Mean-Square Stability of Stochastic Markovian Jumping Systems with Variable Delay

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Abstract: The mean-square stability of stochastic Markovian jumping systems with variable delay is investigated in this paper. Based on Lyapunov-Krasovskii functional (LKF) method, new stability criteria are presented by applying generalized Finsler lemma (GFL). Convex analysis property is applied to use the information of the time-varying delay such that the worst-case enlargement for variable delay is avoided. The result is formulated in terms of a set of linear matrix inequalities (LMIs). Numerical examples are given to show the effectiveness of the approach.

Keywords: Mean-square stability, stochastic systems, Markovian jump, variable delay, convex analysis.

1. INTRODUCTION

A Markovian jump model can be used to describe dynamical systems with random changing structures and parameters, such as repairs of machines in manufacturing systems, component and interconnection failures of fault-tolerant systems, and so on (Bai and Hamilton (2005)). Markovian jump systems have been deeply investigated in the past decades, see for example, Bai and Hamilton (2005); Mao and Yuan (2006); de Souza (2006); Wu et al. (2010), and the references therein.

The effect of delay to the systems has attracted considerable attention in the past decades (Gu et al. (2003); He et al. (2007); Park and Ko (2007); Shao (2009); Suplin et al. (2004); Xu and Lam (2008)), since the delay in real systems will degrade the performance of the systems and even lead to oscillation and instability. When describing practical systems with Markovian models, delays should also be considered (Chen et al. (2009b)). Cao and Lam (2000) has discussed delay-independent robust $H_\infty$ control for uncertain Markovian jumping systems with constant delay by Lyapunov-Krasovskii functional (LKF) method and linear matrix inequality (LMI) technique. The delay-dependent method in Boukas et al. (2001) has been established by using model transformation approach. To obtain some delay-dependent criteria for Markovian jumping systems with time-varying delay, Xu et al. (2007) has introduced some free-weighting matrices.

Many practical systems are actually affected by random noises with the form of Wiener process. Thus, Itô stochastic delay systems have been deeply studied (Chen et al. (2009a, 2010a, 2008, 2010b); Gao et al. (2006); Gershon et al. (2007); Li et al. (2008); Rodkina and Basin (2006); Xue et al. (2008); Zhang et al. (2009)). Recently, LMI-based delay-independent results for stochastic delay systems have been proposed in Gao et al. (2006); Xue et al. (2008) by LKF method. By means of convergence theorem and degenerate functionals, Rodkina and Basin (2006) has dealt with delay-dependent almost surely global asymptotic stability for nonlinear stochastic delay systems. Input-output approach has been applied to obtain delay-dependent results in Gershon et al. (2007). Delay-dependent mean-square stability conditions for stochastic delayed systems have been presented by descriptor model transformation in Li et al. (2008). Applying free-weighting matrix technique, which has been widely used in deterministic systems with delay (see He et al. (2007) etc.), some less conservative criteria for stochastic time-delay systems have been established in Chen et al. (2009a, 2010a, 2008); Zhang et al. (2009).

Furthermore, when considering both delay and stochastic noises in a Markovian jumping system, there appear stochastic delay Markovian jumping systems (Guan et al. (2005); He et al. (2010); Mao and Yuan (2006); Wei et al. (2007); Yue and Han (2005)). Model transformation approach has been employed to obtain delay-dependent results in Guan et al. (2005). Moreover, free matrix method has been extended to study stochastic delay systems with Markovian jumping parameters in He et al. (2010); Wei et al. (2007); Yue and Han (2005). If the delay in stochastic systems is time-varying, by invoking Jensen’s inequality the term $\int_{t-d(t)}^{t} sR(x(s))ds$ has been estimated as $-\int_{t-d(t)}^{t} sR(x(s))dsR \int_{t-d(t)}^{t} s(x(s))dsds$, see Chen et al. (2008); Guan et al. (2005); He et al. (2007); Wei et al. (2007); Yue and Han (2005); Zhang et al. (2009)). This estimation enlarges the time-varying delay ‘d(t)’ by its upper bound ‘\theta’. Park and Ko (2007); Shao (2009) have shown that the usage of convex analysis can remain the information of ‘d(t)’ and avoid estimating the delay by its lower and upper bounds. These drive us to deal with stability of stochastic systems with time-varying delay by the same idea as Park and Ko (2007); Shao (2009).

