) = [e_{F1}(t), e_{F2}(t), \ldots, e_{FN}(t)]$ and $\beta = [\beta_1, \beta_2, \ldots, \beta_N]^T$. As in Section 3, choosing $N = n+1$, $\beta \in \mathbb{R}^{n+1}$, and since $\sum_{i=1}^{n+1} \beta_i = 1$, equation (12) can be recast as a matrix equation similar to (8) i.e. if $e_{Fi}(t) - e_{Fn+1}(t) = \bar{e}_i(t)$ and $\bar{E}(t)$ is a matrix whose i^{th} column is $\bar{e}_i(t)$

$$\bar{E}(t)\bar{\beta} = -e_{Fn+1}(t). \qquad (13)$$

Equation (13) may be made the starting point for the adaptive identification of $\hat{\bar{\beta}}(t)$, the estimate of the constant vector $\bar{\beta}$. As in Section 3 $\hat{\bar{\beta}}(t)$ provides an estimate of θ_p and can be used to determine a feedback gain which will assure stability of the overall system.

5. STABILITY

In Section 3 and 4 second level adaptation procedures for estimating the constant vector α as $\hat{\alpha}(t)$ and β as

$\hat{\beta}(t)$were described, using adaptive and fixed models. This, in turn, yields an estimate $\hat{\theta}_p(t)$ of θ_p from equations (7) or (11). The next step is to determine how the overall system can be stabilized using a control input of the form $u(t) = k^T(t)x_p(t)+r(t)$, where the parameter vector $k(t)$ is adjusted adaptively. Expressing $k(t)$ as $k(t) = [\theta_m - \hat{\theta}_p(t)]$, the input $u(\cdot)$ can be expressed as

$$u(t) = [\theta_m - \sum_{i=1}^{n+1} \hat{\alpha}_i(t)\theta_i(t)]^T x_p(t) + r(t). \qquad (14)$$

Several stable adaptive schemes for adjusting $k(t)$ can, in theory, be determined. However, as stated earlier, only some of them result in significantly improved performance. This is discussed in Section 6.

Using a control input $u(t)$ as given by equation (14), we obtain the error equation

$$\begin{aligned} \dot{e}_c(t) &= A_m e_c(t) - b[\sum_{i=1}^{n+1} \tilde{\alpha}_i(t)\theta_i(t)]^T x_p(t) \\ &= A_m e_c(t) - b\tilde{\alpha}^T(t)\Theta^T(t)x_p(t) \end{aligned} \qquad (15)$$

where $\Theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_{n+1}(t)]$.

The principal question to be addressed concerns the adaptive law for adjusting $\hat{\alpha}(t)$ (or alternatively $\tilde{\alpha}(t)$) so that the overall system is stable, and the performance of the system is satisfactory. In Section 5.1 methods based on conventional adaptive control are described. The new adaptive law proposed for adaptive models in (Han and Narendra, 2010b) are extended in this paper to fixed models and is described in Section 5.2.

5.1 Conventional Adaptive Control

Following well established methods in adaptive control, a Lyapunov function candidate

$$V(e_c, \tilde{\alpha}) = e_c^T P e_c + \tilde{\alpha}^T \tilde{\alpha} \qquad (16)$$

can be chosen, whose time derivative yields $\dot{V}(e_c, \tilde{\alpha}) = -e_c^T Q e_c + 2e_c^T P b\tilde{\alpha}^T\Theta^T x_p + 2\tilde{\alpha}^T\dot{\tilde{\alpha}}$. Hence, an adaptive law $\dot{\hat{\alpha}} = \dot{\tilde{\alpha}} = -e_c^T P b \Theta^T x_p$ results in $\dot{V}(e_c, \tilde{\alpha}) = -e_c^T Q e_c \le 0$ from which it can be concluded (using standard arguments) that $\lim_{t\to\infty} e_c(t) = 0$.

Simulation studies using such an adaptive law have indicated a response that is similar to that obtained in classical adaptive control, which is generally not satisfactory.

