Smooth transitions via quasi-periodic paths

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Abstract: In this paper we consider a problem of connecting two different periodic orbits with a ‘quasi-periodic’ trajectory and present two methods of constructing such a trajectory. Furthermore, we provide a detailed derivation of an algorithm to generate this and discuss various numerical issues as well. The problem is relevant to certain robotics applications where the gait transitions need to be conducted in a quasi-stationary fashion.

Keywords: Optimal control, optimal trajectory, algorithms.

1. INTRODUCTION

Periodic phenomena play a very important role in our daily lives. From locomotion to the beating of the heart many biological process exhibit periodic characteristics. In modern technology one only has to mention the pervasive use of oscillators in the telecommunication industry.

The question of the existence of periodic orbits has attracted a lot of attention in the mathematical community. In the control community the idea of stabilizing a system to a periodic orbit has also been considered extensively.

In this paper, however, we will consider a different aspect and extend several ideas presented in Verriest and Yeung (2008); Yeung and Verriest (2010). Specifically, we are interested in connecting two periodic orbits via a quasi-periodic path. This is of interest in applications, such as the gradual change of locomotion in robotics, where ‘smooth’ transitions are desired. This is reminiscent of the quasi-stationary processes that are often alluded to in thermodynamics, where a system undergoes slow and smooth changes as if at each step it remains stationary. Of course, the system does not remain stationary or there would not be a state transition! Related but yet different ideas appeared in Sultan (2007).

The outline of the paper is as follows: in section 2 we formulate the problem considered; in section 3 we derive an algorithm to generate the ‘smooth’ trajectory; an example is also presented. In section 4 present an alternative method. Finally, in section 5 we conclude and offer various possible extensions.

2. PROBLEM FORMULATION

To fix ideas let us consider the nonlinear system

$$\dot{x} = f(x, u), \quad x \in X \subset \mathbb{R}^n, \quad u \in U \subset \mathbb{R}^m. \tag{1}$$

It is well known that for a fixed $T$–periodic input $u_T$ and an initial condition the state response may contain 0, 1 or multiple periodic orbits. Even when the response is periodic, the period of $x$ might be different from the period of the input $u$. In order to make the following discussion meaningful we consider a subclass of nonlinear system that is relatively well-behaved. Specifically, we will consider the class of convergent systems, and this is extensively studied in Pavlov et al. (2006) in terms of the output regulation problem. For convenience the notion of convergent system is reviewed in appendix A. Roughly speaking, when a system is uniformly convergent in a subset $X$ of the state space, it exhibits behaviors similar to the ones possessed by asymptotically stable linear systems. Indeed, from theorem (2) in the Appendix we see that for any $T$-periodic input $u_T$, there exists a unique $T$-periodic response $x_T$. In other words the periodic behavior is uniquely determined by the input.

In the context of the problem considered in this paper, we assume that there are two periodic orbits $x_i, i = 0, 1$ induced by two periodic inputs $u_i, i = 0, 1$ with possibly two different periods $T_i, i = 0, 1$. We denote the periodic behaviors by $w_i(t) = (x_i(t), u_i(t)), i = 0, 1$, where $x_i(t) = f(x_i(t), u_i(t)), i = 0, 1$. The problem that we consider in this paper is as follows: find an input $u(t), t \in [0, 1]$ that steers the system from the initial orbit $w_0(t), t < 0$ to the final orbit $w_1(t), t > 1$ in a ‘quasi-periodic’ fashion.