This paper is concerned with the mean-square stability of stochastic Markovian jumping systems with variable delay. Applying generalized Finsler lemma (GFL, Chen et al. (2010b)), which is an extension of the standard Finsler lemma (Suplin et al. (2004)), new stability conditions are obtained. Moreover, with the aid of convex analysis property, the proposed results...
can be derived without estimating the time-varying delay ‘\(d(t)\)’ by its worst-case bounds. Numerical examples are provided to show the effectiveness of the method.

**Notations:** \(E\{\cdot\}\) denotes the mathematical expectation; \(Pr\{\cdot\}\) is the probability. For a given matrix \(B \in \mathbb{R}^{n_2 \times n_1}\) satisfying \(\text{rank}(B) = n_0 < n_1\), let \(B^\perp \in \mathbb{R}^{n_1 \times (n_1 - n_0)}\) be the right orthogonal complement of \(B\) with \(BB^\perp = 0\) and \(B^\perp B^\perp^T > 0\). \((\Omega, \mathcal{F}, \mathcal{P})\) is a complete probability space, where \(\Omega\) is the sample space, \(\mathcal{F}\) is the \(\sigma\)-algebra of subsets of \(\Omega\), and \(\mathcal{P}\) is the probability measure on \(\mathcal{F}\). The symmetric term in a symmetric matrix is denoted as \(\ast\).

### 2. Problem Statement and Preliminaries

Consider the following system

\[
\begin{align*}
\frac{dx(t)}{dt} &= [A(t, r_1)x(t) + A_1(t, r_1)x(t - d(t))] + f(x(t), x(t - d(t)), r_1) \quad \text{(1)} \\
&+ g(x(t), x(t - d(t)), r_1)^T dw(t)
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector; \(A(t, r_1)\) and \(A_1(t, r_1)\) are matrix functions of the random jump process \(r_i := r(t)\), where \(r(t)\) is a finite-state Markovian jump process representing the system mode, \(i.e., r(t)\) takes discrete values in a given finite set \(S = \{1, 2, \ldots, n_r\}\); \(d(t)\) is the time-varying delay satisfying \(0 < d(t) \leq h, \ d(t) \leq \mu\) where \(h, \mu\) are scalars; the initial condition of system (1) is given by the real-valued function \(\phi(t)\), which is continuously differential on \([-h, 0]\); \(w(t)\) is an \(m\)-dimensional Wiener process defined on the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) satisfying \(E\{dw(t)\} = 0, \ E\{dw^2(t)\} = dt\). \(f(x(t), x(t - d(t)), r_1), g(x(t), x(t - d(t)), r_1)\) are abbreviated as \(A_i, A_{i1}, f_i, g_i\), respectively.

The transition probability matrix system of (1) is given as

\[
Pr(r_{t+\Delta} = j | r_i = i) = \begin{cases} 
\pi_{ij} \Delta + o(\Delta), & j \neq i \\
1 + \pi_{ii} \Delta + o(\Delta), & j = i
\end{cases}
\]

where \(\Delta > 0, \lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0, \pi_{ij} \geq 0, \forall j \neq i\) is the transition rate from mode \(i\) at time \(t\) to mode \(j\) at time \(t + \Delta\), and

\[
\pi_{ii} = -\sum_{j=1, j \neq i}^{n_r} \pi_{ij} < 0.
\]

In this paper, it supposes that the nonlinear functions \(f_i\) and \(g_i\) satisfy the following assumption.

**Assumption 1.** Assume that \(f_i\) and \(g_i\) satisfy

\[
\left| f_i(t) \right| \leq F_{i1} x(t) + F_{i2} x(t - d(t))
\]

and

\[
\left| g_i(t) \right| \leq G_{i1} x(t) + G_{i2} x(t - d(t))
\]

where \(F_{i1}, F_{i2}, G_{i1}, G_{i2} \in \mathbb{R}^{m \times n}\) and \(G_{i1}, G_{i2} \in \mathbb{R}^{m \times n}\) are known constants matrices with compatible dimensions.

We now present some important preliminaries as follows.

