Input to Output Finite-Time Stabilization of Discrete-Time Linear Systems

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Abstract: Bounded–Input Bounded–Output (BIBO) stability is usually studied when only the input-output behavior of a dynamical system is of interest. The present paper investigates the analogous concept in the framework of Finite Time Stability (FTS), namely the Input–Output FTS (IO–FTS). FTS has been already investigated in several papers in terms of state boundedness, whereas in this work we deal with the characterization of the input-output behavior. A system is said to be input-output finite stable if, assigned a class of input signals and some boundaries in the output signal space, the output never exceeds such boundaries over a prespecified (finite) interval of time. This paper provides some sufficient conditions for the analysis of IO–FTS and for the design of a static state feedback controller guaranteeing IO–FTS of the closed loop system. The effectiveness of the proposed results is eventually illustrated by means of two numerical examples.

Keywords: Discrete-time linear systems, finite-time stability, input-output stability, Linear Matrix Inequalities.

1. INTRODUCTION

The concept of finite-time stability (FTS) dates back to the Sixties, when this idea was introduced in the control literature (Dorato, 1961). A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts; indeed a system can be FTS but not LAS, and vice versa. While LAS deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the state variables do not exceed a given threshold (for example to avoid saturations or the excitation of nonlinear dynamics) during the transients. Sufficient conditions for FTS and finite time stabilization (the corresponding design problems) have been provided in Amato et al. (2001, 2006); Shen (2008) in the context of linear systems, and in Zhao et al. (2008); Ambrosino et al. (2009); Amato et al. (2010) in the context of impulsive and hybrid systems. In this work we focus on the input-output behavior of discrete-time linear systems over a finite time interval.

In this work we focus on the input-output behavior of discrete-time linear systems over a finite time interval. Roughly speaking, and consistently with the definition of FTS given in Dorato (1961); Amato et al. (2001), a system is defined IO-FTS if, given a class of norm bounded input signals over a specified time interval, the outputs of the system do not exceed an assigned threshold during the same time interval.

In order to correctly frame our work in the current literature, we recall that a system is said to be IO \( L_p \)-stable (see Khalil (1992) Ch. 4; Desoer and Vidyasagar (1975); Abdallah et al. (2001)) if for any input of class \( L_p \), the system exhibits a corresponding output which belongs to the same class. The main differences between classic IO stability and IO-FTS are that the latter involves signals defined over a finite time interval, does not necessarily require the inputs and outputs to belong to the same class, and that quantitative bounds on both inputs and outputs must be specified. Therefore, IO stability and IO-FTS are independent concepts.

For the sake of completeness, it should be mentioned that a different concept of IO-FTS for nonlinear systems has been introduced in Hong et al. (2008) extending the definition of finite-time stability given in Bhat and Bernstein (2000) to nonautonomous systems. In these works, the authors focus on the Lyapunov stability analysis of nonlinear systems whose trajectories converge to an equilibrium point in finite time and on the characterization of the associated settling-time. According to this definition of FTS, in Hong et al. (2008) a different concept of finite-time input-output stability is introduced. In particular, the authors consider the case of nonautonomous system with...
a norm bounded input signal over the interval \([0, +\infty]\) and an initial condition \(x(0) = x_0\). The finite time input-output stability is related to the property of a system to have a norm bounded output whose bound, after a finite time interval, does not depend anymore on the initial state. Hence, we can conclude that the concept of IO-FTS introduced in this paper and the one in Hong et al. (2008) are different.

Finite-time stabilization of time-varying system is also tackled in Shaked and Suplin (2001). However, as for classic IO stability, their concept of IO-FTS does not give explicit bounds on input and output signals, and does not allow the input and output to belong to different classes.

The main contributions of this paper are two sufficient conditions which guarantee that a given system is IO-FTS over a specified time interval, for two different input classes. Furthermore the problem of IO finite-time stabilization via state feedback is also solved.