How do we quantify the notion of ‘quasi-periodicity’? If $f$ is a $T$–periodic function, then by definition $f(t) - f(t - T) = 0, \forall t \in \mathbb{R}$. Thus, if $f$ is not $T$-periodic, then

$$\int_I \|f(t) - f(t - T)\|^2 \, dt$$

measures the non-periodicity of $f$ over the interval $I$. Therefore, in connecting two periodic behaviors of the same period (i.e., $T_1 = T_0$) via a quasi-periodic path, it is reasonable to pose the ‘quasi-periodic’ transition problem as follows:

$$\min_u J_0 = \int_0^1 \|w(t) - w(t - T)\|^2_W \, dt \tag{2}$$

$$= \int_0^1 \|x(t) - x(t - T)\|^2_P + \|u(t) - u(t - T)\|^2_R \, dt,$$

subject to the dynamics (1), $w_0(t) = (x_0(t), u_0(t)), t < 0$ and $w_1(t) = (x_1(t), u_1(t)), t > 1$. We may interpret this as a search for a behavior $w(t), t \in [0, 1]$, that minimizes the
non-periodicity of \( w(t)\). Observe that the optimal solution is similar to an amplitude modulation as the periodicity is unchanged.

For the case \( T_0 \neq T_1 \) one may instead consider the optimization problem

\[
\min_{u,T} J_0 = \int_0^1 \| x(t) - x(t - T(t)) \|^2_P + \| u(t) - u(t - T(t)) \|^2_Q dt + \| \dot{T}(t) \|^2 dt.
\]

subject to the same constraints as in (2). The last term is added to avoid large deviations in the periodicity as it is changed from \( T_0 \) to \( T_1 \). We may consider this a frequency modulation as the period is modified in a ‘smooth’ manner. The previous problem on the other hand is equivalent to

\[
\min_u J_0 = \int_0^1 \| x(t) - x(t - T(t)) \|^2_P + \| u(t) - u(t - T(t)) \|^2_Q dt,
\]

subject to (1), \( w_0(t) = (x_0(t), u_0(t)), t < 0 \) and \( w_1(t) = (x_1(t), u_1(t)), t > 1 \), where \( T(t) \) is the straight line

\[
T(t) = \begin{cases} T_0(1 - t) + T_1 t, & t \in [0, 1] \\ T_1, & t > 1 \end{cases}
\]

Observe that this reduces to (2) if \( T_0 = T_1 \). The problem may be solved for by first bringing it into the standard form via the change of variable \( u(t) = u(t - T(t)) + v(t) \). The combined system with (1) is a delay system with time varying delay. For such systems it is argued in Verriest (2009, 2010) that they exhibit causal behavior only if \( 1 - \dot{T} > 0 \). This is satisfied in general as we require that \( T_0, T_1 < \frac{1}{\epsilon} \) for the quasi-periodic transition. In principle one may now proceed to compute the necessary conditions of optimality and derive the corresponding gradient descent algorithm. However, the quantization effect of the time delay \( T(t) \) in \( u(t - T(t)) \) will accumulate and destroy the computation. Indeed, let \( q(t) \) denote the quantization of \( t \), then \( u(t) = u(t - q(T(t)) + \Delta) \) and \( v(t) \approx u(t - q(T(t)) + \dot{u}(t - q(T(t)) \Delta + v(t)) \), where \( \Delta \) is the quantization error. Since differentiation is a high pass process, the quantization noise is amplified. To avoid this we add the term \( \| \dot{u}(t) - (1 - \dot{T})u(t - T(t)) \|^2_R \) to (3) to regularize the problem. This way the quantization noise is filtered out as \( \dot{u} \) is integrated. Furthermore, as \( R \to 0 \) the modified problem approaches the original problem.

In order to solve the modified problem we now introduce a new variable \( \bar{u}(t) = v(t) \) and the new state variable \( z = (x', u') \). The combined dynamics is \( \dot{z} = F(x, v) \), where \( F(x, v) = \begin{bmatrix} f(x, u) \\ v \end{bmatrix} \), and \( \| s \| = \| s \| _P + \| s \| _Q \). The problem may be formally stated as follows:

\[
\min_v J_0 = \frac{1}{2} \int_0^{1+T_1} \| z(t) - z(t - T(t)) \|^2_S + \| v(t) - (1 - \dot{T})v(t - T(t)) \|^2_R dt
\]

subject to \( \dot{z}(t) = F(z(t), v(t)) \) and the final state constraint \( z(1) = (x(1), u(1)) = z_1 \), which is the target state in the final orbit.