**Definition 2.** System (1) is said to be mean-square stable if for any \(\varepsilon > 0\) there exists a scalar \(\delta(\varepsilon) > 0\) such that

\[
E\left\{ \left| x(t) \right|^2 \right\} < \varepsilon, \forall t \geq 0
\]

when

\[
\sup_{-h \leq s \leq 0} E\left\{ \phi(s) \right\} < \delta(\varepsilon).
\]

Additionally, if

\[
\lim_{t \to \infty} E\{r(t)\} = 0
\]

holds for any initial condition, then system (1) is said to be mean-square exponentially stable.

**Lemma 3.** (Gu et al. (2003)) For any vectors \(x, y \in \mathbb{R}^n\), symmetric matrices \(P > 0, Q\), and matrices \(L, E, F(t)\) with compatible dimensions satisfying \(\Gamma^T F(t) F(t) \leq I, \ \Gamma = L F(t) + E^T F(t) L^T < 0\), if and only if there exists a scalar \(\varepsilon > 0\) such that

\[
Q + \varepsilon L^T L + \varepsilon^T E^T E < 0.
\]

**Lemma 4.** (Generalized Finsler Lemma, GFL, Chen et al. (2010b)) Consider a stochastic vector \(\Theta \in \mathbb{R}^n\), a symmetric matrix \(\Theta \in \mathbb{R}^{n \times n}\) and a matrix \(B \in \mathbb{R}^{n \times n}\) with rank \((B) = n_0 < n_1\). Let \(B^\perp\) be the right orthogonal complement of \(B\), i.e., \(BB^\perp = 0\), then the following four statements are equivalent:

\(T_1\) \(E\{\theta T \Theta | \Theta \} < 0, \ \forall \theta \neq 0, \ t > t_0, \ E\{f_i \theta \Theta \delta \Theta \} = 0\)

\(T_2\) \(B^\perp T \Theta \Theta B^\perp < 0\)

\(T_3\) \(E\{\phi(s) \} < 0, \ \forall \phi(s) > 0\)

\(T_4\) \(E\{\phi(s) \} < 0, \ \forall \phi(s) > 0\)

3. Main Results

The main result of this paper is presented as follows.

**Theorem 5.** For given scalars \(h > 0, \mu, \) system (1) is mean-square asymptotically stable, if there exist scalars \(\varepsilon > 0, \delta(\varepsilon) > 0\), and matrices \(P_i > 0, Q_i > 0, P_i > 0, R_i > 0, R_s > 0, M_{i1}, N_{i2} (k = 1, 2, \ i \in S)\) with appropriate dimensions satisfying

\[
\sum_{i=1}^{n_r} \pi_{ij} Q_j \leq Q
\]

\[
P_i < \delta(\varepsilon)
\]

and

\[
\Theta_{i1} = \begin{bmatrix} \Gamma_i & h\tilde{X}_{i1}^T \\ -h\tilde{X}_{i1} & 0 \end{bmatrix} \leq 0, \ k = 1, 2
\]

\[
\tilde{X}_{i1} = [M_{i1}, N_{i1}, \ v = 0, 0, 0], \ \tilde{X}_{i2} = [0 M_{i2}, N_{i2}, \ v = 0, 0, 0],
\]

\[
\Gamma_i = \begin{bmatrix} \Gamma_{i1} & \Gamma_{i2} & R_i & P_i & hA^T_i (R_i + R_s + P_i) & 0 \\ - & \Gamma_{i3} & \Gamma_{i4} & 0 & hA^T_i (R_1 + R_2 + F_i^2) & 0 \\ - & - & \Gamma_{i5} & 0 & 0 & 0 \\ - & - & - & - & - & -I \\ - & - & - & - & - & -h(R_1 + R_2) \\ - & - & - & - & - & -h(R_1 + R_2) \end{bmatrix}
\]

with

\[
\Gamma_{i1} = P_i A_i + A_i^T P_i + \sum_{j=1}^{n_r} \pi_{ij} P_i + Q_i + hQ + Z - \frac{R_i}{h}
\]

\[
+ (1 + \varepsilon) F_i^2 P_i + \delta_i G_i^T_i G_i + M_{i1} + M_i^T
\]

\[
\Gamma_{i2} = P_i A_i - M_{i1}^T - N_{i1}
\]

\[
\Gamma_{i3} = - (1 - \mu) Z + F_i^2 F_i + \delta_i G_i^T_i G_i
\]

\[
- N_{i1} - N_{i2} + M_{i2} + M_i^T
\]