Our work is organized as follows. In Section 2 the problem we deal with is precisely stated, and some preliminary definitions are provided. Two sufficient conditions which guarantee IO-FTS of a given linear system are introduced in Section 3. These two conditions deal with two different class of input signals. Sufficient conditions to solve the IO finite-time stabilization problem via static state feedback are also provided. In Section 4 two examples illustrating the applicability of the proposed results are discussed. Some concluding remarks are eventually provided.

**Notation.** The symbol \(\mathbb{N}\) denotes the set of natural (i.e. non-negative) number. Given a set \(\Omega \subseteq \mathbb{N}\), a symmetric positive definite matrix \(R\) and a discrete-time signal \(\sigma(\cdot) : \Omega \mapsto \mathbb{R}^{m}\), the weighted norm \(\left( \sum_{k \in \Omega} \sigma(k)R\sigma(k) \right)^{1/2}\) will be denoted by \(\|\sigma\|_{R,\Omega}\).

**2. PROBLEM STATEMENT**

This section introduces the definition of IO-FTS for the class of discrete-time linear systems. First, we introduce some classes of discrete-time signal needed for the definition of IO-FTS.

**2.1 Input signals**

Let \(\mathcal{D}(\mathbb{R}^{n})\) denote the vector space of \(\mathbb{R}^{n}\)-valued sequences on the set \(\mathbb{N}\). The subspace \(\mathcal{L}_{p}\) of \(\mathcal{D}(\mathbb{R}^{n})\), with \(p < +\infty\), consists of all the sequences \(v\) such that

\[
\left( \sum_{k=0}^{\infty} \|v(k)\|^{p} \right)^{\frac{1}{p}} < +\infty \quad (1)
\]

The left-hand side is defined to be the norm in \(\mathcal{L}_{p}\) and it is denoted with \(\|v\|_{p}\).

In this paper, given a subset \(\Omega \subseteq \mathbb{N}\), we indicate with \(\mathcal{L}_{p,\Omega}\) the restriction of \(\mathcal{L}_{p}\) over the interval \(\Omega = \{0, 1, \ldots, N\}\). In particular, all the sequences \(v \in \mathcal{L}_{p,\Omega}\) verify

\[
\left( \sum_{\Omega} \|v(k)\|^{p} \right)^{\frac{1}{p}} < +\infty . \quad (2)
\]

Finally, we indicate with \(\mathcal{L}_{\infty}\) the subspace of \(\mathcal{D}(\mathbb{R}^{n})\) composed by all the sequences \(v\) such that

\[
\|v(k)\|^2 < +\infty \quad \forall k \in \mathbb{N}. \quad (3)
\]

According to the previous definitions, we indicate with \(\mathcal{L}_{\infty,\Omega}\) the restriction of \(\mathcal{L}_{\infty}\) over the set \(\Omega\).

**2.2 Definition of IO-FTS**

Let us consider a discrete-time linear system given by

\[
x(k+1) = A(k)x(k) + B(k)u(k) + G(k)w(k), \quad x(0) = 0 \quad (4a)
\]

\[
y(k) = C(k)x(k) + F(k)w(k), \quad (4b)
\]

where \(A(\cdot) : \mathbb{N} \mapsto \mathbb{R}^{n \times n}, G(\cdot) : \mathbb{N} \mapsto \mathbb{R}^{p \times n}, C(\cdot) : \mathbb{N}+ \mapsto \mathbb{R}^{m \times n}\), and \(F(\cdot) : \mathbb{N}+ \mapsto \mathbb{R}^{m \times r}\), are discrete-time matrix-valued functions.

