Abstract: The problem under consideration is the controllability of a wide class of convolution Volterra systems, namely the class of “diffusive” systems, for which there exists an input-output state realization whose state evolves in the so-called diffusive representation space. We first show that this universal state variable is approximately controllable, and then deduce that such Volterra systems always possess suitable controllability properties, stated and proved. Then, we show how to solve the optimal null control problem in an LQ sense. A numerical example finally highlights these results.

Keywords: controllability, operators, infinite dimensional system, integral equations, diffusive representation, state realization, optimal control

1. INTRODUCTION

There exists a large literature considering the control problem of infinite dimensional systems, and new techniques are available to compute suitable control laws for linear or nonlinear partial differential equations. See e.g. Tucsnak and Weiss [2009] for a recent textbook on the control of linear operators, or Coron [2007] for techniques adapted to nonlinear dynamical equations. The class of systems under consideration here are described by a pseudo-differential operator of diffusive type. These models appear in various applications such as in acoustics (see e.g., Fellah et al. [2001], Polack [1991]), in combustion (see e.g. Rouzaud [2003]), in electrical engineering (see e.g. Bidan et al. [2001]), in biology (see e.g. Topaz and Bertozzi [2004]), etc. The operators under consideration in this paper belong more precisely to the general class of “diffusive operators” introduced in Montseny [2005]. Such operators can be realized by means of input-output dissipative infinite dimensional equations, as very early considered for example in Kirkwood and Fuess [1941], Rouse Jr [1953], Macdonald and Brachman [1956], or more recently in Montseny et al. [1993], Staffans [1994], Montseny [1998], etc. Various aspects have been studied in the literature for this class of systems. A lot of results can be found in Montseny [2005]; we can also mention other works about the identification (Casenave and Montseny [2009]), the numerical simulation (Montseny [2004]), the inversion (Casenave [2009]), or the dissipativity (Matignon and Prieur [2005]) of such systems.

The aim of this paper is to prove the controllability or more precisely the approximate controllability of dissipive operators. In the sequel, we consider the Volterra input-output model of the form:

\[ x(t) = \int_0^t h(t-s) u(s) \, ds, \quad \forall t \geq 0, \quad (1) \]

with \( h \in L^1_{\text{loc}}(\mathbb{R}^+), x \in C^0(\mathbb{R}^+) \). Note that in this case, we necessarily have: \( x(0) = \int_0^t h(t-s) u(s) \, ds = 0 \). Model (1) can be rewritten under the symbolic form:

\[ x = H(\partial_t) u, \quad (2) \]

where \( H(p) \) is the symbol (or transfer function, non necessarily rational) of the operator \( H(\partial_t) \).

Note that model (2) can represent a wide variety of dynamical systems. Let us give the following examples:

- models of the form \( K(\partial_t)x = \lambda x + u, \lambda \in \mathbb{R} \), are a particular case of (2) with \( H(\partial_t) = (K(\partial_t) - \lambda I)^{-1} \).
- SISO models of the form:

\[
\begin{aligned}
X &= AX + Bu, \quad X(0) = 0 \\
x &= CX
\end{aligned}
\]

(3)

with \( X(t) \in \mathbb{R}^n \), can also be rewritten under the form (2), with \( H(p) = C(pI - A)^{-1}B \).

In this paper, we study the controllability of systems of the form (2) with \( H(p) \) non rational. Such systems do not admit any state representation in \( \mathbb{R}^n \). So the notion of controllability, well-defined in the case of systems with finite dimensional state representations, has to be reformulated. For that aim, we use the so-called diffusive representation (Montseny [2005]), which enables to realize the operator \( H(\partial_t) \) by means of a suitable (infinite-dimensional) diffusive state equation with input \( u \), from which we define an approximate controllability in a state space well adapted to the state realization. We then establish that all systems of the form (1), admitting such a diffusive realization, are approximately controllable.

This paper is organized as follows. Some preliminaries and the problem statement are given in Section 2. The main
results are given in Section 3. The numerical simulation and the effective computation of the control are considered in Section 4. In particular an example is introduced and considered in this section. Section 5 contains some concluding remarks and point out some further research lines. Finally, the Appendix A collects the proofs of the main results. It necessitates to develop precisely the framework associated with the diffusive representation.

