Nonlinear Output Regulation With Adaptive Conditional Servocompensator

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Abstract: An output feedback controller with adaptive conditional servocompensator is developed for robust output regulation of a class of nonlinear minimum-phase systems. It is shown that the controller achieves asymptotic regulation and can be tuned to recover the transient performance of a state feedback sliding-mode-controller.

1. INTRODUCTION

The conditional servocompensator was introduced by Seshagiri et al. (2005) in the design of output feedback control for a nonlinear regulation problem. It is different from the traditional servocompensator (or internal model) in that it is designed to act as a servocompensator only in a small neighborhood of a certain manifold. The state of the conditional servocompensator is always small and the trajectories of the closed-loop system under the output feedback controller can be made arbitrarily close to the trajectories that would have been obtained under a state feedback sliding mode controller, by tuning two design parameters, one associated with continuous implementation of the sliding mode control and the other with a high-gain observer. This property shows that the conditional servocompensator has a little effect on the transient behavior of the closed-loop system, which can be shaped through the design of the sliding mode control. Therefore, the controller ensures a desired transient behavior in addition to asymptotic regulation.

The conditional servocompensator includes an internal model that generates the constant and/or sinusoidal signals that must be included in the steady-state control to maintain zero steady-state regulation error. The amplitudes and phases of the sinusoidal signals can be unknown but their frequencies must be known because they define the internal model. Naturally, one question of practical significance is: What happens if there is uncertainty in these frequencies or if they are not known at all? The first case is addressed in Li et al. (2010), where it is shown that if, due to internal model uncertainty, there is an error in the steady-state control that is bounded by a constant δ, then the steady-state regulation error will be of the order of μδ, where μ is a design parameter can be made small. In this paper we deal with the case when the frequencies are unknown, but belong to known intervals. We design the conditional servocompensator with adaptive tuning of the coefficients of the servocompensator states as they enter into the feedback control. The control is designed as a state feedback continuously-implemented sliding model controller, together with a high-gain observer to estimate the states. Consistent with the spirit of the conditional servocompensator, the adaptive law is designed to be active only when the trajectories of the closed-loop system are in a small neighborhood of the sliding manifold. It is shown that all variables of the closed-loop system are bounded and the regulation error converges to zero. Moreover, if all the modes of the internal modes are excited, so that a certain persistence of excitation condition is satisfied, then parameter convergence is achieved as well.

Adaptive output regulation of nonlinear systems, driven by linear neutrally-stable exosystems with unknown frequencies, has drawn much attention in the last ten years. Since the original work of Nikiforov (1998), several interesting results have been presented in Serrani et al. (2001), Ding (2003), Ye et al. (2003), Serrani (2006), and Liu et al. (2009). The work Serrani et al. (2001) is the closest to this paper because it applies to the same class of nonlinear systems and also because of similarities in the control design; in particular, both papers use high-gain observers to implement robust state feedback designs. In the conclusions, we compare the controllers of the two papers as well as their technical results.

2. PROBLEM FORMULATION

Consider a single-input-single-output nonlinear system, modeled by

\[
\begin{align*}
\dot{z} &= \phi(z, e, w, \theta) \\
\dot{e}_i &= e_{i+1}, 1 \leq i \leq \rho - 1 \\
\dot{e}_\rho &= b(z, e, w, \theta) + a(z, e, w, \theta)u \\
y &= e_1
\end{align*}
\]

(1)

where \((z, e) \in R^n\) is the state, \(u \in R\) is the control input, \(y \in R\) is the measured regulation error, \(\theta\) is a vector unknown constant parameters that belongs to a compact set \(\Theta \subset R^l\), \(w\) is a time-varying signal generated by a linear neutrally-stable model (exosystem) \(\dot{w} = S_0w\), in which \(w\) belongs to a compact set \(W\) and \(S_0\) has simple eigenvalues on the imaginary axis. The functions \(\phi(\cdot), a(\cdot)\) and \(b(\cdot)\) are locally Lipschitz in their arguments and satisfy * This work was supported in part by the NSF under the grant number ECCS-0725165
\[ \phi(0, 0, w, \theta) = 0 \]
\[ b(0, 0, w, \theta) = -\chi(w, \theta) a(0, 0, w, \theta) \]
\[ a(z, e, w, \theta) \geq a_1 > 0 \]