**Definition 1.** (IO-FTS of Linear Discrete Time Systems). Given a positive integer \(N \in \mathbb{N}\), a class of input signals \(W\) defined over \(\Omega = \{0, 1, \ldots, N\}\), a positive definite discrete-time matrix-valued function \(Q(\cdot)\), system (4) is said to be IO-FTS with respect to \((W, Q(\cdot), N)\) if

\[
w(\cdot) \in W \Rightarrow y^{T}(k)Q(k)y(k) < 1 \quad \forall k \in \Omega.
\]

Note that in the above definition the set \(W\) not necessarily coincides with an \(\mathcal{L}_p\) space. More precisely, in this paper we shall consider two different cases, since different classes of signals may require different analysis techniques. From now on \(R\) will denote a positive definite symmetric matrix.

i) The set \(W\) coincides with the set of signals with bounded weighted \(\mathcal{L}_2\) norm over \(\Omega = \{0, 1, \ldots, N\}\), i.e.,

\[
W_2(N, R) := \{w(\cdot) \in \mathcal{L}_{2,\Omega} : \|w\|_{\Omega, R} \leq 1\}.
\]

ii) The set \(W\) coincides with the set of the uniformly bounded signals over \(\Omega = \{0, 1, \ldots, N\}\), i.e.,

\[
W_{\infty}(N, R) := \{w(\cdot) \in \mathcal{L}_{\infty,\Omega} : w^{T}(k)Rw(k) \leq 1, k \in \Omega\}.
\]

Although the definitions of \(W_2(N, R)\) and \(W_{\infty}(N, R)\) depend on the choice of \(N\) and \(R\), in the rest of the paper we will drop this dependency in order to simplify the notation.

In Section 3 ad hoc conditions to prove the IO-FTS for the two classes of input i) and ii) are provided. These conditions will be then exploited to deal with the following design problem.

**Problem 1.** (IO-FTS via State Feedback). Consider the linear system

\[
x(k+1) = A(k)x(k) + B(k)u(k) + G(k)w(k), \quad x(0) = 0 \quad (5a)
\]

\[
y(k) = C(k)x(k) + F(k)w(k) \quad (5b)
\]

where \(u(\cdot)\) is the control input and \(w(\cdot)\) is the disturbance. Given a positive scalar \(N\), a class of disturbances \(W\) defined over \(\Omega = \{0, 1, \ldots, N\}\) and a positive definite discrete-time matrix-valued function \(Q(\cdot)\), find a state feedback control law \(u(k) = K(k)x(k)\), where \(K(\cdot)\) is a discrete-time matrix-valued function, such that the closed loop system

\[
x(k+1) = A_{cl}(k)x(k) + G(k)w(k) \quad (6a)
\]

\[
y(k) = C(k)x(k) + F(k)w(k) \quad (6b)
\]

with \(A_{cl}(k) = (A(k) + B(k)K(k))\), is IO-FTS with respect to \((W, Q(\cdot), N)\). ▲
3. MAIN RESULTS

This section provides sufficient conditions for IO-FTS when the two input classes \(W_2\) and \(W_\infty\) are considered. Sufficient conditions are also provided to solve Problem 1.

3.1 \(W_2\) input signals

In order to provide a sufficient condition for IO-FTS of system (4) with respect to \((W_2, Q(\cdot), N)\) we first introduce the following lemma.

**Lemma 1.** Given system (4), a positive definite discrete-time matrix-valued function \(Q(\cdot)\) and a set \(\Omega = \{0, 1, \ldots, N\}\), the condition \(w(\cdot) \in W_2 \Rightarrow y_T(k)q(k)y(k) < 1\) with \(k \in \Omega\), is satisfied if there exists two positive definite matrix-valued functions \(P(\cdot)\) and \(T(\cdot)\) and a scalar \(\theta > 1\) such that

\[
A^T(h)P(h + 1)A(h) - P(h) + \quad + A^T(h)P(h + 1)G(h)T^{-1}(h)G^T(h)P(h + 1)A(h) < 0 \quad h \in \{0, 1, \ldots, k - 1\} \tag{7a}
\]

\[
G^T(h)P(h + 1)G(h) + T(h) < R \quad h \in \{0, 1, \ldots, k - 1\} \tag{7b}
\]

\[
P(k) \geq 2\theta C^T(k)Q(k)C(k) \tag{7c}
\]

\[
\theta R - R \geq 2\theta F(k)^T Q(k)F(k) \tag{7d}
\]

