A Quasi-Linear Filter with Conditional Gaussian Sum Distributions for Nonlinear Dynamical Systems

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Abstract: In this paper, we present a new Gaussian sum filter, Alspach (1972), based on the stochastic equivalent linearization technique, Sunahara (1970), for suboptimal state estimation of discrete time nonlinear systems. The derived quasi-linear filter with Gaussian sum conditional PDFs is expected to be a low computational cost and high quality for recursive state estimation.

Keywords: nonlinear filters, recursive filters, stochastic equivalent linearization, quasi-linear filters, discrete-time stochastic systems, Gaussian sum distribution.

1. INTRODUCTION

The famous Kushner’s nonlinear filter for stochastic differential equations was developed in the late 1960, Kushner (1967); Jazwinski (1970). Recently, again, many attentions have been focusing on nonlinear filtering theory and its approximated filters, Sugimoto (2009a), such as the unscented Kalman filter, Julier (2004), the particle (or Monte Carlo) filter, Kitagawa (1996); Gersh (1997), and the Gaussian filter, Ito (2000); Arasaratnam and Haykin (2007). In this paper, we present a new Gaussian sum filter, Alspach (1972), based on the stochastic equivalent linearization technique, Sunahara (1970), for discrete time nonlinear systems.

Let us consider the following discrete-time nonlinear state space models.

\[ x_{t+1} = f_t(x_t) + w_t \]
\[ y_t = h_t(x_t) + v_t \]

where \( x_t \) is a \( n \)-dimensional state vector and \( y_t \) is \( m \)-dimensional observation vector. \( w_t \) and \( v_t \) are \( n \)-dimensional and \( m \)-dimensional Gaussian white noises, respectively. \( f_t(x_t) \) and \( h_t(x_t) \) are \( n \)-dimensional and \( m \)-dimensional vector-valued nonlinear functions. The means and covariance matrices of \( w_t \) and \( v_t \) are given by

\[ E[w_t] = 0, \quad E[v_t] = 0 \]

\[ E\left[\begin{bmatrix} w_t & v_t \\ w_s & v_s \end{bmatrix} \right] = \begin{bmatrix} Q_t & 0 \\ 0 & R_t \end{bmatrix} \delta_{t-s} \]

where \( \delta_t \) is Kronecker’s \( \delta \)-function and due to whiteness of noises, we have the relations:

\[ E[w_t x_s^T] = 0, \quad E[v_t x_s^T] = 0, \quad t \geq s. \]  

Also the initial state \( x_0 \) is independent of \( w_t \) and \( v_t \).

2. STOCHASTIC EQUIVALENT LINEARIZATION WITH GAUSSIAN SUM DISTRIBUTIONS

The stochastic equivalent linearization technique, Sunahara (1970), is applied to obtain the suboptimal linear estimates for continuous time systems by assuming the Gaussian conditional probability density functions (CPDFs):

\[ p(x_t|Y^{t-1}), p(x_t|Y^t) \]

We now extend the equivalent linearization technique to discrete time systems under the assumption of Gaussian sum CPDFs as follows:

\[ p(x_t|Y^{t-1}) = \sum_{i=1}^{r_t} \alpha_{i,t|t-1}(2\pi)^{-\frac{n}{2}} |P_{i,t|t-1}|^{-\frac{1}{2}} \]

\[ \times \exp \left\{ -\frac{1}{2} (x_t - \mu_{i,t|t-1})^T P_{i,t|t-1}^{-1} (x_t - \mu_{i,t|t-1}) \right\} , \]

\[ p(x_t|Y^t) = \sum_{i=1}^{r_t} \alpha_{i,t|t}(2\pi)^{-\frac{m}{2}} |P_{i,t|t}|^{-\frac{1}{2}} \]

\[ \times \exp \left\{ -\frac{1}{2} (x_t - \mu_{i,t|t})^T P_{i,t|t}^{-1} (x_t - \mu_{i,t|t}) \right\} . \]

Namely, these CPDFs are Gaussian sum distribution with means: \( \mu_{i,t|t-1}, \mu_{i,t|t} \): \( i = 1, \cdots, r_t \) and covariances: \( P_{i,t|t-1}, P_{i,t|t} \): \( i = 1, \cdots, r_t \), where \( \alpha_{i,t|t-1} \) and \( \alpha_{i,t|t} \) denote weighting factors of distributions with the property,
\[ \sum_{i=1}^{r} \alpha_{i,t} = 1; \quad \alpha_{i,t} \geq 0. \] Also we express \( Y^t = \{ y_0, \ldots, y_t \} \) as the observation data up to the time \( t \).

