New Special Cases of General Active Change Detection and Control Problem

M. Šimandl ∗ J. Široký ∗∗ I. Punčochář ∗∗

* Department of Cybernetics and Research Centre DAR, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic (e-mails: simandl@kky.zcu.cz (M. Šimandl), ivop@kky.zcu.cz (I. Punčochář)).

** Department of Cybernetics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic (e-mail: jan.siroy@rcware.eu)

Abstract: The paper deals with a general formulation of active change detection and control problem. A thorough systematic analysis of the general formulation shows that it is possible to obtain some theoretically and practically significant special cases concerning the active change detection and control. All special cases are specified within the general formulation, their optimal general solutions are provided and key aspects are highlighted by means of theoretical examples. Since the special cases are treated within the common framework provided by the general formulation, the examination of relationships among them is considerably simplified.

Keywords: Stochastic systems, optimal control, change detection, optimal experiment design.

1. INTRODUCTION

Since enhanced performance and safety are of great importance in many applications, change or fault detection methods have been intensively studied for several years, and many important results have already been brought out. The area of so called active fault detection has recently received a great deal of attention because active fault detection makes it possible to improve the quality of detection by using a wisely designed input signal.

The idea of active approach has originally emerged in parameter estimation (Mehra, 1974). The first attempt to formulate and solve the active fault detection problem for multiple linear Gaussian models can be found in (Zhang, 1989). A detector based on the sequential probability ratio test was used to identify a change, and an auxiliary input signal was designed to speed up detection. This idea was further extended, and a generalized formulation of active fault detection was proposed in (Kerestecioglu, 1993). Active fault detection for systems subject to bounded energy noises was introduced in (Nikoukhah et al., 2000) and thoroughly summarized in (Campbell and Nikoukhah, 2004). In contrast to stochastic approaches the use of a proper auxiliary input signal in the case of systems subject to bounded energy noises allows for error-free decisions.

So far mentioned approaches assume that there are some inputs available for injecting an auxiliary input signal. Nevertheless, all accessible inputs of a system are usually used for control. In such a situation it is possible to inject the auxiliary input signal into the system only during wisely chosen test periods or combine the auxiliary input with a control signal to obtain a reasonable compromise between detection and control quality. Note that these contradictory aims also exist in the dual control (Filatov and Unbehauen, 2004). At the beginning, only relationships and limitations of integrated fault detection and control were studied (Stoustrup et al., 1997; Jacques et al., 2003). Simultaneous fault detection and control was tackled as a mixed $H_2/H_\infty$ optimization problem in (Khosrowjerdi et al., 2004). Due to deterministic setup and robust control design technique, the performance of the detector is independent of the controller as shown there. A similar approach to simultaneous design was considered in (Niemann, 2006), where the same setup for active fault detection to both open loop systems and closed loop systems was applied. However the controller can significantly influence detection quality in the stochastic framework as follows from fundamental results concerning dual control (Feldbaum, 1960–61). A method that allows to incorporate control objectives into active fault detection in stochastic framework was proposed in (Blackmore and Williams, 2006), where control objectives were expressed in terms of equality and inequality linear constraints on input and expected system state trajectories.

A general formulation of active fault detection and control for stochastic systems was introduced in (Šimandl and Punčochář, 2009). This general formulation includes several specific design problems that can be obtained as its special cases by choosing a particular value of a design parameter and by partially fixing the structure and parameters of the designed system. Although main attention has mainly been paid to three special cases so far, a closer look at the general formulation indicates that it covers other theoretically and practically interesting special cases. Therefore, the goal of this paper is to perform...
a deeper theoretical analysis of the general formulation and specify all possible special cases together with their general solutions. The paper is theoretically oriented and focuses on providing a framework for studying several special cases and their mutual relationships in a consistent way. Thus, it neither discusses nor provides particular design algorithms for actual implementation of these special cases.

The structure of the paper is the following. The general formulation of active change detection and control is adopted in Section 2, and the optimal solution is presented there, as well. Since the level of generality is too high, Section 3 introduces a special form of the criterion and possible constraints on the structure of design system are discussed. Section 4 presents all individual special cases where the new ones are presented in more detail. At the end of this section, all special cases are summarized, and a note on computational aspects and approximate techniques is made. Section 5 concludes the paper.