2. PRELIMINARIES

2.1 Problem statement

Let us consider the following Cauchy problem, on which will be based the state realization of (1):

\[\begin{cases}
\partial_t \psi(t, \xi) = \gamma(t) \psi(t, \xi) + u(t), & t > 0, \ \xi \in \mathbb{R}, \\
\psi(0, \xi) = \psi_0(\xi),
\end{cases}\]  

(4)

where \(\gamma \in W_{\text{loc}}^{1,\infty}(\mathbb{R}; \mathbb{C})\) defines an infinite simple arc in \(\mathbb{C}^+ \subset \mathbb{C}\), closed at \(\infty\). The problem is supposed to be well posed in the space \(C^0([0, T]; \Psi)\) of measurable functions with values in a suitable topological state space \(\Psi\). For all \(\psi_0 \in \Psi\) and \(u \in L^2(0, T)\), (4) admits a unique solution, given by:

\[\psi(t, \xi) = e^{\gamma(t) T} \psi_0(\xi) + \int_0^t e^{\gamma(s) T} u(t - s) \, ds, \quad t \in [0, T].\]  

(5)

This solution will be denoted \(\psi(t; \psi_0, u)\) in the sequel. Let \(S_T : L^2(0, T) \to \Psi\) be the operator defined by:

\[S_T(u) = \int_0^T e^{\gamma(s) T} u(T - s) \, ds.\]  

(6)

We introduce the set:

\[\mathcal{R}_T = \{\psi(T; \psi_0, u); u \in L^2(0, T)\};\]

\(\mathcal{R}_T\) is called the reachable set of system (4) at time \(T\). We now consider the following definitions:

Definition 1. System (4) is said to be:

- **controllable** (in \(\Psi\)) on \([0; T]\), if \(\mathcal{R}_T\) is equal to \(\Psi\);
- **approximately controllable** (in \(\Psi\)) on \([0; T]\), if \(\mathcal{R}_T\) is densely embedded in \(\Psi\), that is, \(\mathcal{R}_T \subset \Psi, \mathcal{R}_T \not\subset \Psi\);
- **approximately controllable** (in \(\Psi\)) if it is approximately controllable on \([0; T]\) for any \(T > 0\).

For all \(\psi_0 \in \Psi\), note that \(\mathcal{R}_T = S_T(L^2(0, T)) + e^{\gamma(T) T} \psi_0\). Thus the set \(\mathcal{R}_T\) is dense in \(\Psi\) if and only if \(S_T(L^2(0, T))\) is dense in \(\Psi\). More precisely, we have (see e.g. Tucsnak and Weiss [2009]):

Proposition 2. Consider a topological vector space \(\Psi\) such that \(\psi_0, e^{\gamma(T) T} \psi_0 \in \Psi\). The system (4) is approximately controllable in \(\Psi\) if and only if \(S_T(L^2(0, T))\) is dense in \(\Psi\) for any \(T > 0\).

The problem under consideration in this paper is the approximate controllability of (4) and then of (2).

2.2 Notation and introduction of the control space

Let us first introduce the space:

\[\mathcal{D}_\infty = \left\{\phi \in C^\infty(\mathbb{R}), \forall n \in \mathbb{N}, \sqrt{1 + \left(1 + \partial_x^2 \phi\right)} \in L^\infty(\mathbb{R})\right\};\]

it is classically a Fréchet space with topology defined by the countable set of norms: \(\|\phi\|_n = \|\sqrt{1 + \partial_x^2 \phi}\|_{L^\infty}\).

By denoting \(L^2(\mathbb{R}^+)\) the space of functions of \(L^2(\mathbb{R}^+)\) with compact support and \(\mathcal{L}_\gamma\) the operator:

\[\mathcal{L}_\gamma : u \in L^2(\mathbb{R}^+) \to \mathcal{L}_\gamma u := \int_0^{+\infty} e^{-\gamma s} u(s) \, ds,\]  

(7)

thanks to the property \(u \in L^2(0, T) \iff u(T - \cdot) \in L^2(0, T)\), we have:

\[\bigcup_{T > 0} \mathcal{S}_T(L^2(0, T)) = \mathcal{L}_\gamma(L^2_\gamma(\mathbb{R}^+)).\]  

(8)

Let us introduce the following space:

**Definition 3.** \(\Delta_{\gamma}\) is the completion of \(\mathcal{L}_\gamma(L^2_\gamma(\mathbb{R}^+))\) in \(\mathcal{D}_\infty\).