Here, we consider the linear approximation to the nonlinear functions: \( f_t(x_t), h_t(x_t) \) in (1) and (2) as

\[
\begin{align*}
    f_t(x_t) &= a_t + \sum_{i=1}^{r_t} B_{i,t}(x_t - \mu_{i,t[t]}) + e_t^f, \\
    h_t(x_t) &= c_t + \sum_{i=1}^{r_t} D_{i,t}(x_t - \mu_{i,t[t-1]}) + e_t^h,
\end{align*}
\]

where \( a_t \) and \( c_t \) are n-dimensional and \( m \)-dimensional vectors, respectively, and \( B_{i,t} \) and \( D_{i,t} \) are \( n \times n \) and \( m \times n \) matrices. Further, \( e_t^f \) and \( e_t^h \) denote the linear approximation errors.

Similar to the stochastic equivalent linearization, Suharara (1970), here we assume that the linear approximations are unbiased, namely

\[
E \{ f_t(x_t) | Y^t \} = E \{ a_t + \sum_{i=1}^{r_t} B_{i,t}(x_t - \mu_{i,t[t]} | Y^t \} \}
\]

\[
E \{ h_t(x_t) | Y^{t-1} \} = E \{ c_t + \sum_{i=1}^{r_t} D_{i,t}(x_t - \mu_{i,t[t-1]} | Y^{t-1} \} \}
\]

Furthermore, we consider the following optimization problem for minimizing square error of linear approximation. Then \( a_t, B_{i,t}, c_t \) and \( D_{i,t} \) in (7) and (8) are obtained by minimizing the conditional expectation of the square errors norms of \( e_t^f \) and \( e_t^h \):

\[
E \{ \| e_t^f \|^2 | Y^t \} = J(a_t, \{ B_{i,t} \})
\]

\[
E \{ \| e_t^h \|^2 | Y^{t-1} \} = E \{ h_t(x_t) - c_t - \sum_{i=1}^{r_t} D_{i,t}(x_t - \mu_{i,t[t-1]} | Y^{t-1} \} \}
\]

Then we compute (11) as follows:

\[
J(a_t, \{ B_{i,t} \}) = E \{ \| f_t(x_t) - a_t - \sum_{i=1}^{r_t} B_{i,t}(x_t - \mu_{i,t[t]} | Y^t \} \}
\]

\[
= \text{trace} \left\{ E \left[ \left( f_t(x_t) - a_t - \sum_{j=1}^{r_t} B_{j,t}(x_t - \mu_{j,t[t]} \right)^T | Y^t \right) \right] \right. \]

\[
= \text{trace} \left\{ E \left[ (f_t(x_t) - a_t)(f_t(x_t) - a_t)^T \right] + \sum_{i=1}^{r_t} \sum_{j=1}^{r_t} B_{i,t}(x_t - \mu_{i,t[t]} | Y^t \} \right. \]

\[
- \sum_{j=1}^{r_t} \left( f_t(x_t) - a_t \right)(x_t - \mu_{j,t[t]} | Y^t \} B_{j,t}^T
\]

\[
- \sum_{i=1}^{r_t} B_{i,t}(x_t - \mu_{i,t[t]} | Y^t \} \left[ f_t(x_t) - a_t \right]^T \}
\]

Then from (18), we have the relation, for \( i = 1 \ldots r \)}
\[ \Gamma_{i,t} = E \left\{ (f_t(x_t) - a_t)(x_t - \mu_{i,t|t-1})^T | Y_t \right\} = E \left\{ (f_t(x_t) - a_t)x_t^T | Y_t \right\} = E \left\{ f_t(x_t)x_t^T | Y_t \right\} - a_t \]