\[
\Gamma_{i4} = M_i^T + N_{i2}
\]

\[
\Gamma_{i5} = Q_i - \frac{R_i}{h} - N_{i2} - N_{i2}^T
\]
Proof. The following notation will be used in the sequel,
\[ y_i(t) = A_i x(t) + A_{1i} x(t - d(t)) + f_i(t), i \in S \] (7)
and by this, system (1) is then written as
\[ dx(t) = y_i(t) dt + g_i(t) dw(t). \] (8)
Integrate (8) on the interval \([t-h, t]\) to yield
\[ x(t) - x(t-h) = \int_{t-h}^{t} y_i(s) ds + \int_{t-h}^{t} g_i(s) dw(s). \] (9)
Take mathematical expectation to both sides of (9) to obtain
\[ \mathbb{E}\left( \int_{t-h}^{t} [-x(t) + x(t-h) + h y_i(s)] ds \right) = 0. \] (10)
On the basis of (7) and (10), there is
\[ \mathbb{E}\left( \int_{t-h}^{t} \mathcal{B}_i \xi_i(t, s) ds \right) = 0 \] (11)
where \[ \xi_i(t, s) = \left[ y_i^T(t) x_i^T(t) x_i^T(t-d(t)) x_i^T(t-h) f_i^T(t) y_i^T(s) \right]^T. \] (12)
The orthogonal complement of \( \mathcal{B}_i \) is
\[ \mathcal{B}_i^\perp = \begin{bmatrix} A_i & A_{1i} & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & -I & 0 \\ -I & 0 & 0 & 0 \end{bmatrix}. \] (13)
For the \( i \)-th mode \( i \in S \), select the following LKF
\[ V(t, x, i) = x^T(t) P_{x_i}(t) + \int_{t-d(t)}^{t} x^T(s) Z(s) ds \]
\[ + \int_{t-h}^{t} x^T(s) Q(s) ds \] (14)
\[ + \int_{t-h}^{t} x^T(s) Q(s) ds \]
\[ + \int_{t-h}^{t} y_i^T(s) (R_{1i} + R_{2i}) y_i(s) ds \]
\[ + \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \] (15)
where the weak infinitesimal generator is
\[ \mathcal{L} V(t, x, i) \]
\[ = 2 x^T(t) P_{x_i}(t) x(t) + \sum_{j=1}^{K} \pi_{ij} P_{x_j}(t) \int_{t-d(t)}^{t} x^T(s) Z(s) ds \]
\[ + \int_{t-h}^{t} x^T(s) (Q_{1i} + Q + h Q) x(s) ds \]
\[ + \int_{t-h}^{t} x^T(s) Q x(s) ds \]
\[ + \int_{t-h}^{t} y_i^T(s) (R_{1i} + R_{2i}) y_i(s) ds \]
\[ + \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \] (16)
Observe
\[ \int_{t-d(t)}^{t} \left[ \eta_i^T(t) M_{1i}^T R_{1i}^{1/2} + R_{2i}^{1/2} y_i(s) \right]^2 ds \]
\[ + \int_{t-h}^{t} \left[ \eta_i^T(t) M_{2i}^T R_{2i}^{1/2} + R_{2i}^{1/2} y_i(s) \right]^2 ds \geq 0 \] (17)
where \( \eta_i(t) = [x_i^T(t) x_i^T(t-d(t) x_i^T(t-h) f_i^T(t)]^T, M_{1i} = [M_{1i}, N_{1i}, 0, 0], M_{2i} = [M_{2i}, N_{2i}, 0, 0], \) then we have
\[ - \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \]
\[ = - \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds - \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \]
\[ \leq 2 \eta_i^T(t) M_{1i}^T \int_{t-d(t)}^{t} y_i(s) ds + 2 \eta_i^T(t) M_{2i}^T \int_{t-h}^{t} y_i(s) ds \]
\[ + d_i \eta_i^T(t) M_{1i}^T R_{2i}^{-1} M_{1i} \eta_i(t) \]
\[ + (h - d_i) \eta_i^T(t) M_{2i}^T R_{2i}^{-1} M_{2i} \eta_i(t). \]
It follows from (8) that
\[ \int_{t-d(t)}^{t} y_i(s) ds = x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} g_i(s) dw(s) \]
\[ \int_{t-h}^{t} y_i(s) ds = x(t-d(t)) - x(t-h) - \int_{t-h}^{t} g_i(s) dw(s). \] (19)
Substituting (19) and (20) into (18), we get
\[ - \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \]
\[ \leq 2 \eta_i^T(t) M_{1i}^T [x(t) - x(t-d(t))] \]
\[ + 2 \eta_i^T(t) M_{2i}^T [x(t-d(t)) - x(t-h)] \]
\[ + \eta_i^T(t) d(t) M_{1i}^T R_{2i}^{-1} M_{1i} + (h - d(t)) M_{2i}^T R_{2i}^{-1} M_{2i} \eta_i(t) \]
\[ + \chi(dw(t)) \] (20)
where
\[ \chi(dw(t)) = -2 \eta_i^T(t) M_{1i}^T \int_{t-d(t)}^{t} g_i(s) dw(s) \]
\[ - 2 \eta_i^T(t) M_{2i}^T \int_{t-h}^{t} g_i(s) dw(s). \] (21)
According to (4) and (16)-(21), we have
\[ \mathcal{L} V(t, x, i) \]
\[ \leq 2 x^T(t) P_{x_i}(t) x(t) \sum_{j=1}^{K} \pi_{ij} P_{x_j}(t) + (Q_{1i} + Q + h Q)x(t) \]
\[ + \Delta \left[ g_i^T(t) P_{x_i}(t) \right] - x^T(t-h) Q x(t-h) \]
\[ - (1 - \mu) x^T(t-d(t)) Z x(t-d(t)) \]
\[ + h y_i^T(t) (R_{1i} + R_{2i}) y_i(t) - \int_{t-h}^{t} y_i^T(s) R_{1i} y_i(s) ds \]
\[ + \int_{t-h}^{t} y_i^T(s) R_{2i} y_i(s) ds \] (22)
Furthermore, in light of (3) and (5), there is
\[ \text{tr} \left[ g_i^T(t) P_{x_i}(t) \right] \leq \delta_i \text{tr} \left[ g_i^T(t) g_i(t) \right] \]
\[ \leq \delta_i (| G_{1ix}(t) |^2 + | G_{2ix}(t-d(t)) |^2). \] (23)
It follows from (2) and Lemma 3 that for any scalars $\varepsilon_i > 0$ ($i \in S$)