We denote \(\Omega^-_{\gamma}\) and \(\Omega^+_{\gamma}\) the two open domains delimited by \(\gamma\) such that \(\Omega^+_{\gamma} \supset (a, +\infty)\). Assume there exists \(\alpha_{\gamma} \in (\frac{\pi}{2}, \pi)\) such that:

\[e^{i[-\alpha_{\gamma}, \alpha_{\gamma}]}[a, \infty).\]  

(9)

Given \(\gamma_n\) a sequence of regular functions such that \(\gamma_n(\mathbb{R}) \in \Omega^+_{\gamma_{n+1}}\) and \(\gamma_n \rightarrow \gamma\), the topological vector space \(\Delta_{\gamma}\) is defined as the inductive limit associated with an inductive system \((\Delta_{\gamma_n}, \phi_n)\) where \(\phi_n\) are topological isomorphisms such that \(\phi_n(\Delta_{\gamma_n}) \subseteq \phi_{n+1}(\Delta_{\gamma_{n+1}})\). The topological vector space \(\Delta_{\gamma} := \lim_{\to \gamma} \phi_n(\Delta_{\gamma_n})\) is complete and locally convex, with \(\phi_n(\Delta_{\gamma_n}) \subseteq \Delta_{\gamma}\) continuous and dense. It can be shown (see Montseny [2005]) that for any \(u \in C^0(0, T), (4)\) is well-posed in \(C^0(0, T; \Delta_{\gamma})\).

3. CONTROLLABILITY RESULTS

3.1 Approximate controllability results

We are now in position to state the main results about the controllability of (4), the proofs of which are given in Appendix A.3:

**Theorem 4.** System (4) is approximately controllable in \(\Delta_{\gamma}\).

**Corollary 5.** System (4) is approximately controllable in any topological space \(\Psi\) such that \(\Delta_{\gamma} \to \Psi\) with dense embedding.

In particular, from the dense embedding: \(\Delta_{\gamma} \to L^2 := \prod_{\gamma} L^2_{\gamma}(\mathbb{R})\), (4) is approximately controllable in the Hilbert subspace \(L^2_{\gamma}\).

**Remark 6.** The sector condition (9) appears as the cornerstone in the proof of Theorem 4: it is indeed at the origin of the analyticity of \(\mathcal{L}_\gamma\mu\) on which the result is based; it also expresses the diffusive nature of system (4). Due to \(\mathcal{L}_\gamma(L^2_{\gamma}(\mathbb{R})) \subseteq \Delta_{\gamma}\), system (4) is not exactly controllable; this is a consequence of the analyticity in \(C\) of \(p \to \int_0^{+\infty} e^{\gamma s} u(s) \, ds\) when the support, supp \(u\), of the function \(u\) is compact.

3.2 Controllability of the Volterra problem (2)

Consider an operator \(H(\partial_t)\) admitting a so-called \(\gamma\)-symbol\(^1\) \(\mu \in \Delta_{\gamma}\), that is, such that model (2) admits the input-output state representation (see Appendix A):

\[\begin{cases}
\partial_t \psi = \gamma \psi + u, \quad \psi(0, \xi) = 0 \\
x = (\mu, \psi)_{\Delta_{\gamma}'; \Delta_{\gamma}}.
\end{cases}\]  

(10)

\(^1\) As usual, \(\Delta'\) designates the topological dual of \(\Delta\).\)
Definition 7. The system \( u \mapsto H(\partial_t)u \) is said (approximately) controllable if there exists \( \mu \in \Delta' \gamma \) such that for any \( u \in L^2_{\text{loc}}(R^+) \):

- (10) is approximately controllable,
- for any \( (\psi, x) \) solution of (10), we have: \( H(\partial_t)u = x \).

Then, from Theorem 4, we get:

Corollary 8. If \( H(\partial_t) \) admits a \( \gamma \)-symbol, then the system (2) is approximately controllable.

Example 9. The “fractional” differential model of the form:

\[
\partial_t^\alpha x = \lambda x + u, \quad x(0^+) = 0, \quad 0 < \alpha < 1
\]

is approximately controllable. Indeed, (11) is equivalently written: \( x = (\partial_t^\alpha - \lambda)^{-1}u \) and \( (\partial_t^\alpha - \lambda)^{-1} \) admits a \( \gamma \)-symbol as soon as \( \lambda \in \Omega_\gamma \) (see Appendix A). More generally, for any \( \alpha > 0 \), the system \( x = \partial_t^{-\alpha}(\lambda x + u) \) is approximately controllable.