**Theorem 1.** The system \( \dot{z} = f(z, v) \) with the prescribed final state constraint minimizes the performance index (4) if the input \( v(t) \) is chosen to satisfy:

**Euler-Lagrange equations:**

\[
\dot{\lambda}(t) = S \left( \frac{2 - \dot{T}}{1 - T} \right) z(t) - z(t - T(t))
\]

\[
- \frac{1}{1 - T} \left( \frac{t + T_0}{1 - T} \right) + R^{-1} \frac{\partial F}{\partial v} \lambda(t), \ t \in [0, 1 - T_1),
\]

\[
\dot{\lambda}(t) = S (2z(t) - z(t - T(t)) - z(t + T_1)) + R^{-1} \frac{\partial F}{\partial v} \lambda(t), \ t \in [1 - T_1, 1].
\]

**Boundary conditions:**

\( \lambda(1) = \nu \), where \( \nu \) is chosen such that the final state constraint is satisfied.

**Optimality condition:**

\[
v(t) = \begin{cases} \frac{1}{2 - \dot{T}} \left( - (1 - \dot{T})v(t - T(t)) \right) \ldots \\ - \frac{1}{2 - \dot{T}} \left( (1 - \dot{T})v(t - T(t)) - v(t + T_1) \right) + R^{-1} \frac{\partial F}{\partial v} \lambda(t), \ t \in [0, 1 - T_1) \end{cases}
\]

**Proof.** Before computing the Fréchet derivative

\[
\delta J = \lim_{\epsilon \to 0} \frac{J_\epsilon - J_0}{\epsilon},
\]

we first adjoin the dynamics and the final state constraint to the performance index via the Lagrange multipliers \( \lambda \) and \( \nu \), respectively:

\[
J_0 = \nu'(z(1) - z_1) + \frac{1}{2} \int_0^1 \| z(t) - z(t - T(t)) \|^2_S dt + \ldots
\]

\[
+ \frac{1}{2} \| v(t) - (1 - \dot{T})v(t - T(t)) \|^2_R dt + \int_0^{1+T_1} \frac{1}{2} \| z(t) - z(t - T_1) \|^2_S dt
\]

\[
+ \frac{1}{2} \| v(t) - v(t - T_1) \|^2_R dt + \int_0^1 \lambda(t) (F(z(t), v(t)) - \dot{z}(t)) dt
\]

After letting \( z(t) \to z(t) + \epsilon \eta(t) \), \( v(t) \to v(t) + \epsilon w(t) \)
\[ \delta J = \nu'(\eta(1) - \lambda(t)\eta(t))|_0^1 + \int_0^1 \left( z(t) - z(t - T(t)) \right)' S \eta(t) - \eta(t - T(t)) \right) + \left( v(t) - (1 - \dot{T})v(t - T(t)) \right)' R \times \left( w(t) - (1 - \dot{T})w(t - T(t)) \right) + \left( \lambda'(t) \frac{\partial F}{\partial z} + \dot{\lambda}'(t) \right) \eta(t) + \lambda'(t) \frac{\partial F}{\partial v} w(t) dt \]

\[ + \int_1^{1+T_T} \left( (z_1(t) - z(t - T_1))' S (-\eta(t - T_1)) + (v_1(t) - v(t - T_1))' R (-w(t - T_1)) \right) dt \]

\[ = \int_0^1 \left( \left( \frac{2 - \dot{T}}{1 - \dot{T}} \right) z(t) - z(t - T(t)) \ldots - \frac{1}{1 - \dot{T}} t + T_0 \right)^' \left( \frac{2 - \dot{T}}{1 - \dot{T}} \right) z(t) + \lambda'(t) \frac{\partial F}{\partial z} + \dot{\lambda}'(t) \right) \eta(t) + \left( \left( 2v(t) - (1 - \dot{T})v(t - T(t)) - v(t + T_1) \right)' R \ldots + \lambda'(t) \frac{\partial F}{\partial v} w(t) dt + (\nu - \lambda(1))'\eta(1) + \lambda'(0)\eta(0). \]