The function \( \chi(w, \theta) \) is the steady-state value of the control \( u \) on the zero-error manifold \( \{z = 0, e = 0\} \). The goal is to design an output feedback controller to asymptotically regulate \( y \) to zero while ensuring boundedness of all variables of the closed-loop system. This goal will be reached under the following three assumptions, which restrict the class of nonlinear systems. The first assumption states that the steady-state control \( \chi(w, \theta) \) is generated by a linear model independent of \( \theta \). The linear model accounts for the constant and sinusoidal signals generated by \( \dot{w} = S_0w \) together with a finite number of the sinusoidal harmonics that are induced by nonlinearities. It holds when \( \chi(w, \theta) \) is a polynomial function of the components of \( w \).

The other two assumptions state minimum-phase properties that enable us to focus the control design on ensuring the boundedness and regulation of \( e \). They require the system \( \dot{z} = \phi(z, e, w, \theta) \) to be input-to-state stable, when \( e \) is viewed as the driving input, and the origin \( z = 0 \) of \( \dot{z} = \phi(z, 0, w, \theta) \) to be exponentially stable. Both properties are required to hold uniformly in \((w, \theta)\).

**Assumption 1.** There exist constants \( c_0 \) to \( c_{q-1} \), independent of \( \theta \), such that the matrix

\[
S = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 \\
c_0 & \cdots & c_{q-1}
\end{bmatrix}
\]

has simple eigenvalues on the imaginary axis and \( \chi(w, \theta) \) is generated by the linear internal model

\[
\dot{\tau} = S\tau, \quad \chi = \Gamma\tau \tag{2}
\]

where \( \Gamma^T = [\chi, L_4\chi, \ldots, L_{q-1}\chi] \), \( L_4\chi = \frac{\partial \chi}{\partial w}S_0w \), and \( \Gamma = [1, 0, \ldots, 0]_1 \times q \).

**Assumption 2.** There exist a \( C^1 \) function \( V_z(z, w, \theta) \), class \( K_\infty \) functions \( \alpha_1 \), \( \alpha_2 \) and class \( K \) functions \( \alpha_3 \) and \( \alpha_4 \), independent of \( w \) and \( \theta \), such that

\[
\alpha_1(||z||) \leq V_z(z, w, \theta) \leq \alpha_2(||z||),
\]

\[
\frac{\partial V_z}{\partial z} \phi(z, e, w, \theta) + \frac{\partial V_z}{\partial w}S_0w \leq -\alpha_3(||z||)
\]

for all \(||z|| \geq \alpha_4(||e||)\) and \((w, \theta) \in \Omega \times \Theta \).

**Assumption 3.** There exist a \( C^1 \) function \( V_{zz}(z, w, \theta) \), defined in some neighborhood of \( z = 0 \), and positive constants \( \lambda_1 \) to \( \lambda_4 \), independent of \( w \), such that

\[
\lambda_1 ||z||^2 \leq V_{zz}(z, w, \theta) \leq \lambda_4 ||z||^2
\]

\[
\frac{\partial V_{zz}}{\partial z} \phi(z, 0, w, \theta) + \frac{\partial V_{zz}}{\partial w}S_0w \leq -\lambda_3 ||z||^2
\]

\[
\left\| \frac{\partial V_{zz}}{\partial z} \right\| \leq \lambda_4 ||z||
\]

3. CONTROL AND ADAPTIVE LAWS

The controller of Seshagiri et al. (2005) comprises two components: a continuously-implemented sliding mode controller that brings the trajectories of \((z, e)\) to a small neighborhood of the origin within finite time, and a conditional servocompensator that implements the internal model (2) in some neighborhood of the sliding surface. We describe the controller in the partial state feedback case when \( e \) is measured and then bring in a high-gain observer to replace \( e \) by its estimate \( \hat{e} \).