Remark 10. As in the finite dimensional case, it is interesting to introduce, for the controllability of (2), the notion of minimal state realization. Indeed, in the state realization (10), which is of the same type as (3) but in infinite dimension, we note that only the \( \xi \) in \( \text{supp} \mu \) are involved in the synthesis of \( x \). Assuming that \( \text{supp} \mu \) is a Lebesgue non-negligible set, we then introduce the semi-norm in \( L^2 \) defined by:

\[
p_\mu(\psi) = \sqrt{\int_{\text{supp} \mu} |\psi|^2 d\xi},
\]

and consider the quotient Hilbert space \( \Delta_{\mu, \gamma} := \Delta_{\gamma}/\ker p_\mu \).

From the above, we deduce that system (2) is (a fortiori) approximately controllable in \( \Delta_{\mu, \gamma} \). Moreover, if \( \mu \sim f \in L^1_{\text{loc}}(R) \), the following \( (\gamma) \)-realization is minimal:

\[
\begin{align*}
\partial_t \psi &= \gamma \psi + u, \quad \psi(0, \xi) = 0, \quad \xi \in \text{supp} \mu, \\
x &= (\mu, \psi) \Delta_{\mu, \gamma} = \int_{\text{supp} \mu} f \psi d\xi.
\end{align*}
\]

3.3 About null control of (2)

We now state some results relating to approximative controllability to zero of (2). We suppose that the operator \( H(\partial_t) \) admits a \( \gamma \)-symbol \( \mu \in \Delta'_{\gamma} \). First we state an approximate controllability; it can indeed be shown that this problem is well posed\(^2\), that is there exists a unique solution \( u \), given by:

\[
\psi^\text{opt} = -(S_T^*S_T + \varepsilon I)^{-1}S_T(\gamma T, \psi_0).
\]

4.2 Discrete formulation

Consider a mesh \( \{t_k\}_{k=0}^{K} \) of the time variable \( t \), such that \( t_k = T \). The control \( u \) is computed as:

\[
u = \sum_{k=0}^{K-1} u_k 1_{[t_k, t_{k+1})}.
\]

We have:

\[
(S_T u)(\xi) = \sum_{k} \int_{t_k}^{t_{k+1}} e^{\gamma(t)}(T-s) ds u_k = \sum_{k} S_t^k u_k,
\]

with:

\[
S_t^k = \int_{t_k}^{t_{k+1}} e^{\gamma(t)}(T-s) ds = \frac{e^{\gamma(T)} - e^{\gamma(t_k+1)} - e^{\gamma(t_k)} - e^{\gamma(T-t_k)}}{-\gamma(T)}.
\]

Under numerical approximation, problem (15) is then written:

\[
\min_{u_K \in \mathbb{R}^K} \left\{ \|e^{\gamma T} \psi_0 + S u_K\|_{L^2}^2 + \varepsilon \|u_K\|_{L^2}^2 \right\},
\]

with \( u_K := [u_k]_{k=1}^{K} \), \( e^{\gamma T} \psi_0 = e^{\gamma(T)} \psi_0 e^{\gamma(T)} \psi_0 \) and \( S_t^k \) the matrix of elements \( S_t^k \). The solution is given by:

\[
u^\text{opt} = -(S^* S + \varepsilon I)^{-1}S^* (e^{\gamma T} \psi_0).
\]

4.3 Numerical example

Let \( H(\partial_t) \) be the non rational operator with symbol:

\[
H(p) = \frac{\ln(p)}{p + 200}.
\]

\( H \) is holomorphic in \( \mathbb{C} \backslash \mathbb{R}^+ \) and \( H(p) \rightarrow 0 \) when \( p \rightarrow \infty \) in \( \mathbb{C} \backslash \mathbb{R}^+ \); then \( H(\partial_t) \) admits a diffusive state realization with

\[\frac{2}{\lambda} \text{image of the operator: } u \in L^2(0, T) \rightarrow (u, S_T u) \in L^2(0, T) \times L^2(\mathbb{R}_x) \text{ is closed for the graph norm.}\]
\[ \gamma \text{ defined by } \gamma(\xi) = -|\xi|. \] The \( \gamma \)-symbol of \( H(\partial_t) \) identifies with the distribution \( \mu \) given by:
\[ \mu = \text{pv} \frac{1}{|\xi| - 200} + \ln(200) \delta_{200}. \]