2. PROBLEM FORMULATION

This section introduces the general formulation of the optimal active fault detection and control problem that has been proposed in (ˇSimandl and Punˇcoch´ar, 2009).

2.1 Observed and controlled system

Consider an observed and controlled system $S_1$ described at each time step $k \in T = \{0, \ldots, F\}$ by the nonlinear stochastic discrete-time state space model

$$
\begin{align*}
    x_{k+1} &= f_k (x_k, \mu_k, u_k, w_k), \\
    \mu_{k+1} &= g_k (x_k, \mu_k, u_k, e_k), \\
    y_k &= h_k (x_k, \mu_k, v_k),
\end{align*}
$$

where $F < \infty$ denotes a finite horizon, $f_k (x_k, \mu_k, u_k, w_k)$, $g_k (x_k, \mu_k, u_k, e_k)$ and $h_k (x_k, \mu_k, v_k)$ are known generally nonlinear vector functions. The input and output of the system are denoted as $u_k \in \mathcal{U}_0 \subseteq \mathbb{R}^n_u$ and $y_k \in \mathbb{R}^n_y$, respectively. The discrete or continuous subset $\mathcal{U}_0$ specifies admissible values of the input $u_k$. Such a restriction is required not only because there are always constraints imposed by the system but also because the optimal unconstrained input would otherwise tend to infinity in some special cases. The unmeasured state $x_k = [x_k^T, \mu_k^T]^T$ of the system is composed of the variables $x_k \in \mathbb{R}^{n_x}$ and $\mu_k \in \mathcal{M} \subseteq \mathbb{R}^{n_\mu}$. The variable $x_k$ represents the basic part of the state $x_k$, which is usually driven by the input $u_k$ to a desirable value or region of the state space. The variable $\mu_k$ carries information about changes or faults in the system. It can be a vector representing a signal or a scalar variable indexing a mode of behavior of the system. The initial state $x_0$ is described by a known probability density function (pdf) $p(x_0) = p(x_0|\mu_0)$. The white noise sequences $\{w_k\}$, $\{e_k\}$ and $\{v_k\}$ are described by known pdf’s $p(w_k)$, $p(e_k)$ and $p(v_k)$, respectively. The initial state $x_0$ and the noise sequences $\{w_k\}$, $\{e_k\}$, $\{v_k\}$ are mutually independent. Note that the function $g_k (x_k, \mu_k, u_k, e_k)$ represents a stochastic model of changes or faults. This function together with the function $f_k (x_k, \mu_k, u_k, w_k)$ can be used for predicting the future behavior of the system.

2.2 Detector and controller

Within the general formulation the aim is to design a causal deterministic system $S_2$ that generates a decision $d_k$ and an input $u_k$ based on complete available information at each time step $k \in T$. Therefore, the system $S_2$ can be described at each time step $k \in T$ by the relation

$$
    d_k = p_k (u_k^T),
$$

where $d_k \in \mathcal{M}$ is a decision providing information about the variable $\mu_k$. The complete available information, which has been received up to the time step $k$, is stored in the information vector $I_k^0 = [y_0^T, u_0^T, d_0^T, v_0, \ldots, d_{k-1}, y_k, u_k]^T$. The notation $v_k^j$ is used for expressing the time sequence of the variables or functions $z_k$ from the time step $i$ up to the time step $j$. If it happens in an expression that $i$ is greater that $j$, then the sequence $v_k^j$ is empty, and the corresponding variable or function is simply left out from the expression. According to this rule, the information vector for the time step $k = 0$ is defined as $I_0^0 = I_0 = y_0$.