**Proof.** Let us consider the quadratic function

\[
V(h) = x^T(h)P(h)x(h),
\]

it follows that

\[
\Delta V(h) = V(h + 1) - V(h) = x^T(h + 1)P(h + 1)x(h + 1) - x^T(h)P(h)x(h)
\]

\[
= x^T(h)(A^T(h)P(h + 1)A(h) - P(h))x(h) + \quad + w^T(h)G^T(h)P(h + 1)G(h)w(h) + \quad + w^T(h)A^T(h)P(h + 1)G(h)w(h).
\]

From now on the dependence from the time will be omitted for brevity. Moreover the matrix \(P(h + 1)\) will be indicated with \(P(+)\). Thus, condition (7a) implies that for all \(h \in \{0, 1, \ldots, k - 1\}\)

\[
\Delta V(h) < w^TGT(+)P(+)Gw + w^TGT(+)Ax + \quad + x^TAT(+)P(+)Gw - x^TAT(+)GTG^T P(+)Ax.
\]

Let \(v_1 = \left( T^{1/2} - T^{-1/2}GT(+)Ax \right) \), then

\[
v_1^T v_1 = w^T T w + x^TAT(+)GTG^T P(+)Ax + \quad - x^TAT(+)Gw - w^T GT(+)Ax.
\]

It follows that for all \(h \in \{0, 1, \ldots, k - 1\}\)

\[
\Delta V(h) < w^TGT(+)Gw - v_1^T v_1 + w^T T w
\]

\[
< w^T(GTG^T P(+) + G)w.
\]

Taking into account condition (7b), we have that for all \(h \in \{0, 1, \ldots, k - 1\}\)

\[
\Delta V(h) < w^T Rw. \tag{8}
\]

Summing (8) over the set \(\{0, 1, \ldots, k - 1\}\) and taking into account that \(x(0) = 0\) and that \(u(\cdot)\) belongs to \(W_2\), we obtain

\[
V(k) = x(k)^T P(k)x(k) \leq \sum_{h=0}^{k-1} w^T(h)Rw(h) \tag{9}
\]

\[
= \|w\|^2_{[0,1,\ldots,k-1]} < \|w\|^2_{\Omega,R} \leq 1.
\]

Finally, let us consider the output weighted norm \(y(k)^T Q(k)y(k)\). According to the output equation (4b), we have

\[
y(k)^T Q(k)y(k) = (C(k)x(k) + F(k)w(k))^T Q(k)(C(k)x(k) + F(k)w(k)) \tag{10}
\]

Let \(y_2(k) = \left( Q(k)^{1/2} C(k)x(k) - Q(k)^{1/2} F(k)w(k) \right)\), then

\[
v_2^T v_2 = x^T C^T Q C x + w^T F^T F Q F w
\]

and hence

\[
x^T C^T Q F w + w^T F^T C F x = x^T C^T Q C x + w^T F^T Q F w - v_2^T v_2. \tag{11}
\]

Substituting (11) in (10) we have

\[
y^T Q y = 2x^T C^T Q C x + 2w^T F^T Q F w - v_2^T v_2 < 2 \left( x^T C^T Q C x + w^T F^T Q F w \right) \tag{12}
\]

Exploiting the conditions (7c)–(7d) it follows that

\[
y^T Q y < 2 \left( \frac{1}{2} \theta T P x + \frac{\theta - 1}{2\theta} w^T R w \right). \tag{13}
\]

and hence, taking into account that \(x^T(k) P x(k) < 1\), \(w(k) R u(k) < 1\), since \(w(\cdot) \in W_2\) and \(\theta > 1\), we can conclude that

\[
y(k)^T Q(k)y(k) < 1, \tag{14}
\]

for all \(k \in \Omega\).