For the finite dimensional approximate state realization \(^3\) of \( H(\partial_t) \), we consider a mesh \( \{\xi_l\}_{l=1:1.1} \) of \( L = 70 \) discretization points geometrically spaced between \( \xi_1 = 10^{-3} \) and \( \xi_{70} = 10^4 \). For \( t = 0 \) to \( t = t_0 = 5 \), we apply the control \( u = \sin(\frac{\pi}{2} \xi)^2 \); we get:
\[ \psi_0(\xi) = (\partial_t - \gamma(\xi))^{-1} \sin(\frac{\pi}{2} (t_0)^2) \bigg|_{t=t_0}. \] Then, from \( t = t_0 \) to \( t = T' = 25 \), we apply the control:
\[ u = u_{\text{opt}}^{K,\varepsilon}(t_0, T), \] (A.1)
where \( T = 15 \) and \( u_{\text{opt}}^{K,\varepsilon} \) is obtained by (18) with \( \varepsilon = 10^{-4} \) and \( K = 10^4 + 1 \) (i.e. \( \Delta t = 10^{-3} \)).

Figure 1 gives the time-evolution of the control \( u \), of the output \( x \) and of the state \( \psi \). It can be checked that the input \( u \) succeeds to control the state from its initial value to the final value \( 0 \) within time \( T \) and that the evolution after \( T \) confirms the result of Theorem 12.

5. CONCLUSION

In this paper, the controllability problem for a wide class of Volterra (scalar) systems has been studied. By considering the diffusive representation approach, the result is that any diffusive Volterra system \( x = H(\partial_t)u \) is approximately controllable. This result can be trivially extended to general diffusive non linear and/or non \( t \)-invariant Volterra systems of the form \( x = H(t, \partial_t)f(t, x, v) \) with \( f \) an invertible function \(^4\), simply by replacing \( \mu(\xi) \) by \( \mu(t, \xi) \).

In that sense, this controllability result can be viewed as a consistent extension of controllability of scalar differential systems \( \partial_t x = f(t, x, v), x(0) = 0 \). It seems now to be interesting to tackle the controllability problem for vector systems of the form \( x = H(t, \partial_t)u \) with \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \).

Appendix A. DIFFUSIVE REPRESENTATION AND PROOF OF THE MAIN RESULTS

A complete statement of diffusive representation can be found in Montseny [2005]; a shortened one is presented in Casenave and Montseny [2010].

A.1 Basic principle of diffusive representation

We consider a causal convolution operator defined, on any continuous function \( u : \mathbb{R}^+ \rightarrow \mathbb{R} \), by:
\[ u \mapsto \left( t \mapsto \int_0^t h(t-s)u(s) ds \right). \]
(A.1)

We denote \( H \) the Laplace transform of \( h \) and \( H(\partial_t) \) the convolution operator defined by (A.1).

Let \( u' \) be the restriction of \( u \) to its past and \( u_0(s) = u'(t-s) \) the so-called “history” of \( u \). From causality of \( K(\partial_t) \), we deduce:
\[ [H(\partial_t)(u-u')]|t=0 = 0 \quad \text{for all } t; \]
(A.2)

\(^3\) Details about the numerical approximation of \( \mu \) and \( \psi \) can be found in Montseny [2005], Casenave and Montseny [2010].

\(^4\) in the sense that there exists \( y \) such that \( v := g(t, x, u) \) with \( u := f(t, x, v) \).

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Fig. 1. Numerical results. Time-evolution of the control \( u \) (top), of the output \( x \) (middle) and of the state \( \psi \) (down) for \( t \) in \([0, 25]\) then, we have for any continuous function \( u \):
\[ [H(\partial_t)u](t) = [\mathcal{L}^{-1}(H\mathcal{L}u)](t) = [\mathcal{L}^{-1}(H\mathcal{L}u')](t); \]
(A.3)

where \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) are the Laplace and the inverse Laplace transforms defined by \( (\mathcal{L}f)(p) = \int_0^\infty e^{-pt}f(t) dt \) and by \( (\mathcal{L}^{-1}F)(t) = \int_{b-i0}^{b+i0} e^{pt}F(p) dp \).