The control is given by

\[
u = -k \text{sat}(\frac{s}{\mu}) \tag{3}
\]

where

\[
s = \Lambda^T \sigma + k_1 e_1 + \sum_{i=2}^{p-1} k_i e_i + e_\rho \tag{4}
\]

\(k_1 \rightarrow k_{p-1} \) and \( \mu \) are positive constants, and \( \sigma \) is the state of the conditional servocompensator

\[
\dot{\sigma} = F\sigma + \mu G \text{sat}(\frac{s}{\mu}) \tag{5}
\]

The constants \( k_1 \rightarrow k_{p-1} \) are chosen such that the polynomial

\[
\lambda^{p-1} + k_{p-1}\lambda^{p-2} + \cdots + k_2 \lambda + k_1
\]

is Hurwitz; \( F \) and \( G \) are chosen such that \( F \) is Hurwitz and the pair \((F, G)\) is controllable: \( \Lambda \) is the unique vector that assigns the eigenvalues of \((F + GA^T)\) at the eigenvalues of \( S; \mu \) is chosen as a small constant that defines the boundary layer \(||s|| \leq \mu||\) around the sliding surface \( s = 0; k \) is chosen large enough to ensure that \( s < -a_{00}|s| \) over a compact set of interest whenever \(||s|| \geq \mu||\). This choice ensures that the trajectories enter the layer \(||s|| \leq \mu||\) in finite time. Inside this layer, we invoke Assumption 2 to show that \((z, e)\) is ultimately bounded by a class \( K \) function of \( \mu \). The conditional servocompensator (5) is an exponentially stable linear system driven by \( O(\mu) \) bounded input; hence, by choosing \( \sigma(0) \) to be \( O(\mu) \) we can ensure that \( \sigma(t) \) will be \( O(\mu) \) for all \( t \geq 0 \). Inside the layer \(||s|| \leq \mu||\), \( \text{sat}(s/\mu) = s/\mu \) and the conditional servocompensator reduces to

\[
\dot{\sigma} = (F + GA^T)\sigma + G \left( \sum_{i=1}^{p-1} k_i e_i + e_\rho \right) \tag{6}
\]

This equation ensures the existence of a zero-error invariant manifold. In particular, let \( P \) be the unique nonsingular matrix such that \( P^{-1}(F + GA^T)P = S \) and \( P^{-1}G = [0, \ldots, 0, 1]^T \). It follows from Nikiforov (1998) that the equation

\[
MS - SM = -P^{-1}G(\Gamma + \Lambda^T PM)
\]

has a unique solution \( M \), which is nonsingular. It is shown in (Knobloch et al., 1993, pp. 34-36) that a matrix satisfying the foregoing equation actually satisfies the simultaneous equations

\[
SM = MS, \quad -\Lambda^T PM = \Gamma
\]

Setting

\[
\tilde{\sigma} = \frac{\mu}{k} PM \tau
\]

it can be seen that

\[
\Lambda^T \tilde{\sigma} = -\frac{\mu}{k} \Gamma \tau = -\frac{\mu}{k} \chi
\]

and \((z = 0, e = 0, \sigma = \tilde{\sigma})\) is an invariant manifold. By invoking Assumption 3, it can be shown that the manifold is exponentially attractive.

The state \( \sigma \) of the conditional servocompensator is \( O(\mu) \) for all \( t \geq 0 \). Therefore, the trajectories of \((z, e)\) under the
control (3) can be made arbitrarily close to the trajectories under the sliding mode control
\[ u = -k \text{ sgn} \left( \sum_{i=1}^{p-1} k_i e_i + e_\rho \right) \] (8)
by choosing \( \mu \) small enough; hence, the transient response of the system is not degraded due to the inclusion of the internal model.