In principle, in order to assess IO-FTS of system (4) wrt \((W_2, Q(\cdot), N)\), we should check that for all \(k \in \Omega\) the hypotheses of Lemma 1 are satisfied. In other words the feasibility of \(N\) sets of Linear Matrix Inequalities (LMIs) should be checked over the matrix-valued optimization functions \(P(\cdot) \in \mathbb{R}^{n \times n}\) with \(i = 0, \ldots, N\). However by means of Lemma 1 it is possible to prove in a straightforward way the following theorem, which requires to check the feasibility of a single set of LMIs over the matrix-valued optimization functions \(P(\cdot) \in \mathbb{R}^{n \times n}\).

**Theorem 2.** Assume that the following Difference LMIs

\[
\left( A^T(k)P(k + 1)A(k) - P(k) A^T(k)P(k + 1)G(k) \right) < 0 \quad k \in \{0, 1, \ldots, N - 1\} \tag{13a}
\]

\[
G^T(k)P(k + 1)G(k) + T(k) < R, \quad k \in \{0, 1, \ldots, N - 1\} \tag{13b}
\]

\[
P(k) \geq 20(k)^2 C^T(k)Q(k)C(k), \quad k \in \{0, 1, \ldots, N\} \tag{13c}
\]

\[
\theta(k) R - R \geq 20(k)^2 F(k)^T Q(k)F(k), \quad k \in \{0, 1, \ldots, N\} \tag{13d}
\]

\[
\theta(k) > 1, \quad k \in \{0, 1, \ldots, N\} \tag{13e}
\]

admit a positive definite discrete-time solution \(P(\cdot), T(\cdot)\) and \(\theta(\cdot)\), then system (4) is IO-FTS with respect to \((W_2, Q(\cdot), N)\).

**Proof.** Once it has been noticed that, by using Schur complements, inequality (13a) is equivalent to (7a), it is
straightforward to check that a matrix functions $P(\cdot)$ and a function $\theta(\cdot)$ satisfying (13) also satisfies (7).

It is straightforward to notice that Theorem 2 introduces some conservatism with respect to Lemma 1. This is due to the fact that in Theorem 2 we reduce the number of optimization variables.

It is now possible to state the following sufficient condition to solve Problem 1 for the $W_2$ input class.

**Theorem 3.** Given the class of disturbances $W_2$, Problem 1 is solvable if there exist positive definite discrete-time matrix-valued functions $P(\cdot)$ and $T(\cdot)$, a positive discrete-time scalar function $\sigma(\cdot)$, and a discrete-time matrix-valued function $L(\cdot)$ such that

$$
\begin{bmatrix}
-P & A_{cl}^T & -P \quad A_{cl}^T P(+) A_{cl} - P \quad -T \\

-G(k)^T A_{cl} & 0 & -T \\

0 & G(k)^T & -T
\end{bmatrix}
$$

is solvable if there exist positive definite discrete-time matrices $\Pi(\cdot)$ and $H(\cdot)$, with $\Pi(\cdot)$ of compatible dimensions. We will show that (15a) and (16) are equivalent to

$$
\begin{bmatrix}
-P & A_{cl}^T & -P \quad A_{cl}^T P(+) A_{cl} - P \quad -T \\

-G(k)^T A_{cl} & 0 & -T \\

0 & G(k)^T & -T
\end{bmatrix}
$$

Again using Schur complements, (16) is equivalent to

$$
\begin{bmatrix}
-P & A_{cl}^T & -P \quad A_{cl}^T P(+) A_{cl} - P \quad -T \\

-G(k)^T A_{cl} & 0 & -T \\

0 & G(k)^T & -T
\end{bmatrix}
$$

It is straightforward to check that, under the assumption that $P > 0$, (17a) is equivalent to (18a). Moreover, in the inverse that appears in (18b) we are only interested in the (1,1) term. Using the block matrix inversion formulas, this term is equal to $(A_{cl}^T P(+) A_{cl} - P)^{-1}$. As a consequence (17b) is equivalent to (18b). Therefore, we have shown that (15a) and (16) are equivalent.