We define \( \Psi_u(t, p) := e^{pt}(\mathcal{L}u')(p) = (\mathcal{L}u)(-p) \); by computing \( \partial_t\mathcal{L}u \), Laplace inversion and use of (A.3):

**Lemma 13.** \( \Psi_u \) is solution of the differential equation:
\[ \partial_t\Psi_u(t, p) = p\Psi(t, p) + u, \quad t > 0, \quad \Psi(0, p) = 0, \]
(A.4)
and:
\[ [H(\partial_t)u](t) = \frac{1}{2\pi i} \int_{b-i0}^{b+i0} H(p) \Psi_u(t, p) dp, \forall b \geq 0. \]
(A.5)
Now, let’s consider $\gamma, \Omega^+ \gamma, \Omega^- \gamma$ as defined in Section 2, with $\gamma$ regular. By use of standard techniques (Cauchy theorem, Jordan lemma), it can be shown:

**Lemma 14.** For $\gamma$ such that $H$ is holomorphic in $\Omega_{\gamma}$, if $H(p) \to 0$ when $p \to \infty$ in $\Omega_{\gamma}$, then:

$$[H(\partial_t) u](t) = \frac{1}{2\pi i} \int_{\gamma} H(p) \psi_u(t, p) \, dp.$$  \hspace{1cm} (A.6)

Under assumptions of Lemma 14, we have (we use the notation $(\mu, \psi) = \{ \mu \psi \})$:

**Proposition 15.** Denoting $\mu = \frac{\partial}{\partial \psi} H \circ \gamma$ and $\psi(t, \cdot) = \Psi_u(t, \cdot) \circ \gamma$, we have, for all $t \geq 0$,

$$H(\partial_t) u(t) = \langle \mu, \psi(t, \cdot) \rangle.$$  \hspace{1cm} (A.7)

where $\psi$ is the solution on $(t, \xi) \in \mathbb{R}^{+} \times \mathbb{R}$ of:

$$\partial_t \psi(t, \xi) = \gamma(\xi) \psi(t, \xi) + u(t), \quad \psi(0, \xi) = 0.$$  \hspace{1cm} (A.8)

**Definition 16.** The function $\mu$ defined in Proposition 15 is called $\gamma$-symbol of operator $H(\partial_t)$. The function $\psi$ solution of (A.8) is called the $\gamma$-representation of $u$.

**Proposition 17.** The impulse response $h = L^{-1} H$ of operator $H(\partial_t)$ is given by:

$$h(t) = \langle \mu, e^{\gamma t} \rangle = \mathcal{L}_{\gamma} \mu;$$  \hspace{1cm} (A.9)

furthermore, $h$ is holomorphic in $\mathbb{R}^{+}$.  \hspace{1cm} (A.10)

**Proof.** (A.9) is obtained by setting $u$ as the Dirac measure in (A.8). Analyticity of $h$ is a consequence of the sector condition (9) (which makes equation (A.8) of diffusive nature).

**A.2 General topological framework**

The results of Proposition 15 can be extended to a wide class of operators, provided that the associated $\gamma$-symbols are extended to suitable distribution spaces we will introduce in the sequel (so, the expression $(\mu, \psi)$ will refer to a topological duality product).

Let consider the spaces $D_{\infty}$ and $\Delta_{\gamma}$ as defined in Section 2.2, and first note that we have the continuous and dense embeddings:

$$D \hookrightarrow D_{\infty} \hookrightarrow L^2(\mathbb{R}) \hookrightarrow D' \hookrightarrow D'.$$  \hspace{1cm} (A.10)

We can show:

**Proposition 18.** $S_T(L^2(0, T)) \subset D_{\infty}$.

**Proof.** $\forall u \in L^2(0, T), ||S_T u|| \leq \int_0^T \left| \mathcal{L}_{\gamma}(T-t) u(t) \right| \, dt$

$$\leq \|u\|_L^2 \sqrt{\int_0^T e^{2\Re(\gamma(T-t))} \, dt} = \|u\|_L^2 \sqrt{\frac{e^{2\Re(\gamma(T-t))} - 1}{2\Re(\gamma(T))}}.$$  \hspace{1cm} (A.10)

Under the hypotheses made on $\gamma$, we can show from simple analysis that there exists $c > 0$ such that, for all $\xi > 0$:

$$\sqrt{\frac{e^{2\Re(\gamma(T-t))} - 1}{2\Re(\gamma(T))}} \leq \frac{c}{\sqrt{1 + \xi^2}}.$$  \hspace{1cm} (A.10)

The extension to $\partial_T^c(S_T u)$ is then obtained by induction.

**Proposition 19.** $\Delta_{\gamma}$ is a strict subspace of $D_{\infty}$.

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5 That is: there exists an open domain $D \subset \mathbb{C}$ such that $\mathbb{R}^{+} \subset D$ and $h$ admits an analytical continuation on $D$.