2.3 General criterion

Similarly to the optimal stochastic control (Bar-Shalom, 1981), a suitable criterion is needed for evaluating the behavior of the closed loop system. Design of the optimal system $S_2$ is then based on the minimization of such a criterion. Since the general formulation should cover as many situations as possible, the criterion is a function of all variables of interest (i.e., $x_k^F, \mu_k^F, u_k^F, d_k^F$). An additive criterion is considered in the following form

$$
    J (\rho_k^F) = \mathbb{E} \left\{ \sum_{k=0}^F L_k (x_k, \mu_k, u_k, d_k) \right\},
$$

where $\mathbb{E}\{\cdot\}$ is the expectation operator with respect to all random variables. The cost function $L_k (x_k, \mu_k, u_k, d_k)$ is a non-negative real-valued function chosen by a designer. To conclude this section note that the optimal system $S_2$ described by the sequence of the functions $\rho_k^* = [\rho_k^0, \rho_k^1, \ldots, \rho_k^F]$ is determined by minimization of the criterion (3) subject to the constraints represented by the system $S_1$ and system $S_2$.

2.4 General solution

Before the optimal solution to the stated optimization problem can be provided, it is necessary to stress how important role the availability of information plays in dynamic optimization problems. When a dynamic optimization problem is formulated in the stochastic framework, there are three basic information processing strategies (IPSs), namely open loop, open loop feedback and closed loop IPS, that impose different assumptions on information availability at each time step of the optimization horizon (Bar-Shalom and Tse, 1974; Šimandl and Punˇcoch´ar, 2009). Only the closed loop (CL) IPS is considered in this paper because it is superior to other two IPSs as demonstrated e.g. in the context of detection in (Šimandl and Herejt, 2003). The usage of the CL IPS
results into dynamic programming where the optimization is performed backward in time (Bertsekas, 1995).

The optimal system $S_2$ is determined by solving the following backward recursive equation

$$V_k^* (I_0^k) = \min_{d_k \in M, u_k \in C_k} E \{ L_k (x_k, \mu_k, u_k, d_k) + V_{k+1}^* (I_0^k) | I_0^k, u_k, d_k \}$$

where $k = F, F-1, \ldots, 0$, $E[\cdot]$ stands for the conditional expectation operator and the cost-to-go function, sometimes also called Bellman function, $V_0^*$ is an estimate of the minimum cost that will be incurred from the time step $k$ up to the final time step $F$ given the information vector $I_0^k$. The initial condition for the backward recursive equation (4) is $V_{F+1}^* = 0$, and it is easy to show that the optimal value of the criterion (3) can be computed as $J^* = J (\rho^*_k)$.

The optimal decision $d_k^*$ and the optimal input $u_k^*$ are given as

$$d_k^* = \rho_k^* (I_0^k) = \arg \min_{d_k \in M} E \{ L_k (x_k, \mu_k, u_k, d_k) + V_{k+1}^* (I_0^k) | I_0^k, u_k, d_k \}$$

where $\rho_k^* (I_0^k)$ is a vector function describing the optimal system $S_2$. The backward recursive equation (4) and the optimal system $S_2$ described by (5) represent the general solution to the given problem. Note that a suitable nonlinear estimation method has to be used for obtaining the probability density functions $p(x_k|I_0^k, u_k, d_k)$ and $p(y_{k+1}|I_0^k, u_k, d_k)$ that are needed for evaluating the conditional expectations in the backward recursive equation. Note that based on the properties of the considered system, the following two identities hold (Simandl and Puncochář, 2009)

$$p(x_k|I_0^k, u_k, d_k) = p(x_k|y_0^{k-1}, u_0^{k-1})$$

$$p(y_{k+1}|I_0^k, u_k, d_k) = p(y_{k+1}|y_0^{k-1}, u_0^{k-1})$$

The presented general formulation of active change detection and control has been analyzed, and some additional assumptions have been made to allow examining essential special cases covered by this general formulation. The specifications of these essential special cases together with the corresponding optimal solutions will be given in the following sections, and the special cases will be also discussed from practical applicability point of view.

3. SPECIFICATION OF COST FUNCTION, DETECTOR AND CONTROLLER

On the one hand the generality of the presented formulation is appealing from theoretical point of view, but on the other, it hinders its applicability in particular cases. Therefore, some additional assumptions concerning the structure of the designed system $S_2$ and the cost function $L_k (x_k, \mu_k, u_k, d_k)$, that make it possible to analyze the general problem to a deeper extent and define several special cases, are given in this section.