In order to avoid evaluating \( \eta(t) \) we choose \( \lambda(1) = \nu \) and \( -\lambda(t) = S \left( \frac{2 - \dot{T}}{1 - \dot{T}} \right) z(t) - z(t - T(t)) \]

\[ - \frac{1}{1 - \dot{T}} t + T_0 \right)^' \frac{\partial F}{\partial z} \lambda(t), \quad t \in [0, 1 - T_1), \]

\[ - \lambda(t) = S (2z(t) - z(t - T(t)) - z(t + T_1)) + \frac{\partial F}{\partial z} \lambda(t), \quad t \in [1 - T_1, 1], \]

Hence, the gradient is

\[ \delta J = \int_0^1 c'(t)\delta v(t) dt, \]

where

\[ c(t) = \begin{cases} R \left( \frac{2 - \dot{T}}{1 - \dot{T}} \right) v(t) - (1 - \dot{T}) v(t - T(t)) \ldots - v \left( \frac{t + T_0}{1 - \dot{T}} \right) + \frac{\partial F}{\partial v} \lambda(t), & t \in [0, 1 - T_1) \\ R \left( 2v(t) - (1 - \dot{T}) v(t - T(t)) - v(t + T_1) \right) + \frac{\partial F}{\partial v} \lambda(t), & t \in [1 - T_1, 1]. \end{cases} \]

from which the optimality condition follows immediately.

Observe from the optimality condition that in order to compute the input one needs both the past and the future values of \( v(t) \).

### 3. Algorithm

The previous necessary conditions are in general difficult to apply even for a fairly simple system. Therefore, in this section we derive a gradient descent algorithm to compute the optimal input. Unlike the free endpoint problems the algorithm needed is substantially more involved. The algorithm is adapted from Bryson and Ho (1975). It is based on the observations that the original problem consists of two subproblems and that the Euler-Lagrange equations are linearized equations.

The first step is to compute the Fréchet derivative of the boundary condition \( J_i = z_i(1) = \epsilon_i'(z(1)) \). This gives the boundary mismatch at each iteration of the algorithm. Indeed, one may obtain the necessary conditions by the following substitution in theorem (1): \( R = 0, S = 0, \nu \rightarrow \epsilon_i'(z(1)) \) and \( \lambda \rightarrow \lambda_i'(z(1)) \). The Fréchet derivative of the boundary condition is

\[ \delta z_i(1) = \int_0^1 (\lambda_i'(z(1)) - \lambda_i'(z(t))) \frac{\partial F}{\partial v} \delta v(t) dt, \]

where

\[ \lambda_i'(z) = -\frac{\partial F}{\partial z} \lambda_i'(z), \]

\[ \lambda_i'(1) = \begin{cases} 0; & i \neq j \\ 1; & i = j, j = 1, \ldots, N, \end{cases} \]

where \( N = \text{dim} \ z \). (In fact \( N = n + m \) where \( \text{dim} x = n \), \( \text{dim} u = m \).)

Subsequently, we compute the Fréchet derivative in the absence of the boundary conditions. In other words, we compute \( \delta J_v \) where \( J_v \) is the performance index in (4). This gradient is easily obtained by setting \( \nu = 0 \) in theorem (1). Indeed, the gradient is

\[ \delta J_v = \int_0^1 c'(t)\delta v(t) dt, \]

with the same costate equations as in (5), except that the boundary condition is now \( \lambda(1) = 0 \)