$$ | f_i(t)^2 | \leq | F_{1ix}(t)^2 | + | F_{2ix}(t - d(t))^2 |$$

$$+ 2 | F_{1ix}(t) | | F_{2ix}(t - d(t)) |$$

$$+ \varepsilon_i | F_{1ix}(t)^2 + \varepsilon_i^{-1} | F_{2ix}(t - d(t))^2 |$$

$$= (1 + \varepsilon_i) | F_{1ix}(t)^2 |$$

$$+ (1 + \varepsilon_i^{-1}) | F_{2ix}(t - d(t))^2 |$$

and hence

$$ - | f_i(t)^2 | + (1 + \varepsilon_i) | F_{1ix}(t)^2 |$$

$$+ (1 + \varepsilon_i^{-1}) | F_{2ix}(t - d(t))^2 | \geq 0.$$ (24)

Integrating both sides of (22) from ‘$t$’ to ‘$T$’, and observing $E\{X(dt)\} = 0$ and (15), (21)-(24), it’s easy to obtain

$$E\{dV(t_0, t_1)\} = E\{ZV(t_0, t_1)\}$$

$$\leq \frac{1}{h} E\{ \int_{t_0}^{t_1} [\varepsilon_T \Delta t, \varepsilon(s, t) + d(t) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t)$$

$$+ (h - d(t)) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t)] ds \}

$$\leq \frac{1}{h} E\{ \int_{t_0}^{t_1} [\varepsilon_T \Delta t, \varepsilon(s, t) + d(t) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t)$$

$$+ (h - d(t)) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t)] ds \}

$$= h(R_1 + R_2) P_i 0 0 0 0$$

$$\Lambda_{ii} = \begin{bmatrix} 0 & \Lambda_{1i} & 0 0 0 0 \\ \Lambda_{1i} & 0 0 0 0 0 \\ 0 & 0 0 0 0 0 \\ \Lambda_{1i} & 0 0 0 0 0 \\ 0 & 0 0 0 0 0 \\ \Lambda_{1i} & 0 0 0 0 0 \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_1 \eta_t(t) M_T R_2^{-1} M_t \eta_t(t) \\ (h - d(t)) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t) \end{bmatrix}$$

$$\neq 0$$

$M_1i = [0 M_1i N_1i 0 0 0]$,

$M_2i = [0 0 M_2i N_2i 0 0 0]$.