3.1 Structural constraints

The minimization in (4) and (5) subjected to the system (1) is performed over all inputs and decisions belonging to the Cartesian product $\mathcal{U}_k = \mathcal{M} \times \mathcal{C}_k$. As it was mentioned in the introduction, the set $\mathcal{U}_k$ can express physical limits imposed on the inputs. However, besides these usual constraints there can be some other constraints that further restrict the minimization to a subset of the set $\mathcal{U}_k$. Such constraints typically arise when a part of the designed system is given in advance. The equality constraints represented by a given controller or a given detector are considered in this paper, and the following three situations will be treated

- no additional constraints,
- given detector $d_k = \sigma_k (I_0^k)$,
- given controller $u_k = \gamma_k (I_0^k, d_k)$.

A given controller $\gamma_k (I_0^k, d_k)$ is a function of complete available information $I_0^k$ and the decision $d_k$ at the time step $k$. A typical example is a fault tolerant controller in which based on the decision a control law is selected from a set of predefined control laws. A given detector $\sigma_k (I_0^k)$ is a function of complete available information at the time step $k$. Note that the input $u_k$ is not utilized because without the output $y_{k+1}$, which is not known at the time step $k$, the input $u_k$ does not bring any additional information for state estimation.

3.2 Cost function

Since detection and control objectives should be considered within the general formulation, the cost function $L_k (x_k, \mu_k, u_k, d_k)$ has to express them in a reasonable way. The detection and control objectives can be formulated mathematically using two separate non-negative real-valued cost functions $L^d_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n\mu} \to \mathbb{R}^{+}$ and $L^c_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n\mu} \to \mathbb{R}^{++}$, where $L^d_k (\mu_k, d_k)$ expresses the change detection objective, $L^c_k (x_k, u_k)$ expresses the control objective and $\mathbb{R}^{++}$ denotes the set of non-negative real numbers. In most cases, these two objectives represent opposing aims because the optimal control signal differs from the optimal change detection signal. Therefore, this problem can be regarded as a typical multiobjective optimization problem for which several solutions have been devised (Collette and Siarry, 2003). In this paper, a trade-off between the objectives is expressed as the convex combination of the cost functions $L^d_k (\mu_k, d_k)$ and $L^c_k (x_k, u_k)$

$$L_k (x_k, u_k, d_k) = \alpha_k L^d_k (\mu_k, d_k) + (1 - \alpha_k) L^c_k (x_k, u_k)$$

where the parameter $\alpha_k$ belonging to the closed interval $[0, 1]$ determines the desired importance of these two objectives. Note that in order to be a meaningful cost function, the function $L^d_k (\mu_k, d_k)$ should satisfy the inequality $L^d_k (\mu_k, d_k) \leq L^d_k (\mu_k, \bar{d}_k)$ for all $\mu_k \in \mathcal{M}$, $d_k \in \mathcal{M}$, $d_k \neq \bar{d}_k$ at each time step $k \in T$, and the strict inequality has to hold at least at one time step. Based on the value of the parameter $\alpha_k$, three following situations can be distinguished

- $\alpha_k = 0$ only control objective,
- $\alpha_k \in (0, 1)$ detection and control objectives,
- $\alpha_k = 1$ only detection objective.

In all three situations, it is considered that the parameter $\alpha_k$ satisfies the particular condition at each time step $k \in T$. In the situation with $\alpha_k \in (0, 1)$, the value
of the parameter $\alpha_k$ determines whether more emphasis is put on the detection objective or the control objective.

4. SPECIAL CASES

The discussion concerning the structural constraints and the cost function given in the previous section serves as a basis for deriving individual special cases. Classification according to three different structural constraints discussed in Subsection 3.1 is covered within the first three subsections of this section. In each of these subsections, three special cases differing in the value of the parameter $\alpha_k$ are considered. To highlight some interesting properties of individual special cases, three theoretical examples are presented at the end of each subsection. These theoretical examples share the following specification. Design problems are considered on two step horizon $(F = 1)$, the set $\mathcal{M} = \{1, 2, \ldots, M\}$ is discrete, the cost function $L^d_k(\mu_k, d_k)$ is the zero-one function $(L^d_k(\mu_k, d_k) = 0$ if $d_k = \mu_k$, $L^d_k(\mu_k, d_k) = 1$ otherwise) and the cost function $L^u_k(x_k, u_k)$ is of the quadratic form $L^u_k(x_k, u_k) = x_k^TQ_kx_k + u_k^TR_ku_k$, with $Q_k = Q_k^T \geq 0$, $R_k = R_k^T > 0$.