So, with $\mathcal{L}_{\gamma}$ the adjoint of $\mathcal{L}_{\gamma}$, we have ker$(\mathcal{L}_{\gamma}) \neq \{0\}$ and the dual $\Delta_{\gamma}$ is the quotient space:

$$\Delta_{\gamma} = D_{\infty} / ker(\mathcal{L}_{\gamma}).$$

We have the following continuous and dense embeddings:

$$\Delta_{\gamma} \hookrightarrow L^2_{\gamma} \subset L^2_{\gamma} \hookrightarrow \Delta_{\gamma}.$$  \hspace{1cm} (A.7)

**Remark 20.** Thanks to (A.7), if $\gamma(0) = 0$, then the Dirac distribution $\delta$ is a $\gamma$-symbol of the operator $u \mapsto \int_0^1 u(s) \, ds$, denoted $\delta_{\gamma}$.

Now suppose that $\gamma \in W^{1, \infty}$ (so, $\gamma$ can be non regular), and consider the space $\Delta_{\gamma}$ defined in Section 2.2. The so-defined space $\Delta_{\gamma}$ has the following properties (Montseny [2005]):

- $\Delta_{\gamma} = \bigcup_{n} \phi_n(\Delta_{\gamma})$ and $\Delta_{\gamma}$ is independent of the choice of the sequence $\gamma_n$.
- $L^2_{\gamma}(\mathbb{R}) \ni \phi_n(\Delta_{\gamma}) \forall n$; so, $\forall n \in \mathbb{L}^2_{\gamma}(\mathbb{R}), \psi_n = \phi_n(\Delta_{\gamma})$.
- The dual $\Delta_{\gamma}$ of $\Delta_{\gamma}$ is a complete, locally convex topological vector space; we have $\Delta_{\gamma}^* = \bigcap_{n} \phi_n(\Delta_{\gamma})$.
- With $\psi_n$ defined by (4) and $\mu \in \Delta_{\gamma}$, the symbol $H$ of the convolution operator defined by $u \mapsto (\mu, \psi_n)_{\Delta_{\gamma}}$ is holomorphic in $\Omega_{\gamma}$ and $H(p) \to 0$ when $p \to \infty$ in $\Omega_{\gamma}$.
- The impulse response $h$ of operator $u \mapsto (\mu, \psi_n)_{\Delta_{\gamma}}$ is $h(t) = (\mu, e^{\gamma t})_{\Delta_{\gamma}} = L^2_{\gamma} \mu$; it is holomorphic in $\mathbb{R}^{+}$.
- When $\mu$ can be identified with a locally integrable function (denoted $\mu$), then $\langle \mu, \psi \rangle_{\Delta_{\gamma}}$ can be expressed by the integral (in the Lebesgue sense):

$$\langle \mu, \psi \rangle_{\Delta_{\gamma}} = \int_0^\infty \mu \psi \, d\xi.$$  \hspace{1cm} (A.10)

**A.3 Proof of the main results**

**Proof of Theorem 4.** Thanks to Proposition 2, we have to prove that $S_T(L^2(0, T))$ is dense in $\Delta_{\gamma}$ for any $T > 0$. Let us consider the (continuous) extension of operator $\mathcal{L}_{\gamma}$ to $L^2(\mathbb{R}^\tau)$; we have:

$$\mathcal{L}_{\gamma} : L^2(\mathbb{R}^\tau) \rightarrow D_{\infty} \text{ and } S_T : L^2(0, T) \rightarrow D_{\infty},$$

and, then, by identifying $L^2$ spaces with their duals:

$$\mathcal{L}_{\gamma} : D_{\infty} \rightarrow L^2(\mathbb{R}^\tau) \text{ and } S_T : D_{\infty} \rightarrow L^2(0, T).$$

From Fubini theorem:

$$\langle \mu, S_T u \rangle = \mu \left( \int_0^T e^{\gamma t} u(s) \, ds \right) = \int_0^T u(\mu, e^{\gamma t}) \, ds = \int_0^T u S_T^\tau \mu \, ds = \mathcal{L}_{\gamma}^* \mu = (\mu, e^{\gamma t})_{\Delta_{\gamma}, D_{\infty}} \in L^2(0, T);$$  \hspace{1cm} (A.10)

similarly:

$$\mathcal{L}_{\gamma}^* \mu = (\mu, e^{\gamma t})_{D_{\infty}, D_{\infty}} \in L^2(\mathbb{R}^\tau).$$