5.2 The New Adaptive Law (Second Level Adaptation)

In classical adaptive control where only a single identification model is used, there is little freedom in the choice of methods for proving stability. Over the years Lyapunov's method has become a convenient vehicle for discussing stability. Using such an approach the adaptive law is chosen so that a positive definite candidate function has a time-derivative that is negative semidefinite along any trajectory. While improvement in performance is attempted by the proper choice of the fixed parameters of the system, such improvement is not significant in most cases.

In contrast to the above, when multiple models ($\ge n+1$) are used in the first level, additional information concerning the unknown parameter θ_p is available. In particular $\sum_{i=1}^{n+1} \alpha_i\theta_i(t) = \theta_p$ or $\sum_{i=1}^{n+1} \beta_i\theta_i = \theta_p$ given in equations (7) and (11) for adaptive and fixed models, can be directly used in the second level to estimate $\hat{\alpha}_i(t)$ or $\hat{\beta}_i(t)$.

Alternative methods based on identification errors $e_i(t)$ can also be derived. Convergence of $\hat{\alpha}(t)$ (or $\hat{\beta}(t)$) can be established directly without recourse to an explicit Lyapunov function given in (16). This is described in this section.

In section (3) it was shown that $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ satisfies the algebraic equation $\bar{E}(t)\bar{\alpha} = \ell(t)$, where $\bar{\alpha} \in \mathbb{R}^n$ and $\bar{E}(t) \in \mathbb{R}^{n\times n}$ and $\ell(t) \in \mathbb{R}^n$ are known time-varying matrix and vector respectively.

Similarly, if a matrix $E(t) \in \mathbb{R}^{n\times(n+1)}$ is defined as $E(t) = [e_1, e_2, \ldots, e_{n+1}]$, if follows that $E(t)\alpha = 0$.

From the definitions of $\bar{E}(t)$ and $\ell(t)$, it follows that $\bar{E}(t)$, $\ell(t)$ and $\hat{\ell}(t)$ are bounded. Further the rank of the matrices $\bar{E}(t)$ and $E(t)$ are n, since the initial conditions $\theta_i(t_0)$ chosen for the identification models with $i = 1, 2, \ldots, n$ are linearly independent.

The estimate of $\hat{\bar{\alpha}}(t)$ of $\bar{\alpha}$ can be derived as $\dot{\hat{\bar{\alpha}}} = -\bar{E}^T(t)\tilde{\ell}(t)$ which can also be written as

$$\dot{\tilde{\bar{\alpha}}} = -\bar{E}^T(t)\bar{E}(t)\tilde{\bar{\alpha}}(t) \qquad (17)$$

where $\bar{E}(t)$ are known and bounded. Similarly, the estimate of $\hat{\alpha}(t)$ of α can be derived as $\dot{\hat{\alpha}} = -E^T(t)E(t)\hat{\alpha}(t)$ which can also be written as $\dot{\tilde{\alpha}} = -E^T(t)E(t)\tilde{\alpha}(t)$.

In this case the adaptive law is chosen directly to decrease the norm of the vector $\tilde{\alpha}(t)$ i.e. $\|\tilde{\alpha}\|$. Since a Lyapunov function candidate $V(\tilde{\alpha}) = \tilde{\alpha}^T\tilde{\alpha}$ results in $\dot{V}(\tilde{\alpha}) = -\|E(t)\tilde{\alpha}\|^2 \le 0$, it follows that $\tilde{\alpha}(t)$ is bounded and $\|\tilde{\alpha}(t)\|$ decreases monotonically with time. In view of the comments made earlier concerning $\bar{E}(t)$ and $E(t)$, the adaptive law based on the output errors of the identification models (i.e. (17)) results in $\lim_{t\to\infty} \hat{\alpha}(t) = \alpha$. So that the parameter estimate $\hat{\theta}(t)$ tends to θ_p. This assures the asymptotic stability of the overall system.

To assure asymptotic stability using the conventional approach described in Section 5.1, the two adaptive laws can be combined as

$$\dot{\hat{\alpha}}(t) = -e_c^T(t)Pb\Theta^T(t)x_(t) - E^T(t)E(t)\hat{\alpha}(t). \qquad (18)$$

This assures that $\dot{V}(e_c, \tilde{\alpha}) \le 0$ so that the overall system is stable.