4.1 No additional constraints

The goal is to design the system $S_2$ using the cost function (8) without any additional structural constraints on the designed system. Based on the value of the parameter $\alpha_k$, it is possible to distinguish three special cases (Simandl and Punčochář, 2009)

- $\alpha_k = 0$ active controller,
- $\alpha_k \in (0, 1)$ active detector and active controller,
- $\alpha_k = 1$ active detector.

The first special case ($\alpha_k = 0$) exactly coincides with the stochastic optimal control problem (Filatov and Unbehauen, 2004), and detection problem is not actually considered at all.

In the second special case ($\alpha_k \in (0, 1)$), the aim is to design the active detector and controller without any additional constraints. The backward recursive equation (4) takes the form

\[ V^*_k(y^k_0, u^{-1}_0) = \min_{d_k} \mathbb{E} \left\{ \alpha_kL^d_k(\mu_k, d_k) \left| y^k_0, u^{-1}_0, d_k \right. \right\} \]

\[ + \min_{u_k} \mathbb{E} \left\{ (1 - \alpha_k)L^u_k(x_k, u_k) \left| y^k_0, u^{-1}_0 \right. \right\} \]

\[ + V^*_{k+1}(y^{k+1}_0, u^*_0) \left| y^k_0, u^{-1}_0 \right. \}. \] (9)

This backward recursive equation demonstrates how the problems of change detection and control are interconnected. The optimal decision $d^*_k$ is determined by minimizing the conditional expected value of the cost function $L^d_k(\mu_k, d_k)$

\[ d^*_k = \arg \min_{d_k, u_k} \mathbb{E} \left\{ \alpha_kL^d_k(\mu_k, d_k) \left| y^k_0, u^{-1}_0, d_k \right. \right\} \] (10)

and the optimal input $u^*_k$ is given as the minimum conditional expected value of the weighted sum of the instantaneous cost $L^u_k(x_k, u_k)$ and the future cost $V^*_{k+1}(y^{k+1}_0, u^*_0)$

\[ u^*_k = \arg \min_{u_k} \mathbb{E} \left\{ (1 - \alpha_k)L^u_k(x_k, u_k) \left| y^k_0, u^{-1}_0 \right. \right\} \]

\[ + V^*_{k+1}(y^{k+1}_0, u^*_0) \left| y^k_0, u^{-1}_0 \right. \}. \] (11)

Since the expected future cost $V^*_{k+1}(y^{k+1}_0, u^*_0)$ consists of the detection and control costs, the optimal input $u^*_k$ represents a trade-off between exciting and controlling the system $S_1$.

The solution to the third special case can directly be derived from (9), (10), and (11) by setting $\alpha_k = 1$. In this special case, the backward recursive equation has the form

\[ V^*_k(y^k_0, u^{-1}_0) = \min_{d_k} \mathbb{E} \left\{ L^d_k(\mu_k, d_k) \left| y^k_0, u^{-1}_0, d_k \right. \right\} \]

\[ + \min_{u_k} \mathbb{E} \left\{ V^*_{k+1}(y^{k+1}_0, u^*_0) \left| y^k_0, u^{-1}_0 \right. \right\}, \] (12)

and the optimal decision $d^*_k$ and the optimal input $u^*_k$ are given as

\[ d^*_k = \arg \min_{d_k} \mathbb{E} \left\{ L^d_k(\mu_k, d_k) \left| y^k_0, u^{-1}_0, d_k \right. \right\}, \] (13)

\[ u^*_k = \arg \min_{u_k} \mathbb{E} \left\{ V^*_{k+1}(y^{k+1}_0, u^*_0) \left| y^k_0, u^{-1}_0 \right. \right\}, \] (14)

respectively. The optimal decision $d^*_k$ is determined in the same way as the decision in the case of the active detector and controller ($\alpha_k \in (0, 1)$), but the optimal input $u^*_k$ minimizes the expected future costs caused by incorrect decisions solely.