Now, let us pre-multiply (16) by

$$
\begin{bmatrix}
\Pi & 0 & 0 \\
0 & I & 0 \\
0 & -G^T P(+) & I
\end{bmatrix}
$$

and post-multiply it by the transpose of (19). In this way, we obtain the following equivalent condition:

$$
\begin{bmatrix}
\Pi & 0 & 0 \\
0 & G^T & 0 \\
0 & -G^T P(+) G & -T
\end{bmatrix}
$$

Taking into account condition (15c) with $\theta > 1$, in order to verify inequality (20) it is sufficient that

$$
\begin{bmatrix}
\Pi & 0 & 0 \\
0 & G^T & 0 \\
0 & -2G^T C^T Q C G & -T
\end{bmatrix}
$$

Taking into account that $A_{cl}(k) = (A(k) + B(k)K(k))$, equation (14a) readily follows by letting $L(k) = K(k)\Pi(k)$.

Evaluating (14b) follows by (15b) applying the Schur complements.

Letting $\theta(\cdot) = (\cdot)^{-1}$, equations (14c) follows by (15c) pre and post-multiplied by $\Pi$ and applying the Schur complements. Finally (14d) and (14e) follow respectively by (15d) and (15e) dividing both the terms by $\theta(\cdot) > 0$.

### 3.2 $W_\infty$ Input signals

As for $W_2$ input signals, before introducing a sufficient condition for $IO$-FTS of system (4) with respect to $(W_\infty, Q(\cdot), N)$, we first state the following lemma.

**Lemma 2.** Given system (4), a positive definite discrete-time matrix-valued function $Q(\cdot)$ and a set $\Omega = \{0,1,\ldots,N\}$, the condition $w(\cdot) \in W_\infty \Rightarrow y^T(k)Q(k)y(k) < 1$ with $k \in \Omega$, is satisfied if there exists two positive definite matrix-valued functions $P(\cdot)$ and $T(\cdot)$ and a positive scalar $\theta > 1$ such that (7a)-(7b)-(7d) and

$$
P(k) \geq 2\theta^T C^T Q C G - T
$$

are satisfied, with $Q(k) = k Q(k)$.

**Proof.** By using the same arguments exploited in Lemma 1, it turns out that inequality (8) holds. Since $w(\cdot) \in W_\infty$, it follows that

$$
\Delta V(h) < 1.
$$

Summing (22) over the set $\{0,1,\ldots,k-1\}$ and taking into account that $x(0) = 0$ we obtain $x(k)^T P(k)x(k) < k$.
Now, let us consider equation (12); exploiting the terminal conditions (21)–(7d), it follows that
\[ y^T Q y < 2 \left( \frac{1}{2 \theta k} x^T P x + \theta - \frac{1}{2 \theta} u^T R u \right), \]
and hence, taking into account that \( x(k)^T P x(k) < k \), \( u(k)^T R u(k) < k \) because \( u(\cdot) \in \mathcal{W}_2 \) and \( \theta > 1 \), we can conclude that
\[ y(k)^T Q(k) y(k) < 1, \]
for all \( k \in \Omega \).

By using similar arguments as for Theorems 2 and 3, it is possible to prove the following results for the \( \mathcal{W}_\infty \) input class.

**Theorem 4.** Let \( \tilde{Q}(k) = kQ(k) \) and assume that the Difference LMIs (13a)–(13b)–(13d)–(13e) and
\[ P(k) \geq 2 \theta C^T(k) \tilde{Q}(k) C(k) \]
admit a positive definite discrete-time solution \( P(\cdot), T(\cdot) \) and \( \theta(\cdot) \) for all \( k \in \Omega = \{0,1,\ldots,N\} \), then system (4) is IO-FTS with respect to \( (\mathcal{W}_\infty, \tilde{Q}(\cdot), N) \).