The foregoing summary of the controller of Seshagiri et al. (2005) is valid when the matrix \( S \) of the internal model (2) is known because it is used to calculate \( \Lambda \). When \( S \) is unknown we replace \( \Lambda \) by \( \hat{\Lambda} \) and derive an adaptive law to adjust \( \hat{\Lambda} \) online. The adaptive law is derived by Lyapunov analysis. It uses a Lyapunov function of the form \( V_\nu = W + (\mu/2 \gamma) \hat{\Lambda}^T \hat{\Lambda} \), where \( W \) is a positive definite function of \( (z, e, \sigma) \), \( \gamma \) is a positive constant, and \( \hat{\Lambda} = \Lambda - \hat{\Lambda} \). Because the conditional servocompensator (5) is designed to act as internal model only inside the layer \( \{ |s| \leq \mu \} \), the derivative of \( V_\nu \) is pursued only inside this layer. It is shown that the derivative of \( V_\nu \) satisfies an inequality of the form
\[ \dot{V}_\nu \leq -X^T Q X - (k/\mu) \xi \Lambda^T \sigma + (\mu/\gamma) \hat{\Lambda}^T \hat{\Lambda} \]
where
\[ \xi = \sum_{i=1}^{p-1} k_i e_i + e_\rho \] (9)
and \( X^T Q X \) is a positive definite function of \( (z, e, \sigma) \).

By choosing \( \hat{\Lambda} = (k_\gamma/\mu)^2 \sigma \xi \) we can ensure that \( \dot{V}_\nu \) is negative semidefinite. However, because we plan to add a high-gain observer, we follow Khalil (1996a) in using smoothed parameter projection to ensure \(-k(\mu/\gamma) \hat{\Lambda}^T \sigma + (\mu/\gamma) \hat{\Lambda}^T \hat{\Lambda} \leq 0\) while maintaining \( \hat{\Lambda} \) bounded in a given compact set. Suppose \( \Lambda \) belongs to a convex hypercube \( \Xi \), defined by
\[ \Xi = \{ \Lambda | a_i \leq \Lambda_i \leq b_i \; \; 1 \leq i \leq q \} \]
Let
\[ \Xi_\delta = \{ \Xi | a_i - \delta \leq \Lambda_i \leq b_i + \delta \; \; 1 \leq i \leq q \} \]
where \( \delta > 0 \), and define the projection \( \text{Proj}(\Lambda, \gamma) \) by
\[ \text{Proj}(\hat{\Lambda}, \gamma)_{i} = \begin{cases} \Upsilon_i & \text{if } a_i \leq \hat{\Lambda}_i \leq b_i \\ \Upsilon_i & \text{if } \hat{\Lambda}_i > a_i \text{ and } \Upsilon_i \leq 0 \\ \Upsilon_i & \text{if } \hat{\Lambda}_i < a_i \text{ and } \Upsilon_i \geq 0 \\ \Upsilon_i & \text{if } \hat{\Lambda}_i > b_i \text{ and } \Upsilon_i > 0 \\ \Upsilon_i & \text{if } \hat{\Lambda}_i < b_i \text{ and } \Upsilon_i < 0 \\ \end{cases} \]

The adaptive law \( \dot{\hat{\Lambda}} = \text{Proj}(\Lambda, \gamma) \frac{k \sigma \xi}{\mu^2} \) ensures that \( \hat{\Lambda}(t) \in \Xi_\delta \) for all \( t \geq 0 \). Because the adaptive law is derived based on Lyapunov analysis inside the layer \( \{ |s| \leq \mu \} \), we make another modification that will keep \( \hat{\Lambda} \) constant outside the set \( \{ |s| \leq 2\mu \} \). Let
\[ \beta(s, \mu) = \begin{cases} 0 & \text{if } |s| \geq 2\mu \\ 1 & \text{if } |s| \leq \mu \\ 1 - \frac{|s| - \mu}{\mu} & \text{if } 2\mu > |s| > \mu \\ \end{cases} \]
and take the adaptive law as
\[ \dot{\hat{\Lambda}} = \beta(s, \mu) \text{Proj}(\Lambda, \gamma) \frac{k \sigma \xi}{\mu^2} \]
(10)

Inside the layer \( \{ |s| \leq \mu \} \), \( \beta = 1 \); hence the Lyapunov analysis is not affected by presence of \( \beta \). The transition of \( \beta \) from zero to one is done over an interval to ensure that the adaptive law is locally Lipschitz.