**Theoretical example 1**

This theoretical example provides a deeper insight into the special case with $\alpha_k \in (0, 1)$. At the time step $k = 1$, the optimal decision $d^*_1$ and the optimal input $u^*_1$ are chosen as

\[ d^*_1 = \arg \min_{d_k} \mathbb{E} \left\{ 1 - P(\mu_1 = d_1 | y^1_0, u^0_0) \right\}, \]

\[ u^*_1 = \arg \min_{u_k} u_k^T R_1 u_k. \]

Note that the optimal decision $d^*_1$ minimizes the probability of making a wrong decision and is also optimal in the maximum a posteriori sense. At the time step $k = 0$, the optimal decision $d^*_0$ and the optimal input $u^*_0$ are given as

\[ d^*_0 = \arg \min_{d_k} \mathbb{E} \left\{ 1 - P(\mu_0 = d_0 | y_0) \right\}, \]

\[ u^*_0 = \arg \min_{u_k} \mathbb{E} \left\{ (1 - \alpha_1)u_k^T R_0 u_k \right. \]

\[ + (1 - \alpha_1) E \{ x_k^T Q_1 x_1 | y_0, u_0 \} \]

\[ + \alpha_1 \mathbb{E} \left\{ (1 - \alpha_1)u_k^T R_0 u_k \right. \left. \} \}. \] (18)

The optimal input $u^*_0$ minimizes the weighted sum of the instantaneous cost of the input $u_0$, the expected cost connected with the future state $x_1$ and the expected probability that the optimal decision $d^*_1$ will be incorrect.

4.2 Given detector $d_k = \sigma_k(I^k)$

In some situations, a detector is given in advance, and it has to be taken into account while designing the part of the system $S_2$ that is responsible for generating the input $u_k$. Such a situation typically arises when a suboptimal detector is used because the optimal one is too complex to be implemented, or when the detector has been hardwired into the system. A particular example can be the detection of sensor failure using hardware redundancy, where a physical quantity is measured by several independent sensors, and a suitably designed system input can provide information on the state of each sensor. For this purpose, one needs to consider the future state $x_1$ and the expected probability that the optimal decision $d^*_1$ will be incorrect.

Again, three special cases can be distinguished according to the value of the parameter $\alpha_k$.
\( \alpha_k = 0 \) active controller and passive detector,
\( \alpha_k \in (0, 1) \) active controller for given detector,
\( \alpha_k = 1 \) active controller for given detector.

The first special case (\( \alpha_k = 0 \)) leads to the stochastic optimal control problem. Since the cost function does not express detection objective, the inputs are chosen to satisfy control objectives exclusively. As a result the designed active controller is accompanied by a given detector that can be regarded as a passive one.

In the second special case (\( \alpha_k \in (0,1) \)), the backward recursive equation takes the form
\[
V_k^*(I_0^k) = E\left\{ \alpha_k I_k^2(\mu_k, \sigma_k(I_0^k)) \mid I_0^k \right\} + \min_{u_k \in U_k} E\left\{ V_{k+1}^*(I_{k+1}^0) \mid I_0^k, u_k \right\},
\]
and the decision \( d_k \) and the optimal input \( u_k^* \) are given as
\[
d_k = \sigma_k(I_0^k), \tag{20}
\]
\[
u_k^* = \arg\min_{u_k \in U_k} E\left\{ V_{k+1}^*(I_{k+1}^0) \mid I_0^k, u_k \right\}. \tag{21}
\]

where \( I_{k+1}^0 = I_0^k, y_{k+1}, u_k, \sigma_k(I_0^k) \). Since the cost-to-go function \( V_{k+1}^*() \) also includes the expected cost related to the future wrong decisions, the input \( u_k \) influences not only the future control cost but also the detection cost.