with

$$L_i = \sum_{j=1}^{K} \varepsilon_j P_j + Q_i + hQ + Z + (1 + \varepsilon_i) F_{1ij} F_{ii} + \delta G_{1i} \eta_t(t) + M_{1i} M_{1i}^T$$

$$L_{ii} = -L_{ii}^T N_{ii} N_{ii}$$

$$L_{ii} = -(1 - \mu)Z + (1 + \varepsilon_i^{-1}) F_{2ii} F_{ii} + \delta G_{2i} \eta_t(t)$$

$$- M_{2i} M_{2i}^T$$

$$L_{ii} = -L_{ii}^T N_{ii} N_{ii}$$

It’s clear from (25) that

$$E\{ \int_{t_0}^{t_1} \varepsilon_T \Delta \eta_t(t, s) \Delta t, \eta_t(t) M_T R_2^{-1} M_t \eta_t(t)$$

$$+ (h - d(t)) \eta_t(t) M_T R_2^{-1} M_t \eta_t(t) | ds \} \leq 0$$

then

$$E\{dV(t_0, t_1)\} = 0,$$ (26)

which implies that system (1) is mean-square asymptotically stable.

On the other hand, since the sum of the coefficients of the last two terms in (26) is equal to a constant ‘$h' > 0', of view of the convex analysis property, (26) is feasible if and only if

$$E\{ \int_{t_0}^{t_1} \varepsilon_T \Delta \eta_t(t, s) \Delta t, \eta_t(t) M_T R_2^{-1} M_t \eta_t(t) | ds \} < 0, k = 1, 2.$$ (27)

hold simultaneously, where

$$\Xi_i = \Lambda_i + h \cdot \mu \cdot M_i^{-1} \cdot \Lambda_i, k = 1, 2.$$ (28)

By virtue of (11) and Lemma 4, (27) are equivalent to

$$\Psi_{ii} = \Gamma_i R_{ii}^{-1} \Xi_i R_{ii}^{-1} \Psi_i = 0, k = 1, 2.$$ (28)

Consequently, $\Theta_{ii} < 0$ ($k = 1, 2$) can be obtained directly from (27) by Schur complements. The proof is thus completed. ■

Remark 6. By GFL, the inequalities (27) including vectors in the stochastic scope are equivalently rewritten as deterministic matrices inequalities (28). Thus, it is easy to check the mean-square stability of system (1) by solving the LMIs (4)-(6).

Remark 7. In Chen et al. (2008); Guan et al. (2005); Yue and Han (2005); He et al. (2010), the time-varying delay ‘$d(t)' is enlarged by its worst-case bounds, namely, the lower bound ‘$0' or the upper bound ‘$h'. It is obvious that this approximation is conservative. By using the property of convex analysis, the information of ‘$d(t)' can be reserved without estimating ‘$d(t)' by ‘$0' or ‘$h'. This idea has been applied for systems with time-varying delay in the deterministic scope (Park and Ko (2007); Shao (2009)). In Theorem 4, convex analysis has been successfully extended to systems with time-varying delay in the stochastic context.

Remark 8. The integral term $\mathcal{I}(t) \triangleq - \int_{t-h}^{t} \Psi_i(s) R_{ii}(s) ds$ in (18) is expressed as $\mathcal{I}(t) = - \int_{t-d(t)}^{t} \Psi_i(s) R_{ii}(s) ds - \int_{t-h}^{t} \Psi_i(s) R_{ii}(s) ds$. Subsequently, by virtue of equalities (19) and (20), $\mathcal{I}(t)$ has been estimated as (21). This disposal avoids bounding $\mathcal{I}(t)$ as $- \int_{t-d(t)}^{t} \Psi_i(s) R_{ii}(s) ds$ by Jensen’s inequality as Chen et al. (2008); Guan et al. (2005); He et al. (2010) etc. Moreover, the cross terms in $\mathcal{I}(t)$ satisfying $E\{X(dt)\} = 0$ (see Mao and Yuan (2006)) have not been bounded by Lemma 3 and isometric property as Zhang et al. (2009). Combining this technique with GFL and convex analysis property are useful to reduce the conservatism, which will be shown in Section 4 by illustrative examples.