We can deduce: $\forall \mu \in D_{\infty}$, $\mathcal{L}_{\gamma} \mu \in [0, T]$. Then, due to $\mathcal{L}_{\gamma}$ the adjoint of $\mathcal{L}_{\gamma}$, we have ker$(\mathcal{L}_{\gamma}) \neq \{0\}$ and the dual $\Delta_{\gamma}$ is the quotient space:

$$\Delta_{\gamma} = D_{\infty} / ker(\mathcal{L}_{\gamma}).$$

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to Proposition 17, $L^\gamma_\mu$ is holomorphic on $\mathbb{R}^+$. It follows that:

$$L^\gamma_\mu = 0_{L^2(\mathbb{R}^+)} \iff S^\gamma_\mu = 0_{L^2(0, T)}$$

and so, $\ker L^\gamma_\mu = \ker S^\gamma_\mu$. Consequently, $\Im S^\gamma_\mu = \Im L^\gamma_\mu = \Delta_\gamma$, i.e. $\Im S^\gamma_\mu$ is dense in $\Delta_\gamma$. \hfill $\Box$

**Proof of Proposition 11.** Let $x$ be the trajectory defined on $[0, T]$ by:

$$\forall t \in [0, T], \ x(t) = H(\partial_t^2)u(1(t, 0)) = (\mu, S_T(\psi(1(t, 0)))$$

We obviously have: $x|_{[0,t_0]} = x$. Consider the open set $\mathcal{O}_1 = (-\infty, T - \varepsilon, -\infty, T + \varepsilon) \subseteq \mathbb{R}$. As the operator $\phi \in \Delta_\gamma \mapsto (\mu, \phi)_\Delta_\gamma \in \mathbb{R}$ is continuous, the inverse image of $\mathcal{O}_1$ is an open set of $\Delta_\gamma$, denoted $\mathcal{O}_2$. As $\Im(S_{T-t^0})$ is dense in $\Delta_\gamma$, then, there exists $v \in L^2(0, T-t^0)$ such that $S_{T-t^0}(v) \in \mathcal{O}_2$, and we have

$$\langle \mu, S_{T-t^0}(v) \rangle \in \mathcal{O}_2.$$ \hfill $\Box$

**Proof of Theorem 2.** From the definition of $\Delta_\gamma$, $\mu \in \phi_n(\Delta_\gamma)$ $\forall n$; furthermore, if $\psi$ is solution of (13), then $\psi(t, \cdot) \in \phi_n(\Delta_\gamma)$ $\forall n, t$ (Montseny [2005]). We then have: $(\mu, \psi)_\Delta_\gamma = (\mu_n, \psi_n)_\Delta_\gamma = \mu_n$ with $\mu_n$ and $\psi_n$ regular functions; hence, from properties of $\mu_n$ and $\psi_n$:

$$[x(t)] = [\mu(\psi(t, \cdot))_{\Delta_\gamma}] = \int \mu_n(\xi) \psi_n(t, \xi) \, d\xi$$

$$\leq \left( \int |\psi_n|^{\gamma_n} \right)^{1/|\xi|} \left( \int 1 + |\xi|^2 \right)^{1/2} \left( \int \psi_n(\xi, t)^{\gamma_n} \right)^{1/|\xi|}$$

$$\leq \sup_{\xi, \xi' \in [0, T]} \left| \psi_n(\xi, T) \right| \left( \int 1 + |\xi|^2 \right)^{1/2} \left( \int 1 + |\xi'|^2 \right)^{1/2}$$

$$\leq K \sup_{\xi, \xi' \in [0, T]} \left| \psi_n(\xi, T) \right|.$$ 

Consider now an open ball $B(0, \eta) \subset L^\infty(\mathbb{R})$; then $V = \left( \frac{1}{1+|\xi|^2} B(0, \eta) \right)$ is an open set in the Fréchet space $\Delta_\gamma$, and $V = \phi_n(V_n)$ is an open set in $\Delta_\gamma$. So, let $u \in L^2(0, T)$ such that $\psi(T, \cdot) = S_T u \in V$; it follows:

$$[\langle x \rangle(t)] \leq K \sup_{\xi} \left| \psi_n(\xi, T) \right| \leq K \eta \leq \varepsilon$$

**REFERENCES**