In the output feedback case, when only the regulation error \( y = e_1 \) is measured, the high-gain observer
\[ \dot{\hat{e}}_1 = \hat{e}_{i+1} + g_1 (e_1 - \hat{e}_i) / \epsilon \]
\[ \hat{e}_i = g_\rho (e_1 - \hat{e}_i) / \epsilon^\rho \]
(11)
is used to estimate \( e_\rho \) to \( e_1 \), where \( \epsilon \) is a small positive constant, and \( g_1 \) to \( g_\rho \) are chosen such that the polynomial
\[ \lambda^\rho + g_1 \lambda^{\rho-1} + \cdots + g_\rho - 1 \lambda + g_\rho \]
is Hurwitz. The variables \( \xi \) and \( s \) in the control and adaptive laws are replaced by
\[ \hat{\xi} = k_1 e_1 + \sum_{i=2}^{p-1} k_i e_i + e_\rho \]
and \( \hat{s} = \hat{L}\sigma + \hat{\xi} \)
(12)
respectively. From high-gain observer theory, e.g. Atassi et al. (2001), we know that the control and adaptive laws should be globally bounded functions in \( \hat{e} \) to overcome the peaking of the observer. For the control (3) and conditional servocompensator (5), this condition is satisfied because the dependence on \( \hat{s} \) is through the saturation function \( \text{sat}(\hat{s}) \). To impose this condition on the adaptive law we replace \( \hat{\xi} \) by \( \text{Usat}(\hat{\xi}/U) \) where the saturation level \( U \) is chosen greater than the maximum of \( |\xi| \) under state feedback, where the maximization is taken over the compact set of \( \xi \). Thus the control and adaptive laws under output feedback are given by
\[ u = -k \text{ sat}(\hat{s}/\mu) \]
\[ \hat{s} = F \sigma + \mu G \text{ sat}(\hat{s}/\mu) \]
\[ \hat{\Lambda} = \beta(\hat{s}, \mu) \text{Proj}(\Lambda, \gamma) \frac{k \sigma \xi}{\mu^2} \]
(13)

We conclude this section by examining the persistence of excitation of \( \sigma \), defined by (7), which is needed to prove convergence under output feedback control. As discussed in Serrani et al. (2001), \( \sigma \) is persistently exciting when the solution of (2) contains all the modes of \( S \); that is, all the modes are present in the steady-state control \( \chi \). If for some initial condition, \( w(0), \chi \) does not contain all the modes of \( S \), we can find a nonsingular matrix \( L \), dependent on \( w(0), \) such that
\[ L\sigma = \begin{bmatrix} \tilde{\sigma}_1 \\ 0 \end{bmatrix} \]
(14)
where \( \tilde{\sigma}_1 \) is a \( d \)-dimensional persistently exciting vector. When \( d = q, L \) is the identity matrix and \( \tilde{\sigma} = \tilde{\sigma}_1 \). When \( d = 0, \tilde{\sigma} = 0 \). To see how \( L \) is constructed, let \( H \) be a nonsingular matrix that transforms \( S \) into its real modal form (a block diagonal matrix where the diagonal blocks are either a zero element corresponding to a zero eigenvalue of \( S \) or a \( 2 \times 2 \) matrix corresponding to one of the sinusoidal modes of \( S \)). Rewrite equation (2) as
\[ \dot{\nu} = H^{-1} S H \nu, \quad \chi = H \nu \]
where \( \tau = H \nu \). Assume, without loss of generality, that the elements of \( \nu \) are ordered such that the first \( d \) elements contain the modes that are present in \( \chi \). Partition \( \nu \) as \( \nu^T = [\nu_1^T, \nu_2^T] \), where \( \nu_1 \) is of dimension \( d \). It follows from observability of the pair \( (S, \Gamma) \) that \( \nu_2(t) \equiv 0 \). Hence, from equation (7), we have
\[ \bar{\sigma} = \frac{\mu}{k} PMH \begin{bmatrix} \nu_1 \\ 0 \end{bmatrix} \]

Setting \( \bar{\sigma}_1 = (\mu/k)\nu_1 \) and \( L = (PMH)^{-1} \) yields equation (14).