Similarly, \( c_t, D_t \) are obtained by

\[ c_t = E \left\{ h_t(x_t) | Y_{t-1} \right\} = \tilde{h}_{t|t-1}(x_t) \]

\[ D_t \left[ H_t \left| (\hat{X}_{t|t-1} - \mu_{t|t-1}) \right] = [\Phi_t] \Phi_{0,m \times 1} \right] \]

where

\[ D_t = \begin{bmatrix} D_{1,t} & D_{2,t} & \cdots & D_{r,t} \end{bmatrix} ; m \times nr \text{ matrix} \]

\[ H_t = \begin{bmatrix} H_{11,t} & \cdots & H_{1r,t} \\ \vdots & \vdots & \vdots \\ H_{r1,t} & \cdots & H_{rr,t} \end{bmatrix} ; nr \times nr \text{ matrix} \]

\[ \Phi_t = [\Phi_0, \Phi_{0,1} \cdots \Phi_{0,r}] ; m \times nr \text{ matrix} \]

\[ H_{ji,t} \equiv E \left\{ (x_t - \mu_{ji,t|t-1})(x_t - \mu_{ji,t|t-1})^T | Y_{t-1} \right\} \]

\[ \Phi_{0,t} \equiv E \left\{ h_t(x_t)x_t^T | Y_{t-1} \right\} - c_t \tilde{x}_{t|t-1}. \]

Therefore from (22) – (29), the nonlinear state space model in (1), (2) are approximated by the quasi-linear equations:

\[ x_{t+1} = \sum_{i=1}^{r_t} B_{i,t}x_t + (a_t - \sum_{i=1}^{r_t} B_{i,t}\mu_{i,t|t-1}) + w_t \]

\[ y_t = \sum_{i=1}^{r_t} D_{i,t}x_t + (c_t - \sum_{i=1}^{r_t} D_{i,t}\mu_{i,t|t-1}) + v_t. \]

Then we define the new observation \( \tilde{y}_t \) as

\[ \tilde{y}_t \equiv y_t - (c_t - \sum_{i=1}^{r_t} D_{i,t}\mu_{i,t|t-1}) = \sum_{i=1}^{r_t} D_{i,t}x_t + v_t \]

or we can write

\[ \tilde{y}_t \equiv y_t - c_{S,t} = D_{S,t}x_t + v_t \]

\[ c_{S,t} \equiv c_t - \sum_{i=1}^{r_t} D_{i,t}\mu_{i,t|t-1} \]

\[ D_{S,t} \equiv \sum_{i=1}^{r_t} D_{i,t} : m \times n. \]

### 3. A QUASI-LINEAR FILTER

We now present a new and promising nonlinear filter by combining the Gaussian sum filter and the stochastic linearization technique. Similar to the derivation of the Gaussian sum filter, applying the Bayesian rule to the conditional PDF \( p(x_t | Y^t) \) as follows, Ho and Lee (1964):

\[ p(x_t | Y^t) = \frac{p(x_t, Y^{t-1}, y_t)}{p(Y^t)} = \frac{p(y_t | x_t)p(x_t | Y^{t-1})}{p(y_t | Y^{t-1})} \frac{p(Y^{t-1})}{p(Y^{t-1})} = \frac{p(y_t | x_t)p(x_t | Y^{t-1})}{p(y_t | Y^{t-1})} = \frac{p(y_t | x_t)p(x_t | Y^{t-1})}{p(y_t | Y^{t-1})} \frac{1}{p(y_t | x_t)} \]

\[ = \sum_{i=1}^{r_t} \alpha_{i,t|t-1} N(x_t : \mu_{i,t|t-1}, P_{i,t|t-1})p(y_t - h_t(x_t)) \int \sum_{i=1}^{r_t} \alpha_{i,t|t-1} N(x_t : \mu_{i,t|t-1}, P_{i,t|t-1})p(y_t - h_t(x_t))dx_t \]

where

\[ \xi_t = y_t - \sum_{i=1}^{r_t} D_{i,t}x_t - (c_t - \sum_{i=1}^{r_t} D_{i,t}\mu_{i,t|t-1}) \equiv y_t - D_{S,t}x_t - c_{S,t}. \]