Finally, to obtain a gradient descent algorithm we minimize

\[ \delta J = \delta J_v + \int_0^1 \frac{\|\delta v\|^2}{2k_1} + \sum_{k=1}^N \nu_k \left( c_k'(z(1)) + \int_0^1 \lambda_i'(t) \frac{\partial F}{\partial v} \delta v(t) dt \right), \]

where \( k_1 > 0 \) is a step size parameter. After neglecting the changes in the coefficients, the second variation is

\[ \delta^2 J = \int_0^1 \left( c'(t) + \frac{1}{k_1} \delta v'(t) + \sum_{k=1}^N \nu_k \lambda_i'(t) \frac{\partial F}{\partial v} \right) dt. \]

Hence, (7) is minimal if
\[ \delta v(t) = -k_1 \left( c(t) + \sum_{k=1}^{N} \nu_k \frac{\partial F'}{\partial v} \lambda_k^*(t) \right) \]  

Substitute this into (6)

\[ \delta z_i(1) = -k_1 \int_0^1 \left( \lambda_i' \right)(t) \frac{\partial F'}{\partial v} \left( c(t) + \sum_{k=1}^{N} \nu_k \frac{\partial F'}{\partial v} \lambda_k^*(t) \right) dt,
\]

Express the previous in matrix notation

\[ \delta \mathbf{z}(1) = -k_1 \left( D + \int_0^1 \Lambda(t) \frac{\partial F'}{\partial v} \Lambda(t) dt \right) \mathbf{v}, \]

where

\[ \Lambda(t) = \begin{bmatrix}
\lambda_1^*(t) & \lambda_2^*(t) & \cdots & \lambda_N^*(t)
\end{bmatrix}
\]

and

\[ D = \int_0^1 \Lambda'(t) \frac{\partial F}{\partial v} c(t) dt.
\]

The reachability assumption of the original system implies that \( \Lambda_g \) is invertible and \( \nu \) can be explicitly solved for. Indeed,

\[ \nu = -\Lambda_g^{-1} \frac{\partial \delta z(1)}{\partial k_1}.
\]

Substitute this into (8)

\[ \delta v(t) = -k_1 \left( c(t) - \frac{\partial F'}{\partial v} \Lambda(t) \Lambda_g^{-1} \left( D + \frac{\delta z(1)}{k_1} \right) \right)
\]

\[ = -k_1 \left( c(t) - \frac{\partial F'}{\partial v} \Lambda(t) \Lambda_g^{-1} D \right)
\]

\[ + \frac{\partial F'}{\partial v} \Lambda(t) \Lambda_g^{-1} \delta z(1). \]

Observe that the gradient consists of two parts. The second part involves the mismatch of the boundary conditions.

We summarize the previous computations in the following algorithm.

3.1 Algorithm

(1) Guess an input \( v(t) \).

(2) Compute the dynamics \( \dot{z} = F(z, v) \) forward with the given initial condition \( z(0) = z_0 \).

(3) Let \( z_f \) denote the desired final state constraint. Compute \( \Delta z(1) = z(1) - z_f \) and set \( \delta z(1) = -k_2 \Delta z(1) \), where \( k_2 \geq 0 \) is a small step size parameter.

(4) Compute the costate equation (5) backward with boundary condition \( \lambda(1) = 0 \).

(5) Compute the gradient (9).

\[ v_{\text{new}} = v_{\text{old}} + \delta v. \]

(7) Repeat step 2) with the new input \( v_{\text{new}} \).

Some care needs to be exercised in choosing the parameter \( k_1 \geq 0 \) and \( k_2 \geq 0 \). In the first few iterations one may set \( k_1 = 0 \); in light of (9) this implies that the gradient \( \delta v \) only improves the input in the direction of decreasing the boundary mismatch. Subsequently, one may gradually increase \( k_2 \) to one until a satisfactory candidate is found. Finally, \( k_1 \) may be increased from zero to improve the input in the other direction.

Example Consider the bilinear system

\[ \dot{x}_1 = -ax_1 + ux_2, \]

\[ \dot{x}_2 = -ax_2 + u, a > 0. \]

Due to the special structure the previous system is easily shown to be uniformly convergent. In fact, it is not hard to see that for any periodic input the state response will also be periodic.