Comment: The performance using the adaptive law (18) is seen to be in general not as good as that observed using (17). This can be partially offset by using a gain γ in the adaptive law so that

$$\dot{\hat{\alpha}}(t) = -e_c^T(t)Pb\Theta^T(t)x_(t) - \gamma E^T(t)E(t)\hat{\alpha}(t) \; \gamma > 1 (19)$$

Comment: Using the adaptive law (17) only the convergence of $\hat{\alpha}(t)$ to α can be demonstrated explicitly. However, this assures global asymptotic stability of the overall system implicitly.

6. PERFORMANCE

As stated earlier, conventional adaptive control results in fast and accurate performance when plant parametric uncertainty is small. Switching, and switching and tuning have proved successful when the uncertainty is large. However, when the uncertainty is significantly larger, and the error equations are distinctly nonlinear, the performance is found to be not satisfactory. This also implies that the methods cannot be directly used when the unknown plant parameters vary with time. It is with this class of problems that we are concerned with in this paper.

The advantage of the methods proposed in this paper over those currently used lies in the fact that the unknown parameter vector θ_p can be expressed in terms of output errors which are known to be bounded and which are also accessible. While estimating α or θ_p can be considered to be equivalent, the latter is carried out indirectly using a candidate Lyapunov function, while the former is updated directly to decrease the norm of the error $\tilde{\alpha}(t)$. We comment successively on several cases to provide qualitative reasons for the rapid convergence and significantly improved performance observed using the new adaptive laws (i.e. second level adaptation).

Rapid Convergence of The Identification Models:

This corresponds to the simplest case. Since the estimate of θ_p used in second level adaptation is merely a convex combination of the estimates $\theta_i(t)$, the rapid convergence of $\hat{\theta}(t) = \sum_{i=1}^{n+1} \hat{\alpha}_i(t)\theta_i(t)$ is also assured.

Slow Convergence of The Identification Models:

Since $\tilde{\alpha}(t)$, the error in the estimate of $\hat{\alpha}(t)$ is decreased according to the equation

$$\dot{\bar{\tilde{\alpha}}} = -\bar{E}^T(t)\bar{E}(t)\bar{\tilde{\alpha}}(t) \tag{20}$$

or

$$\dot{\tilde{\alpha}} = -E^T(t)E(t)\tilde{\alpha}(t). \tag{21}$$

Slow convergence of $\bar{E}(t)$ of $E(t)$ assures the rapid convergence of $\tilde{\alpha}(t)$ to zero or of $\hat{\alpha}(t)$ to α and consequently $\hat{\theta}(t)$ to θ_p.

Comment: If a combination of the above two situations prevails (i.e. some of the parameters converge rapidly while others converge slowly) the fast convergence of $\tilde{\theta}(t)$ to zero can be demonstrated by considering the convergence in the two subspaces separately.

6.1 Simulation Studies

Extensive simulation studies have shown that second level adaptation proposed in this paper is far superior to:

a) conventional adaptive control using a single model,

b) switching, and

c) switching and tuning,

with multiple models being used in the last two cases.

Due to space limitations we include only one specific study which is quite conclusive. The plant to be controlled is second order (refer to Section 3) and has two unknown parameters θ_{p1} and θ_{p2}. These switch between two values $[10, 2]$ and $[5, -10]$, both of which correspond to unstable plants. The objective is to estimate the parameter and control it to follow the output of a stable reference model with parameters $[-6, -5]$ and a reference input $r(t) = 6\sin(0.2\pi t)$ (i.e. a period of 10).

The reference output, the plant output, and the control output error $e_c(t)$ are shown for the four cases, a), b), c) mentioned earlier for comparison with second level adaptation. Figure 1(a) shows the response using a single model. In Figure 1(b), switching between 17 models is indicated. Figure 1(c) shows switching and tuning with 9 fixed models and 2 adaptive models.

All three responses are seen to be unsatisfactory with large control error $e_c(t)$. In comparison, second level adaptation using four models initialized at the corners of a (rectangular) region of uncertainty $\mathcal{S}_\theta = [-15, 15] \times [-15, 15] \in \mathbb{R}^2$ is seen to have a very rapid and accurate response.