Meanwhile, if the parametric perturbations $f_i(t)$ satisfy

$$f_i(t) = A_i x(t) + A_i x \varepsilon_i (t - d(t))$$ (29)

then system (1) becomes

$$dx(t) = [(A_i + A_i x) x(t) + (A_i + A_i x) (t - d(t))] dt$$

$$+ g_i(t) dw(t)$$ (30)

where $A_i$ and $A_i x$ are norm-bounded parametric uncertainties and can be described by

$$\Delta i, \Delta A_i = L_i F_i[t] E_{ii} E_{ii}$$ (31)

where $L_i$, $E_{ii}$, $E_{ii}$ are known real-valued matrices with compatible dimensions and $F_i(t)$ are unknown time-varying matrices satisfying $F_i^T(t) F_i(t) \leq 1, i \in S$. We then can obtain the following result for system (30).

Theorem 9. For given scalars $h > 0$, $\mu$, system (30) is mean-square asymptotically stable, if there exist scalars $\varepsilon_i > 0$, $\delta_i > 0$, and matrices $P_i > 0$, $Q_i > 0$, $Q_i > 0$, $R_i > 0$, $R_i > 0$, $M_i$, $N_i$ ($k = 1, 2, i \in S$) with appropriate dimensions satisfying (4), (5) and

$$\Theta_{ii} = \begin{bmatrix} \Gamma_i & h \varepsilon_i \eta_t(t) R_{ii}^{-1} \end{bmatrix} < 0, k = 1, 2$$ (32)

where $\Gamma_i = [M_i, N_i, 0 0 0 0]$, $\Gamma_{ii} = [0 M_i, N_i, 0 0 0]$, and
\[
\dot{\Gamma}_i = \begin{bmatrix}
\Gamma_1 & \Gamma_2 \\
\pi_i \Gamma_1 & 0 \\
0 & \Gamma_i \\
-h(r_1 + r_2) & h(r_1 + r_2)L_i \\
\end{bmatrix}
\]

with
\[
\Gamma_1 = P_{A_i} + A_i^T P_i + \sum_{i=1}^{N} \pi_i P_j + Q_i + hQ + Z - \frac{R_1}{h} + e_i E_1 E_1 + \delta_i G_i^T G_i + M_i + M_i^T \\
\Gamma_2 = P_{A_i} - M_i^T + N_i + e_i E_2 E_2 + \delta_i G_i^T G_i \\
\Gamma_3 = - (1 - \mu) Z + e_i E_2 E_2 + \delta_i G_i^T G_i \\
\Gamma_4 = - N_i - N_i^T + M_i + M_i^T.
\]

**Proof.** Denote
\[
y(t) = A_i x(t) + A_i(t) x(t - d(t)), i \in \mathcal{S}
\]
where
\[
A_i(t) = A_i + \Delta A_i, A_i(t) = A_i + \Delta A_i,
\]
then we obtain (10) and (11), where \(\xi_i(t, s) = [y_i^T(t) x_i^T(t) x_i^T(t - d(t)) x_i^T(t - h) y_i^T(s)]^T\), and
\[
\mathcal{B}_i = \begin{bmatrix}
-I & A_i(t) & A_i(t) & 0 & 0 \\
0 & -I & 0 & 1 & hI
\end{bmatrix}
\]

then we obtain (10) and (11), where \(\xi_i(t, s) = [y_i^T(t) x_i^T(t) x_i^T(t - d(t)) x_i^T(t - h) y_i^T(s)]^T\), and
\[
\mathcal{B}_i = \begin{bmatrix}
-I & A_i(t) & A_i(t) & 0 & 0 \\
0 & -I & 0 & 1 & hI
\end{bmatrix}
\]

Following the similar lines as in the proof of Theorem 1, we have
\[
\Psi_{ki} = \mathcal{B}_i^T \zeta_{ki} \mathcal{B}_i < 0, k = 1, 2
\]

where
\[
\zeta_{ki} = \Lambda_i + h \Lambda_i^T R_i^{-1} \Lambda_i
\]

with
\[
\Lambda_i = \begin{bmatrix}
h(r_1 + r_2) & P_i & 0 & 0 & 0 \\
* & A_i & 0 & 0 & 0 \\
* & * & A_i & 0 & 0 \\
* & * & * & A_i & 0 \\
* & * & * & * & -hR_1
\end{bmatrix}
- \text{diag}(0, (1 + \epsilon_1) F_1^T F_1, (1 + \epsilon_1) F_2^T F_2, 0, 0)
\]