4. MAIN RESULTS

The main properties of the proposed controller are stated in the following two theorems. The first theorem shows semiglobal regulation because for any compact set of initial conditions, the controller can be designed to ensure that all state variables are bounded and \((z,e)\) converges to zero as \(t\) tends to infinity; hence, \(\lim_{t \to \infty} g(t) = 0\). The second theorem shows the advantage of the conditional servocompensator because the trajectories of \((z,e)\) under the proposed output feedback controller can be made arbitrarily close to the trajectories under a state feedback sliding mode controller. Thus, the servocompensator and adaptation have a little effect on the transient behavior of the system.

Theorem 1. Suppose Assumptions 1 to 3 are satisfied. Consider the system (1) under the output feedback controller (11)–(13). Given any compact sets \( \Omega_1 \subset R^n \), \( \Omega_2 \subset R^p \) and any positive constant \( r \), there exists a constant \( k^* > 0 \) such that for each \( k \geq k^* \) there is \( \mu^* > 0 \) and for each \( \mu \in (0, \mu^*) \), there is \( e^* = e^*(\mu) > 0 \) such that for every \( \mu \in (0, \mu^*) \) and \( e \in (0, e^*) \) and for all initial conditions \((z(0), e(0)) \in \Omega_1, e(0) \in \Omega_2, ||(\sigma(0)|| \leq \mu^r, \lambda(0) \in \Xi, \) the variables \((z, e, \epsilon, \delta, \Lambda)\) are bounded for all \( t \geq 0 \) and \( \lim_{t \to \infty} (z(t), e(t)) = 0 \). Moreover, if \( \bar{\sigma} \) is persistently exciting, then the closed-loop system has an exponentially stable equilibrium point at \((z = 0, e = 0, \bar{\sigma} = 0, \delta = 0, \lambda = 0)\), where \( \bar{\sigma} = \sigma - \bar{\sigma} \).

Theorem 2. Suppose the conditions of Theorem 1 hold and let \((z^*, e^*)\) be the state of the system (1) under the state-feedback sliding-mode controller (8) with initial conditions \((z(0), e(0)) = (z^*(0), e^*(0))\). Then, for every \( \tau > 0 \), there is \( \mu^* > 0 \) and for every \( \mu \in (0, \mu^*) \) there is \( e^* = e^*(\mu) > 0 \) such that for every \( \mu \in (0, \mu^*) \) and \( e \in (0, e^*) \), \( ||(z(t), e(t)) - (z^*(t), e^*(t))|| \leq \tau \) for all \( t \geq 0 \).

The proof of Theorem 1 is carried out in two steps. First, we show that the trajectories reach a small neighborhood of the origin in finite time. The neighborhood is described by a compact set \( \Gamma_k \times \Sigma_2 \), which is parameterized in \( \mu \) and \( e \) in such a way that the set shrinks to zero as \( \mu \) and \( e \) tend to zero. This part of the proof makes use of two key properties of the control and adaptive laws; namely, \( \sigma = O(\mu) \) all the time and both \( \Lambda \) and \( \hat{\Lambda} \) are bounded uniformly in \( \mu \) and \( e \). In the second part of the proof we study the closed-loop system inside the set \( \Gamma_k \times \Sigma_2 \) to show convergence of the regulation error. The proof of Theorem 2 is based on the high-gain-observer property that the trajectories under output feedback approach those under state feedback as \( e \to 0 \). The state feedback is a continuous implementation of sliding mode control, in which \( \text{sgn}(s/\mu) \) replaces the sliding variable \( \sigma = O(\mu) \); therefore the trajectories of the two systems come arbitrarily close to each other as \( \mu \to 0 \).

Due to space limitations, the full proofs are omitted. They will appear in a full version of this paper, which is under review in Automatica.