For evaluating (41), we use the relation:

\[ N(x_t : \mu_{i,t|t-1}, P_{i,t|t-1}) N(\xi_t : 0, R_t) \]

\[ = e^{-\frac{1}{2} \left[ (x_t - \mu_{i,t|t-1})^T P_{i,t|t-1}^{-1} (x_t - \mu_{i,t|t-1}) + \xi_t^T R_t^{-1} \xi_t \right]} \]

\[ = (y_t - c_{S,t} - D_{S,t}x_t)^T (R_t + D_{S,t}^T D_{S,t})^{-1} (y_t - c_{S,t} - D_{S,t}x_t). \]
where we have applied the matrix inversion lemma
\[ R^{-1} - R^{-1} D_S W_i D_S^T R^{-1} = (R + D_S P_i D_S^T)^{-1}, \]
and the relations:
\[
\begin{align*}
(R_1 - R_1 D_S W_i D_S^T R_1^{-1})^{-1} &= R_1^{-1} - R_1^{-1} D_S W_i D_S^T R_1^{-1} \\
(R_1 - R_1 D_S W_i D_S^T R_1^{-1})^{-1} &= (R + D_S P_i D_S^T R_1^{-1})^{-1} \\
(D_1 + D_1 R_1 - D_1 D_1^T) &= D_1,
\end{align*}
\]
\[ P_i^{-1} = P_i^{-1} - P_i^{-1} D_S W_i D_S^T R_1^{-1} D_S P_i^{-1} \\
= P_i^{-1} - P_i^{-1} (P_i^{-1} + D_S^T R_1^{-1} D_S) P_i^{-1} \\
= P_i^{-1} - P_i^{-1} (P_i^{-1} + D_S^T R_1^{-1} D_S) P_i^{-1} D_S^T R_1^{-1} D_S \\
= P_i^{-1} P_i^{-1} (P_i^{-1} + D_S^T R_1^{-1} D_S)^{-1} D_S^T R_1^{-1} D_S \\
\]
\[ = P_i^{-1} = P_i^{-1} D_S^T R_1^{-1} D_S \]
(49)

Therefore, from (43), (44) and (47), we have the relation:
\[
N(x_t: \mu_{i,t|t-1}, P_{i,t|t-1}) N(x_t: \mu_{i,t}, \Sigma_t) = k_{i,t} \times N(x_t: (P_{i,t|t-1} + D_S^T R_1^{-1} D_S)^{-1} (P_{i,t|t-1} - P_{i,t|t-1} D_S^T R_1^{-1} D_S), \Sigma_t)
\]
(51)

where
\[
k_{i,t} \equiv \frac{|P_{i,t|t-1} + D_S^T R_1^{-1} D_S|^{1/2}|R_1 + D_S P_i^{-1} D_S^T|^{1/2}}{|P_{i,t|t-1}|^{1/2}|R_1|^{1/2}}
\]

Here we assume the Gaussian sum form for (41):
\[
p(x_t|Y^t) = \sum_{i=1}^{r_t} \alpha_{i,t} N(x_t: \mu_{i,t|t}, P_{i,t|t})
\]
then, from (41), (51), and the matrix inversion lemma, we have
\[
\begin{align*}
\mu_{i,t|t} &= (P_{i,t|t-1} + D_S^T R_1^{-1} D_S)^{-1} \\
\times \left[ P_{i,t|t-1} \mu_{i,t|t-1} + (P_{i,t|t-1} + D_S^T R_1^{-1} D_S)^{-1} D_S^T R_1^{-1} (y_t - c_S) \right] \\
\end{align*}
\]
\[
\begin{align*}
\mu_{i,t|t} &= \mu_{i,t} \pm \frac{1}{2} \left( \frac{1}{P_{i,t|t-1} + D_S^T R_1^{-1} D_S} \right) \left( \frac{1}{D_S^T R_1^{-1} D_S} \right) \\
\end{align*}
\]