\(\Lambda_{ki} = [0, M_i, N_i, 0, 0]
\)

\(\Lambda_{ki} = [0, 0, M_i, N_i, 0, 0]
\)

By employing GFL and following the routine technique to handle norm-bounded parametric uncertainties by Lemma 3 as Chen et al. (2010a); Gu et al. (2003), then the conclusion can be derived immediately. This completes the proof.

4. NUMERICAL EXAMPLES

This section will prove the validity of the method by some numerical examples.

**Example 1.** Consider the Markovian jumping system (30) with

<table>
<thead>
<tr>
<th>Methods</th>
<th>(\mu = 0.1)</th>
<th>(\mu = 0.5)</th>
<th>(\mu = 0.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xu et al. (2007)</td>
<td>2.0035</td>
<td>1.1603</td>
<td>0.4632</td>
</tr>
<tr>
<td>Theorem 9</td>
<td>2.0315</td>
<td>1.3407</td>
<td>0.8895</td>
</tr>
</tbody>
</table>

**Example 2.** Consider the stochastic system (1) with

\[
A_1 = \begin{bmatrix}
-2 & -1.5 \\
-1 & -2.8
\end{bmatrix}, A_{11} = \begin{bmatrix}
-0.5 & 1.2 \\
0 & -1.6
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
-1 & 0.8 \\
0.5 & -1.0
\end{bmatrix}, A_{12} = \begin{bmatrix}
1.0 & 0 \\
-0.1 & 0.8
\end{bmatrix},
\]

\[
L_1 = 0.2I, E_{11} = E_{21} = 0.1I, \\
L_2 = 0.1I, E_{12} = E_{22} = 0.2I,
\]

\[
\pi_{11} = 1, \pi_{22} = -2.
\]

The upper bounds of delay for system (37) by Xu et al. (2007) and Theorem 9 of this paper are provided in Table 1. It can be seen that the presented method is less conservative than Xu et al. (2007).

**Example 3.** Consider the stochastic system (30) with

\[
A = \begin{bmatrix}
-2 & 0 \\
1 & -1
\end{bmatrix}, A_1 = \begin{bmatrix}
-1 & 0 \\
-0.5 & -1
\end{bmatrix},
\]

\[
F_1 = F_2 = 0.1I, G_1 = G_2 = \sqrt{0.1I},
\]

This example has been widely studied in the literature. For constant delay, the maximal upper bound of delay by Gershon et al. (2007) is \(h_M = 1.56\). When the delay is time-varying, then the values of \(h_M\) by some other existing methods and Theorem 5 of this paper are listed in Table 2.

When \(\mu = 0.9\), system (38) is mean-square stable for any delay smaller than 1.2467 by Theorem 5 of this paper. It is 70% larger than the most recent result by He et al. (2010).

**Example 4.** Consider the stochastic system (30) with

\[
A = \begin{bmatrix}
-2 & 0 \\
1 & -1
\end{bmatrix}, A_1 = \begin{bmatrix}
-1 & 0 \\
-0.5 & -1
\end{bmatrix},
\]

\[
L = I, E_1 = E_2 = 0.1I, G_1 = G_2 = \sqrt{0.1I},
\]

When the method in Chen et al. (2008) and Theorem 9 in this paper are used to compute the maximal allowable delays of system (39), the obtained results are given in Table 3. It can be found that for time-varying delay the presented method shows more efficient than Chen et al. (2008). And the difference between Chen et al. (2008) and the presented method in this paper becomes more and more remarkable as \(\mu\) increases.
5. CONCLUSIONS

This paper has considered mean-square stability of stochastic Markovian jumping systems with time-varying delay. By using GFL and LKF method, new results have been presented via LMIs. The convex analysis property has been employed to avoid estimating the time-varying delay \( \dot{d}(t) \) by its upper or lower bounds such that the conservatism can be reduced. Illustrative examples have been provided to show the effectiveness of the method.

REFERENCES