5. SIMULATION EXAMPLE

To facilitate comparison with Serrani et al. (2001), we consider the same example; namely, the controlled van der Pol equation

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \theta_1 x_2 - \theta_2 [x_2^3 + x_1 x_2^2] + u \\
e &= x_1 - w_1
\end{align*}
\]

where \( \theta_1 \) and \( \theta_2 \) are unknown constant parameters that satisfy \( |\theta_1| \leq 3, 0 \leq \theta_2 \leq 5 \), and \( w_1 \) is generated by the harmonic oscillator

\[
\begin{align*}
\dot{w}_1 &= \omega_0 w_2 \\
\dot{w}_2 &= -\omega w_1
\end{align*}
\]

whose frequency \( \omega_0 \) is unknown but satisfies \( 0.5 \leq \omega \leq 5 \). The initial conditions of each of \( x_1, x_2, w_1 \) and \( w_2 \) are in the compact set \([0 1]\). The steady-state control \( \chi \) is given by

\[
\chi = (1 - \omega^2) w_1 - \theta_1 \omega w_2 + \theta_2 \omega^3 w_2^3 + \theta_2 \omega^2 w_1 w_2^2
\]

and satisfies the internal model (2) with

\[
S = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-9\omega^4 & -10\omega^2 & 0 & 0
\end{bmatrix}
\]

whose eigenvalues are \( \pm i\omega \) and \( \pm 3i\omega \). The third harmonic is due to the nonlinear term \( x_2 x_1 x_2^2 \) in the state equation. We start by choosing \((F, G)\) in the controllable canonical form as

\[
F = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-8640 & -3624 & -566 & -39
\end{bmatrix}, G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

In these coordinates the vector \( \Lambda \) that yields \( F + G A^T = S \) is given by

\[
A^T = \begin{bmatrix}
-9\omega^4 & 8640 & -3624 & -10\omega^2 & 566 & 39
\end{bmatrix}
\]

Following Serrani et al. (2001) we transform \((F, G, \Lambda_0)\), where \( \Lambda_0 \) corresponds to \( \omega = 1 \), into its balanced realization \((F, G, \Lambda)\) for numerical stability. In the new coordinates, \( F \) and \( G \) of (5) are given by

\[
F = PFP^{-1}, G = PG, \Lambda = \Lambda P^{-1}
\]

where \( P \) is the constant transformation matrix. From the expressions of \( \Lambda \) and \( \Lambda \) along with the fact that the frequency \( \omega \) belongs to \([0.5 \ 5]\), \( \delta, a_1 \), and \( b_1 \) in the adaptive law are chosen as \( \delta = 0.1, a_1 = -30 \) and \( b_1 = 30 \).

Case 1 We use (11)~(13) to control the van der Pol equation. For the observer parameters in (11) we choose \( g_1 = 2, g_2 = 3, \epsilon = 10^{-5} \). The gain \( k_1 \) and \( k_2 \) of (12) are chosen as \( 1 \) and \( 0.5 \), respectively. The parameters of (13) are taken as \( k = 200, \mu = 0.5, \gamma = 50, \) and \( U = 30 \). In the simulation, the initial conditions for \((x_1, x_2)\) and \((w_1, w_2)\) are \([1 \ 0]\) and \([0 \ 1]\), respectively, and \( \theta_1 = 3, \theta_2 = 4 \). We run the simulation with \( \omega = 3.5 \text{ rad/sec} \). Equation (14) is satisfied with \( \epsilon = 0.4 \). In the simulation, the adaptation is not connected to the system until 20s. We can see from Fig. 1 that after 20s, the regulation error becomes much smaller and converges to zero and \( \hat{\Lambda} \) converges to a constant limit. The eigenvalues of \( F + G A^T \) at this limit are \( \pm 3.5j \).
and \( \pm 10.5j \), which are the eigenvalues of \( S \). At \( t = 50 \), we change \( \omega \) to \( 2.5 \text{ rad/sec} \). We can observe that, after a spike, the regulation error converges to zero and \( \dot{\Lambda} \) converges to a constant limit. Two of the eigenvalues of \( F + GA^T \) are \( \pm 2.5j \) and \( \pm 7.5j \).