For the following example we will connect two orbits induced from a sinusoidal input. The final orbit has a period \( T_1 = 0.05 \) and the magnitude of the input sinusoid is \( A_1 = 20 \); the initial orbit, on the other hand, has a period \( T_0 = 0.95T_1 \) while the magnitude of the input is \( A_0 = 0.5A_1 \). The trajectory is being slowed down in the process. We apply the gradient descent algorithm in (3.1) for \( a = 20 \). The smooth transition is shown in figure (1).

Fig. 1. Direct method: trajectory of smooth transition

4. THE INDIRECT METHOD

The technique of seeking the ‘smoothest’ transition in the previous section shapes directly the behavior \( w(t) \). On the other hand, since we are concerned with periodic behaviors, another method would be to shape the frequency content of the behavior. Let \( x \in L_2[t, t - T] = \{ x \int_{t-T}^{t} \| x(t) \|^2 dt < \infty \} \) and denote the sliding Fourier coefficient by

\[ \langle x \rangle_k(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau) e^{jk\omega \tau} d\tau, \quad \omega = \frac{2\pi}{T}. \]

Furthermore, denote the initial and target orbit by \( w_0 = (u_0, x_0) \) and \( w_1 = (u_1, x_1) \) with period \( T_0 \) and \( T_1 \), respectively. In Yeung and Verriest (2010) we considered the indirect method for \( T_0 = T_1 \) and showed the connection between the direct and indirect method. In this paper we assume that \( T_0 \neq T_1 \), and unlike the former case, it is not clear what the fundamental period \( T \) in the Fourier...
integral needs to be. This impasse may be circumvented by writing $T_0 = \frac{p}{q} + r$ for some $p, q \in \mathbb{N}, q \neq 0, r \in ...$ SYSTEMS

We first review some stability notions related to an autonomous system. Consider the autonomous system

$$\langle \dot{\theta}(t), t \rangle = f(x(t), \theta(t)), t \in [0, t_f].$$

subject to (13), also minimizes (12), where $\theta_p(t) = \theta_p(0(1 - \frac{t}{T_0}) + \theta_p,1 \frac{t}{T_0}, t \in [0, t_f]$). This is a minimum energy problem where the new input is $\theta_d$; the corresponding dynamics is $\ddot{\tau}(t) = f(x(t), u_p(t, \theta_p(t)) + \theta_d(t))$. When $t_f$ is large or $t_f \rightarrow \infty$, the changes in the parameters $\theta_p$ will be small, and the input $\theta_d$ ensures that the final orbit is attained. The performance index forces $\theta_d$ to be as steady as possible.

As an example we apply the previous setup to the nonlinear system (10). For simplicity we consider two orbits generated by the inputs $u(t) = A_i \cos(\omega_i t) + c_i, i = 0, 1$, hence the parameters are $\theta_i = (c_i, \omega_i), i = 0, 1$. Specifically, according to (14) the smooth transition is constructed as follows: $\min_{\theta_d} \int_0^{t_f} ||\frac{d\theta_d}{dt}||^2 dt$, subject to the dynamics (10) and $\theta_d(t) = \phi_0, x(t) = x_0(t), t \leq 0, \theta_d(t) = \phi_1, x(t) = x_1(t), t \geq t_f, u(t) = A(t) \cos(\omega(t) + \theta_d(t))$, where $A(t) = A_0(1 - \frac{t}{T_0}) + A_1 \frac{t}{T_0}, \omega(t) = \omega_0(1 - \frac{t}{T_0}) + \omega_1 \frac{t}{T_0}$ and $\phi(t) = \phi_0(1 - \frac{t}{T_0}) + \phi_1 \frac{t}{T_0}$. For the results in figure (2) we have chosen $t_f = 1, A_0 = 150, T_0 = 0.95T_1, A_1 = 175, T_1 = 0.05$ and apply the gradient descent algorithm as expounded in Bryson and Ho (1975).