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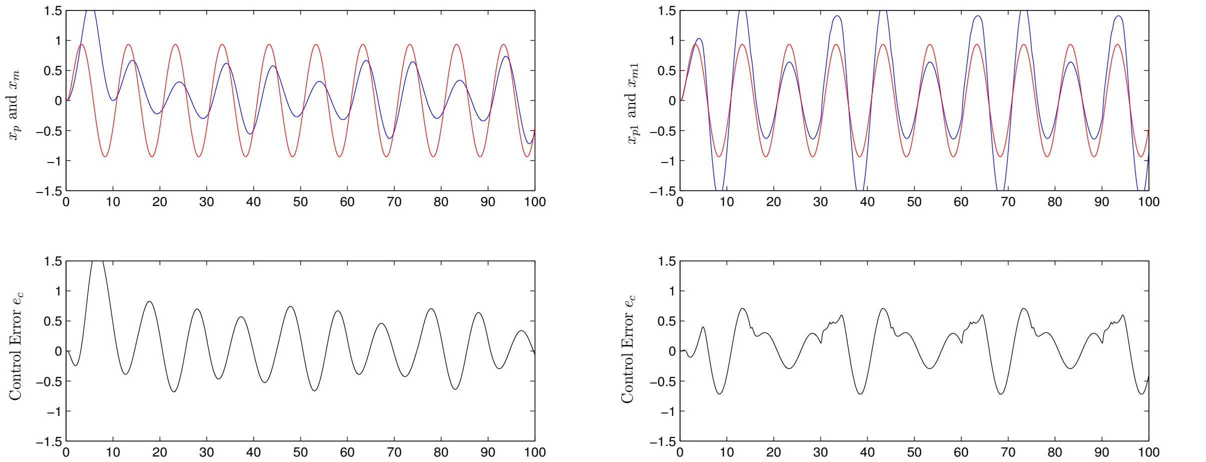
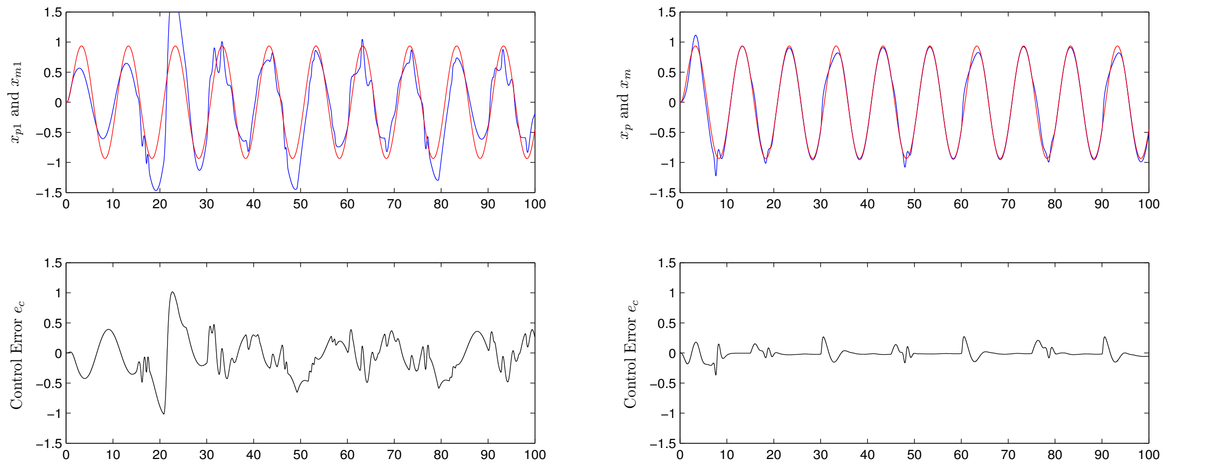


(a) Classical Adaptive Control with a Single Model, Plant Switches Every 15 Units of Time.

(b) Switching with 17 Models, Plant Switches Every 15 Units of Time.



(c) Switching and Tuning with 11 Models, Plant Switches Every 15 Units of Time.

(d) Second Level Adaptation with 4 Models, Plant Switches Every 15 Units of Time.

Fig. 1. Reference Output, Plant Output and Control Error For The Three Methods.

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