**Theorem 5.** Given the class of disturbances \( \mathcal{W}_\infty \), Problem 1 is solvable if there exist positive definite discrete-time matrix-valued functions \( \Pi(\cdot) \) and \( \Theta(\cdot) \), and a discrete-time matrix-valued function \( L(\cdot) \) such that (14b)–(14d)–(14e) and
\[
\begin{pmatrix}
-\Pi(k) & 0 \\
\Psi(k)^T - \Pi(k+1) & 0 \\
0 & G(k)^T \Xi(k)
\end{pmatrix} < 0
\]
\[
\begin{pmatrix}
\Pi(k) \\
0 & \Pi(k) C(k)^T \\
C(k) \Pi(k) & 1/2 \sigma(k)(kQ(k))^{-1}
\end{pmatrix} \geq 0
\]
hold for all \( k \in \Omega = \{0,1,\ldots,N\} \), with
\[ \Psi(k) = \Pi(k) A(k)^T + L(k)^T B(k)^T \]
\[ \Xi(k) = -2k G(k)^T C^T(k+1) Q C(k+1) G(k) - T(k). \]
In this case a controller gain which solves Problem 1 for the input class \( \mathcal{W}_\infty \) is \( K(k) = L(k) \Pi(k)^{-1} \).

4. EXAMPLES

Two numerical examples are presented so as to illustrate the applicability of the proposed results.

**Example 1**

Consider the linear discrete system with two inputs and one output defined by
\[ A = \begin{pmatrix} 0.6 & -0.7 \\ 0.08 & 0.9 \end{pmatrix}, \ G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \]
Given \( R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( N = 10 \), Theorem 4 is exploited to evaluate which is the maximum value \( q_{\text{max}} \in \mathbb{R} \) such that system (24) is IO-FTS wrt \( (\mathcal{W}_\infty, q_{\text{max}}, N) \). Note that \( q_{\text{max}} \) gives an upper bound for the maximum value of \( y(k)^T y(k) \) in the time set \( \omega = \{0,1,\ldots,N\} \) when the input signal \( u(\cdot) \) belongs to \( \mathcal{W}_\infty(N, R) \). In particular this upper bound is equal to \( 1/q_{\text{max}} \).

Using the Matlab LMI Toolbox Qahinet et al. (1995) it turns out that \( q_{\text{max}} \) is equal to 1.35. Therefore it is possible to conclude that system (24) is IO-FTS wrt \( (\mathcal{W}_\infty, q = 1.25, N = 10) \), and the upper bound for \( y(k)^T y(k) \) is 0.7407.

**Example 2**

Consider the linear time-varying system with disturbances
\[ x(k+1) = \begin{pmatrix} 0.6 & -0.7 \\ 0.1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1/1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1/0 \end{pmatrix} w(k) \]
\[ y(k) = \begin{pmatrix} 0 & 1.5 + 0.5 \sin(k) \end{pmatrix} x(k) + 0.1 u(k) \]
with \( x(0) = 0 \) and let \( R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, q = 1, N = 15 \). If we set \( u(k) = 0 \) and we choose \( w(k) = 0.9 \in \mathcal{W}_2(N, R) \), it is straightforward to verify that system (24) is not IO-FTS wrt \( (\mathcal{W}_2, q, N) \). Hence, Theorem 3 is exploited to design a state feedback controller \( u(k) = K(k) x(k) \), with \( K(k) \in \mathbb{R}^{1 \times 2} \), which guarantees the IO-FTS of the closed-loop system. By using the Matlab LMI Toolbox, it is possible to find three piecewise affine matrix-valued functions \( \Pi(k) \), \( T(k) \) and \( L(k) \) which verify the conditions (14). In Fig. 1 the two components of the gain \( K(k) \) are shown.

**CONCLUSIONS**

The results presented in this work are useful to deal with the input-output behavior of dynamical discrete-time linear systems, when the focus is on the boundedness of the output signal over a finite interval of time, as opposed to BIBO stability, which considers infinite time intervals. Both the analysis and state–feedback synthesis problems have been tackled, providing sufficient conditions which can be solved through efficient off-the-shelf numerical optimization tools. The applicability of the devised results has been illustrated by means of two examples. Given a specific input class, IO-FTS permits to provide a quantitative estimation of the maximum output value achieved in finite time.

**REFERENCES**