Also we have the relation:
\[
P_{i,t|t} = (P_{i,t|t-1} + D_S^T R_1^{-1} D_S)^{-1}
\]

Then finally we have the following equations:
\[
\begin{align*}
\mu_{i,t} &= \mu_{i,t} \pm \frac{1}{2} \left( \frac{1}{P_{i,t|t-1} + D_S^T R_1^{-1} D_S} \right) \left( \frac{1}{D_S^T R_1^{-1} D_S} \right) \\
\end{align*}
\]

For evaluating the above integral, Sugimoto (2009b), we can show the relations:
\[
\begin{align*}
\alpha_{i,t} \equiv \frac{\alpha_{i,t|t-1}}{\sum_{j=1}^{r_{i,t}} \alpha_{j,t|t-1} \beta_{j,t}} \\
\beta_{j,t} \equiv k_{j,t} N(y_t: c_S, D_S, \mu_{j,t|t-1}, D_S, P_{j,t|t-1} D_S^T + R) \\
\end{align*}
\]
(57)

\[
\begin{align*}
\alpha_{i,t} \equiv \frac{\alpha_{i,t|t-1}}{\sum_{j=1}^{r_{i,t}} \alpha_{j,t|t-1} \beta_{j,t}} \\
\beta_{j,t} \equiv k_{j,t} N(y_t: c_S, D_S, \mu_{j,t|t-1}, D_S, P_{j,t|t-1} D_S^T + R) \\
\end{align*}
\]
(58)

Therefore the filtered estimate can be computed by
\[
\hat{x}_{t|t} = E[x_t|Y^t] = \sum_{t=1}^{r_t} \alpha_{i,t} N(x_t: \mu_{i,t|t}, P_{i,t|t}) dx_t
\]
(59)

Also the filtered error covariance matrix \( P_{i|t} \) is given by
\[
P_{i|t} \equiv E \left\{ (x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T \right\}
\]
(60)

Furthermore, if the system noise \( w_t \) has the Gaussian sum distribution, Alspach (1972), namely, assume
\[
p(w_t) = \sum_{i=1}^{q_t} \gamma_{i,t} N(w_t: \eta_{i,t}, Q_{i,t})
\]
(61)

then applying (61), the conditional PDF is given by
\[
p(x_{t+1}|x_t) = \frac{p(x_{t+1})}{p(x_t)}
\]
(62)

Then the conditional PDF of the one-step ahead predictor is given by
\[
p(x_{t+1}|Y^t) = \int p(x_{t+1}|x_t)p(x_t|Y^t)dx_t
\]
(63)

For evaluating the above integral, Sugimoto (2009b), we can show the relations:
\[ p(x_{t+1}|Y^t) = \sum_{k=1}^{r_{t+1}} \alpha_{k,t+1|t} N(x_{t+1} : \mu_{k,t+1|t}, P_{k,t+1|t}), \]

where
\[ r_{t+1} = r_t q_t \equiv r \]
\[ \alpha_{k,t+1|t} = \alpha_{j,t} q_{j,t} \]
\[ \mu_{k,t+1|t} = a_{S,t} + \eta_t + B_{S,t} \mu_j \]
\[ P_{k,t+1|t} = B_{S,t} P_{j,t} B_{S,t}^T + Q_{t,t}. \]

The scheme to suppress the increase of the terms in the Gaussian sum distribution is, to resemble several similar terms into one term, to neglect the term when the weighting factor \( \alpha_i \) is sufficiently small: \( \alpha_i \approx 0 \), and so on.

For the simplest case of the system noise \( w_t \) with assuming \( q_t = 1, \eta_t = 0 \), namely
\[ p(w_t) = N(w_t : 0, q_t), \]
then the time-updating formulae in (64) - (67) become
\[ r_{t+1} = r_t \]
\[ \alpha_{i,t+1|t} = \alpha_{i,t} \]
\[ \mu_{i,t+1|t} = a_{S,t} + B_{S,t} \mu_i \]
\[ P_{i,t+1|t} = B_{S,t} P_{i,t} B_{S,t}^T + Q_{t,t}. \]

Therefore the quasi-linear filter with Gaussian sum CPDFs is summarized as the following formula of updating:

**Measurement updating formula**
\[ \hat{x}_{i|t} = \sum_{i=1}^{r} \alpha_{i,t} \mu_{i,t|t}, \]
with (55)-(58).

**Time updating formula**
\[ \hat{x}_{i+1|t} = \sum_{i=1}^{r} \alpha_{i+1,t} \mu_{i+1,t|t}, \]
with (64)-(67) (or with (69)-(72)).