**Case 2**: All the parameters are chosen as those in Case 1 except for \( \theta_2 = 0 \) and \( \omega = 3.5 \text{ rad/sec} \) throughout the simulation time. In the current case, the steady state control is \( \chi = (1 - \omega^2)w_1 - 3\omega w_2 \) which can be generated by a second order internal model. In other words no matter what we set for the initial conditions of \( w_1 \) and \( w_2 \), they fail to excite all modes of \( F + GA^T \). Hence, equation (14) is satisfied with \( d = 2 < q \). From Fig. 2, we observe that the regulation error (RE) converges to zero while \( \dot{\Lambda} \) converges to a constant limit. Two of the eigenvalues of \( F + GA^T \) converge to \( \pm 3.5j \).

**Case 3**: We compare the performance of the output feedback controller (11)\textasciitilde(13) with the state feedback sliding mode controller (8) for the same values of \( k = 200 \), and the same initial conditions \( z_0 = [0 \ 1] \), \( w_0 = [1 \ 0] \), with \( \omega = 3.5 \text{ rad/sec} \) throughout the simulation time. For the controller (11)\textasciitilde(13), the parameters are retained from Case 1. By decreasing \( \mu \) and \( \varepsilon \), we can see from Fig. 3 that the difference between the regulation error in the two cases is decreased.

6. CONCLUSION

We have presented an adaptive version of the conditional servocompensator of Seshagiri et al. (2005) that allows the frequencies of the internal model to be unknown. The adaptive model includes parameter projection, based on the assumption that the unknown frequencies belong to known intervals, and is designed such that adaptation is active only when the trajectories of the closed-loop system are in a small neighborhood of the sliding manifold. The transient performance of the closed-loop can be tuned to be arbitrarily close to the trajectories under state feedback sliding-mode control; hence the transient performance is almost independent of the conditional servocompensator as well as the adaptive law.

The controller of this paper shares a number of similarities with the controller of Serrani et al. (2001). The two controllers apply to the same class of nonlinear systems, they are based on the design of robust state feedback controllers with adaptation of the internal model, the adaptive laws are Lyapunov based, and the output feedback controllers use high-gain observers. There are, however, three differences. First, the state feedback control used here is a continuous implementation of sliding mode control, while the controller of Serrani et al. (2001) uses high-gain feedback. Both designs drive the trajectories towards a certain manifold, with the trajectories under high-gain feedback reaching the manifold faster but at the expense of a large initial spike in the control signal. The continuously-implemented sliding mode control saturates during the reaching phase.
Second, the adaptive law of Serrani et al. (2001) is designed to be active all the time, while in the current paper the adaptive law is active only when the trajectories reach an $O(\mu)$ neighborhood of the sliding manifold. Finally, the adaptive law of the current paper includes parameter projection, which is left out of Serrani et al. (2001) so as not to complicate the presentation. There are also two differences in terms of the proved technical properties of the controller. First, the current paper proves a property about the transient behavior that is not proved for the controller of Serrani et al. (2001). It is worthwhile to mention that in Ma et al. (2010) it is shown that the trajectories under the controller of Serrani et al. (2001), without adaptation, approach the trajectories of a closed-loop system under high-gain state feedback control without internal model, as the gain becomes high enough. This shows that the transient behavior of the controller of Serrani et al. (2001) can be tuned to be independent of the internal model. The result of Ma et al. (2010), however, is limited to the controller without adaptation. It remains to be seen if a similar property holds with adaptation. Second, the proof of asymptotic regulation in Serrani et al. (2001) is done under the assumption that all modes of the internal model are excited. No such condition is needed in the current paper. This is significant because designing the internal model to deal with certain constant and sinusoidal signals does not require all these signals to be present during the operation of the system. Suppose, for example, that the internal model is designed to guard against two sinusoidal disturbances, resulting in an internal model of dimension 4. When both signals are present, both modes of the internal model will be excited. However, if in some run of the system, only one of the two sinusoidal signals is present, the requirement that all modes be excited will be violated. The result of this paper ensures asymptotic regulation in such scenario while that of Serrani et al. (2001) does not do so. This is not to say that the controller of Serrani et al. (2001) does not possess a property similar to the one shown here. It might have such property but it has not been shown.

REFERENCES
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