4.1 An alternative method

The indirect method in the previous section suffers from several shortcomings. The choice of the fundamental period $T$ seems somewhat ad-hoc. Ideally, a Fourier analysis with a time-varying window would be tailor-made for the problem, however, such a technique is non-existent to the best of the author’s knowledge.

In practice many orbits are generated by inputs $u$ that may be parameterized by a finite number of parameters, $\theta \in \mathbb{R}^d$. Denote the input explicitly by $u(t, \theta)$ and the resulting trajectory $x(t, \theta)$. In the present context it is desired to connect the orbits $w_i(t) = (x(t, \theta_i), u(t, \theta_i))$, $i = 0, 1$ in a smooth manner. Recall that if the vector field is Lipschitz, then small changes in $\theta$ lead to small changes in $x$. This motivates us to rephrase the Gluskabi raccordation as follows:

$$\min_{\theta} \int_0^{t_f} ||\frac{d\theta}{dt}||^2 dt, \quad (12)$$

subject to the dynamics $\dot{x} = f(x, u)$ and the boundary conditions

$$\theta(t) = \theta_0, x(t) = x(t, \theta_0), t \leq 0, \quad \theta(t) = \theta_1, x(t) = x(t, \theta_1), t \geq t_f. \quad (13)$$

Without loss of generality we will work with the horizon $[0, t_f]$ instead of the unit interval as in the previous sections.

In order to facilitate the discussion we decompose the periodic input that generates the periodic orbits: $u(t, \theta) = u_p(t, \theta_p) + \theta_d, \theta_d$ is the DC part, $u_p$ is periodic in the absence of the DC component and $\theta = (\theta_p, \theta_d)$. Furthermore, we will also assume throughout that the period of $u_p$ is much smaller than $t_f$. Since a geodesic with a uniform speed minimizes the energy of its derivative along the trajectory Milnor (1997), the optimal solution of

$$\min_{\theta_d} \int_0^{t_f} ||\frac{d\theta_d}{dt}||^2 dt, \quad (14)$$

Fig. 2. Indirect method: trajectory of smooth transition

5. CONCLUSION

In this paper we considered the problem of steering a nonlinear system from one periodic orbit to another in a quasi-periodic fashion. The transition is smooth in the sense that it is as inconspicuous as possible, and we presented two methods to achieve this transfer: the direct and indirect method. This work is a continuation of Verriest and Yeung (2008); Yeung and Verriest (2010). We also provide a detailed derivation of a gradient descent algorithm to generate the corresponding transfer.

Appendix A. CONVERGENT SYSTEMS

We first review some stability notions related to autonomous system. Consider the autonomous system
\[ \dot{x} = f(x, t) \quad (A.1) \]
\[ x \in \mathbb{R}^n, \ t \in \mathbb{R} \] where \( f(x, t) \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \).

**Definition A.1.** (Pavlov et al. (2006)). A solution \( \mathfrak{T}, t \in (t^*, \infty) \) of (A.1) is
- **stable** if for any \( t_0 \in (t^*, \infty) \) and \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( \|x(t_0) - \mathfrak{T}(t_0)\| < \delta \) implies \( \|x(t) - \mathfrak{T}(t)\| < \epsilon \) for all \( t \geq t_0 \).
- **uniformly stable** if it is stable and the number \( \delta \) is independent of \( t_0 \).
- **asymptotically stable** if it is stable and for any \( t_0 \in (t^*, \infty) \) there exists a \( \delta = \delta(t_0) > 0 \) such that \( \|x(t_0) - \mathfrak{T}(t_0)\| < \delta \) implies that \( \lim_{t \to \infty} \|x(t) - \mathfrak{T}(t)\| = 0 \).
- **uniformly asymptotically stable** if it is uniformly stable and there exists a \( \delta > 0 \), independent of \( t_0 \), such that for any \( \epsilon > 0 \) there exists \( T = T(\epsilon) > 0 \) so that \( \|x(t_0) - \mathfrak{T}(t_0)\| < \delta \) for \( t_0 \in (t^*, \infty) \) implies that \( \|x(t) - \mathfrak{T}(t)\| < \epsilon \) for all \( t \geq t_0 + T \).