However we need to obtain \( \alpha_{i,t}, \{ B_{i,t}, c_t, \{ D_{i,t}\} \) such that we evaluate the values of
\[ E[f_t(x_t)|Y^t], \quad E[f_t(x_t)x_t^T|Y^t] \]
\[ E[h_t(x_t)|Y^{t-1}], \quad E[h_t(x_t)x_t^T|Y^{t-1}] \]
(75) in (22), (23), (28) and (29).

**Evaluation of Expectation**

Our equivalent linearization method requires to evaluate the expectation of (75) under the assumption of the conditional Gaussian (or, Gaussian sum) PDF. For this purpose, we show a method to evaluate the higher moments by applying the moment generating function as follows.

The CPDF for \( n \)-dimensional Gaussian random vector \( x = [x_1, x_2 \cdots x_n]^T \) is described by
\[ p(x|Y) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^\frac{1}{2}} \times \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right\}, \]
where \( \mu \) and \( \Sigma \) are conditional mean and conditional covariance matrix, respectively, as follows

\[ \mu = E[x|Y] \]
\[ \Sigma = E[(x - \mu)(x - \mu)^T|Y] = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}. \]

Let \( s = [s_1 \ s_2 \ \cdots \ s_n]^T \), then it is well known that the moment generating function for the multivariate normal distribution is given by
\[ M(s) = E[e^{s^T X|Y}] = \exp \left\{ s^T \mu + \frac{1}{2} s^T \Sigma s \right\} \]
(78)

For example, we can evaluate the conditional expectation of a third moment \( E[x_{i_1} x_{i_2} x_{i_3}|Y] \) as follows. Namely, applying the relation,
\[ \frac{\partial^3 M(s)}{\partial s_{i_1} \partial s_{i_2} \partial s_{i_3}} \bigg|_{s=0} = E[x_{i_1} x_{i_2} x_{i_3}|Y]. \]
(79)

Thus, we have
\[ E[x_{i_1} x_{i_2} x_{i_3}|Y] = \mu_{i_1} \sigma_{j_{i_2} j_{i_3}} + \mu_{i_2} \sigma_{j_{i_1} j_{i_3}} + \mu_{i_3} \sigma_{j_{i_1} j_{i_2}} + \mu_{j_{i_1} j_{i_2} j_{i_3}} \]
(80)

By the similar computation, we can show
\[ E[x_{i_1} x_{i_2} x_{i_3} x_{i_4}|Y] = \sigma_{j_{i_1} j_{i_2}} \sigma_{j_{i_3} j_{i_4}} + \sigma_{j_{i_1} j_{i_3}} \sigma_{j_{i_2} j_{i_4}} + \sigma_{j_{i_2} j_{i_1}} \sigma_{j_{i_3} j_{i_4}} + \sigma_{j_{i_2} j_{i_3}} \sigma_{j_{i_1} j_{i_4}} + \sigma_{j_{i_3} j_{i_1}} \sigma_{j_{i_2} j_{i_4}} + \sigma_{j_{i_3} j_{i_2}} \sigma_{j_{i_1} j_{i_4}} + \sigma_{j_{i_3} j_{i_4}} \sigma_{j_{i_1} j_{i_2}} + \sigma_{j_{i_2} j_{i_3}} + \sigma_{j_{i_1} j_{i_4}} \sigma_{j_{i_2} j_{i_3}} + \sigma_{j_{i_1} j_{i_2}} \sigma_{j_{i_3} j_{i_4}} + \sigma_{j_{i_1} j_{i_4}} \sigma_{j_{i_2} j_{i_3}} + \sigma_{j_{i_2} j_{i_4}} \sigma_{j_{i_1} j_{i_3}} + \sigma_{j_{i_2} j_{i_3}} \sigma_{j_{i_1} j_{i_4}}. \]
(81)

Thus we can evaluate the conditional expectation of the polynomials of \( x_t \) explicitly for the nonlinear functions appeared in (75).