For the third special case (\( \alpha_k = 1 \)), the backward recursive equation can be obtained from (19) in the form
\[
V_k^*(I_0^k) = E\left\{ I_k^2(\mu_k, \sigma_k(I_0^k)) \mid I_0^k \right\} + \min_{u_k \in U_k} E\left\{ V_{k+1}^*(I_{k+1}^0) \mid I_0^k, u_k \right\}. \tag{22}
\]
The decision \( d_k \) and the optimal input \( u_k^* \) are given as
\[
d_k = \sigma_k(I_0^k), \tag{23}
\]
\[
u_k^* = \arg\min_{u_k \in U_k} E\left\{ V_{k+1}^*(I_{k+1}^0) \mid I_0^k, u_k \right\}. \tag{24}
\]
The optimal input \( u_k^* \) is designed such that the expected future decision cost is minimized.

**Theoretical example 2** The optimal active generator for a given detector (\( \alpha_k = 1 \)) is designed. At the time step \( k = 1 \), the decision \( d_1 \) and the optimal input \( u_1^* \) are given as follows
\[
d_1 = \sigma_1(I_0^1), \tag{25}
\]
\[
u_1^* = \arg\min_{u_1 \in U_1} E\left\{ V_2^*(I_0^2, u_1) \right\}. \tag{26}
\]

Since the initial condition \( V_2^* \) is the zero function, the optimal input \( u_1^* \) can be chosen arbitrarily. At the time step \( k = 0 \), the decision \( d_0 \) and the optimal input \( u_0^* \) are given by the following equations
\[
d_0 = \sigma_0(y_0), \tag{27}
\]
\[
u_0^* = \arg\min_{u_0 \in U_0} \left\{ \left[ 1 - E\left\{ P(\mu_1 = \sigma_1(I_0^1) \mid y_0^0, u_0) \right\} \right] y_0, u_0 \right\}. \tag{28}
\]

It can clearly be seen that the optimal input \( u_0^* \) minimizes the expected probability that the decision \( d_1 \), generated by the given detector \( \sigma_1(I_0^1) \), will be incorrect.

### 4.3 Given controller \( u_k = \gamma_k(I_0^k, d_k) \)

When a controller is given, the goal is to design the part of the system \( S_2 \) that generates decisions. Contrary to the given detector where the decision \( d_k \) is a function of \( I_0^k \) in the case of a given controller the input \( u_k \) is generally a function of \( I_0^k \) and also the decision \( d_k \). A multimodel controller consisting of several controllers that are switched according to the decision \( d_k \) can be considered as a typical example of such a situation.

Again, three following special cases are considered

- \( \alpha_k = 0 \) active generator for given controller,
- \( \alpha_k \in (0, 1) \) active detector for given controller,
- \( \alpha_k = 1 \) active detector for given generator.

The optimal solution for all three special cases can be derived by substituting the particular value of the parameter \( \alpha_k \) into the backward recursive equation
\[
V_k^*(I_0^k) = \min_{d_k \in \Gamma_k} E\left\{ \alpha_k L_k^2(\mu_k, d_k) \right\} + (1 - \alpha_k) L_k^2(\mu_k, d_k) \tag{29}
\]

The optimal decision \( d_k^* \) and the input \( u_k \) are given as
\[
d_k^* = \arg\min_{d_k \in \Gamma_k} E\left\{ \alpha_k L_k^2(\mu_k, d_k) \right\} + (1 - \alpha_k) L_k^2(\mu_k, d_k)
\]
\[
u_k^* = \min_{k \in \Gamma_k} E\left\{ V_{k+1}^*(I_{k+1}^0) \mid I_0^k, d_k \right\}. \tag{30}
\]

The optimal decision \( d_k^* \) and the input \( u_k^* \) are given as
\[
d_k^* = \arg\min_{d_k \in \Gamma_k} E\left\{ \alpha_k L_k^2(\mu_k, d_k) \right\} + (1 - \alpha_k) L_k^2(\mu_k, d_k)
\]
\[
u_k = \gamma_k(I_0^k, d_k^*). \tag{31}
\]

where \( \Gamma_{k+1} = \left\{ I_0^k, y_{k+1}, d_k, \gamma_k(I_0^k, d_k) \right\} \). With no additional constraints, the optimal decision \( d_k^* \) is given by (10) and minimizes just the cost at the current time step. However, when the input \( u_k \) depends on the decision \( d_k \), the future cost plays a role in determining the optimal decision \( d_k^* \).