We now define stability of a solution in a pre-defined subset of the state space, instead of a neighborhood of a solution.

**Definition A.2.** (Pavlov et al. (2006)). A solution \( \mathfrak{T}(t), t \in (t^*, \infty) \) of (A.1) is
- **asymptotically stable in a set** \( \mathcal{X} \subset \mathbb{R}^n \) if it is asymptotically stable and a solution of (A.1) starting at \( x(t_0) \in \mathcal{X}, t_0 \in (t^*, \infty) \) implies that \( \|x(t) - \mathfrak{T}(t)\| \to 0 \) as \( t \to \infty \).
- **uniformly asymptotically stable in a set** \( \mathcal{X} \subset \mathbb{R}^n \) if it is uniformly stable and it attracts solutions of system (A.1) starting at \( x(t_0) \in \mathcal{X}, t_0 \in (t^*, \infty) \) uniformly over \( t_0 \). In other words, for any compact set \( K \subset \mathcal{X} \) and any \( \epsilon > 0 \) there exists \( T(\epsilon, K) > 0 \) such that if \( x(t_0) \in K, t_0 \in (t^*, \infty) \), then \( \|x(t) - \mathfrak{T}(t)\| < \epsilon \) for all \( t \geq t_0 + T(\epsilon, K) \).

We now turn our attention to convergent systems.

**Definition A.3.** (Pavlov et al. (2006)). The system (A.1) is
- **convergent in a set** \( \mathcal{X} \subset \mathbb{R}^n \) if there exists a solution \( \mathfrak{T}(t) \) with the following properties:
  (i) \( \mathfrak{T}(t) \) is defined and bounded for all \( t \in \mathbb{R} \).
  (ii) \( \mathfrak{T}(t) \) is asymptotically stable in \( \mathcal{X} \).
- **uniformly convergent in** \( \mathcal{X} \) if it is convergent in \( \mathcal{X} \) and \( \mathfrak{T}(t) \) is uniformly asymptotically stable in \( \mathcal{X} \).

If system (A.1) is (uniformly) convergent in \( \mathcal{X} = \mathbb{R}^n \), then it is called globally (uniformly) convergent.

\( \mathfrak{T}(t) \) is a steady state solution defined for all \( t \in \mathbb{R} \).

Finally, we define the convergence properties for the controlled vector field

\[ \dot{x} = f(x, u) \quad (A.2) \]

with state \( x \in \mathbb{R}^n \) and input \( u \in \mathbb{R}^m \). Let \( PC_m \) denote a class of piecewise continuous function.

**Definition A.4.** (Pavlov et al. (2006)). System (A.2) is (uniformly) convergent in a subset \( \mathcal{X} \subset \mathbb{R}^n \) for a class of input \( \mathcal{N} \subset PC_m \) if it is (uniformly) convergent in \( \mathcal{X} \) for every input \( u \in \mathcal{N} \).

Thus, for any \( u(t), t \in \mathbb{R} \) we may define a steady state solution \( x_u(t), t \in \mathbb{R} \) parameterized by \( u \). Notice that these functions are defined over the whole real line and not just the semi infinite interval \((t_0, \infty)\) as in definitions (A.1) and (A.2).

The following theorem is important.

**Theorem 2.** (Pavlov et al. (2006)). Suppose system (A.2) with a given input \( u(t) \) is uniformly convergent in \( \mathcal{X} \). If the input \( u(t) \) is constant (periodic with period \( T \)), then the corresponding steady state solution \( \mathfrak{T}_u(t) \) is also constant (periodic with period \( T \)).

**REFERENCES**


5554