The conditional expectation of the other nonlinear functions of \( x_t \) in (75) can be numerically computed by applying Gauss-Hermite quadrature, Ito (2000); Arabasaratnam and Haykin (2007), or by applying the unscented transformation, Julier (2004).

**4. NUMERICAL EXPERIMENTS**

The parameter estimation problem for noisy 2nd order Auto-Regression Models (AR(2)-models) is considered by applying 4 types of nonlinear filtering methods; the extended Kalman filter (EKF), the unscented Kalman filter (UKF), the quasi-linear filter (QLF), and the proposed Gaussian sum quasi-linear filter (GSQLF).

Let us consider noisy AR(2)-models \( y_t \) as follows:
\[ z_{t+1} = a_1 z_t + a_2 z_{t-1} + w_t, \]
\[ y_t = z_t + v_t, \]
where \( E[w_t] = E[v_t] = 0, \) \( \text{Var}[w_t] = q_w, \) \( \text{Var}[v_t] = r_v. \) Let \( x_{1,t} \equiv z_t, \quad x_{2,t} \equiv z_{t-1}, \quad x_{3,t} \equiv a_1, \quad x_{4,t} \equiv a_2. \) (82)
then we have the following nonlinear state equation.
\[ x_{1,t+1} = x_{3,t} + x_{1,t} + w_{t} \]
\[ x_{2,t+1} = x_{1,t} + w_{t} \]
\[ x_{3,t+1} = x_{3,t} \]
\[ x_{4,t+1} = x_{4,t} \]
\[ y_t = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (83) \]

The filtered estimates of \( a_1 (\equiv x_2) = -1.6 \) and \( a_2 (\equiv x_4) = -0.89 \) are shown in Figs. 1 and 2, respectively, where \( q_w = 5 \) and \( r_v = 40 \). In Tables 3 and 4, the root mean square (RMS) errors of \( a_1 \) and \( a_2 \) are shown for \( r_v = 0.1, 1, 10, 20, 40 \). Namely, the following quantities are listed in Tables 1 and 2,

\[
\sqrt{\frac{1}{250} \sum_{t=1}^{250} (a_1 - \hat{x}_{3,t(t)})^2}, \quad \sqrt{\frac{1}{250} \sum_{t=1}^{250} (a_2 - \hat{x}_{4,t(t)})^2}. \quad (84)
\]

The S/N ratios are lower \( (r_v \text{ are larger}) \), QLF and GSQLF show superior estimation results than EKF and UKF methods, where \( r_v \equiv 5 \) assumed in GSQLF.

![Fig. 1. Estimates of \( a_1 \)](image1)

![Fig. 2. Estimates of \( a_2 \)](image2)

Table 1. RMS errors of \( a_1 \)

<table>
<thead>
<tr>
<th>( r_v )</th>
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<th>UKF</th>
<th>QLF</th>
<th>GSQLF</th>
</tr>
</thead>
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<td>0.1</td>
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<td>0.309</td>
<td>0.309</td>
<td>0.304</td>
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<td>40</td>
<td>1.018</td>
<td>0.867</td>
<td>0.783</td>
<td>0.729</td>
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</tbody>
</table>

5. CONCLUSIONS

We have presented a new quasi-linear filter based on the stochastic equivalent linearization with Gaussian sum CPDFs for discrete time nonlinear systems. The new quasi-linear filter will be very practical and useful in the aspects from accuracy of estimation and from the computational cost. Numerical experimental results are shown; for (Example 1) AR(2) parameter estimation problems by using noisy AR measurement processes where the AR coefficients are assumed as state variables and (Example 2) The experimental results for practical and important real problems occurred in GNSS/INS (Global Navigation Satellite System/Inertial Navigation System) integrated algorithm, Sugimoto (2009a), is not shown in this paper, due to the limited space of the paper, and which will be shown in other occasions.

REFERENCES