**Theoretical example 3** The active generator for a given controller (\( \alpha_k = 0 \)) is presented to illustrate the situation in which only the control objective is considered. The optimal decision \( d_k \) and the input \( u_1 \) at the time step \( k = 1 \) are given as
\[
d_k^* = \arg\min_{d_k \in \Gamma_k} \gamma_1(I_0^1, d_k)^T R_1 \gamma_1(I_0^1, d_k), \tag{32}
\]
\[
u_k = \gamma_1(I_0^1, d_k^*). \tag{33}
\]

At the time step \( k = 0 \), the optimal decision \( d_0^* \) and the input \( u_0 \) are given as follows
\[
d_0^* = \arg\min_{d_0 \in \Gamma_0} \left\{ \gamma_0(y_0, d_0)^T R_0 \gamma_0(y_0, d_0) \right\}
\]
\[
+ E\left\{ x_1^T Q_1 x_1 \mid y_0, d_0 \right\}
\]
\[
+ E\left\{ \min_{d_0 \in \Gamma_0} \gamma_1(I_0^1, d_1)^T R_1 \gamma_1(I_0^1, d_1) \mid y_0, d_0 \right\}, \tag{34}
\]
\[
u_0 = \gamma_0(y_0, d_0^*). \tag{35}
\]

It can clearly be seen that the optimal decision \( d_0^* \) minimizes the current and future control costs through the given controller. Therefore, in this special case, the optimal decisions do not necessarily match the decisions that are optimal from estimation point of view (e.g. decisions in the maximum a posteriori sense) as demonstrated in Punčochář et al. (2010) by means of a numerical example. This observation may seem counterintuitive at first, but it just reflects the fact that only the control objective is considered in this special case.
4.4 Discussion

In this section, the additional assumptions on the structure of the designed system $S_2$, and cost function were employed for deriving nine special cases included in the general formulation of active change detection and control. All special cases are transparently summarized in Tab. 1. Some of them (e.g. AC and AC PD) represent well known change detection or control problems. For example the AC is known as the dual adaptive control problem. The other special cases represent design problems that have been treated in a limited context (AGGD and AGGC) or have not been considered at all (ACGD and ADGC).

<table>
<thead>
<tr>
<th>Special Case</th>
<th>$\alpha_k = 0$</th>
<th>$\alpha_k \in (0,1)$</th>
<th>$\alpha_k = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>no constraints</td>
<td>AC</td>
<td>ADAC</td>
<td>AD</td>
</tr>
<tr>
<td>given detector</td>
<td>AC PD</td>
<td>ACGD</td>
<td>AGGD</td>
</tr>
<tr>
<td>given controller</td>
<td>ACGC</td>
<td>ADGC</td>
<td>ADGG</td>
</tr>
<tr>
<td>AC</td>
<td>Active Controller</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ADAC</td>
<td>Active Detector and Active Controller</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AD</td>
<td>Active Detector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AC PD</td>
<td>Active Controller and Passive Detector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ACGD</td>
<td>Active Controller for Given Detector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AGGD</td>
<td>Active Generator for Given Detector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AGGC</td>
<td>Active Generator for Given Controller</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ADGC</td>
<td>Active Detector for Given Controller</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ADGG</td>
<td>Active Detector for Given Generator</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The optimal solutions to all special cases involve the conditional expectation and minimization operators that make the solutions computationally intractable, especially in the case of the discrete set $\mathcal{M}$. Therefore, some approximate techniques have to be employed to obtain viable design procedures, see e.g. (Šimandl et al., 2005; Blackmore and Williams, 2006; Punčochář and Šimandl, 2008) for particular techniques.

5. CONCLUSION

The increasing complexity of modern control systems requires a rigorous approach to problem of change detection and control because a deep understanding of possible interactions between a detector and a controller is vital for successful design. The paper dealt with the simultaneous change detection and control problem from a general perspective and provided a systematic overview of nine different special cases of change detection and control problem. Some of these special cases match known design problems whereas the others are new. To highlight the key aspects of particular problems, three selected special cases were discussed in more details through the theoretical examples.

REFERENCES


